U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

SOME EXPERIMENTAL GAMES

Merrill M. Flood

RM-789-1

Revised 20 June 1952

Assigned to ____________________

The RAND Corporation

1700 MAIN ST. • SANTA MONICA • CALIFORNIA

46

Summary: Results are reported of a few experimental games conducted over the past few years to test the applicability and usefulness of the axiomatic structures developed by von Neumann-Morgenstern and by others.

SOME EXPERIMENTAL GAMES

Merrill M. Flood

1. Introduction

The non-constant-sum case of the theory of games [1]* remains incomplete. Several authors have considered the problem of extending the initial theory. Some very interesting contributions have been made recently [2,4,5]. The approach has been to add new axioms, and to modify old ones, in an effort to obtain a set that is at once quite acceptable on a priori grounds of reasonableness and also strong enough to determine each player's moves. This effort has not yet been very successful.

It has too often been forgotten that a theory is simply something to be tested experimentally, and to be rejected for any application where it fails to fit the conditions. Of course, it may be accepted tentatively if it appears to work well in many cases and to fail in none.** The theory of games has not really been put to this test even in the two-person

---

* Bracketed numerals refer to the Bibliography.
** These epistemological remarks are really sheer nonsense in light of the vast literature on philosophy of science and on the scientific method. I include them as a hint at my own point of view.

constant-sum case. The theory itself is entirely acceptable, as such, since it is mathematically rigorous and non-contradictory, but the question remains concerning its applicability.

The two-person zero-sum case seems to be useful in understanding certain parlor games. Even here there is room for doubt since the theory neither predicts the outcome of a chosen method of play, nor describes how persons will actually play, in even the simplest games. For example, in matching pennies the theory argues that a player can ensure zero-expectancy if he chooses his face each time by a random device—and I believe this to be the case—but it does not predict the outcome if a player chooses his strategy somehow in terms of his observations of past plays of his opponents. I am confident that a better theory for the two-person zero-sum case will offer a solution for this dynamic problem good enough for me to use at least against a child of average intelligence.* After all, one pays a good deal of attention in real life conflict situations to the identification of flaws in an opponent's habits of action that can be turned to one's advantage in subsequent plays. Even in two-person constant-sum games with perfect information, where the present theory offers any one of many pure strategies as equally good, there is this same deficiency since in real life some pure strategies will bring forth error on the part of an opponent more often than will others—checkers would almost always end in a draw between

* Some of my friends insist that they can do well against their children in matching pennies, or its variants, and others have relatively little success and claim that their children go quickly to a random method of play; I have not yet found a careful experimental study reported in the literature but suspect that such does exist.

experts if they were careless about this feature of play. None of these remarks is intended to detract from the great value of two-person constant-sum game theory; they are made in order to show that the theory as it stands is decidedly limited in it's usefulness and is liable to extension, refinement, and improvement.

The utility concept, as it enters into game theory, can be criticized on the basis that suitable operational measures often cannot be found in real life applications. Indeed, in most cases, it is far more difficult to construct an acceptable accounting procedure for recording utilities in the attempted application than it would be to find a reasonably good strategy in the absence of game theory. Quite often, in fact, the uncertainty about the measure of utility is so great that it dominates any differences that might be met in choosing between available strategies. Again, these remarks are not intended to detract from the great value of game theory but are made in order to show that there is room for great improvement as regards the applicability of the theory in real situations.

I have long felt that the axiomatic structures developed by von Neumann-Morgenstern, and by others, should be tested for applicability and usefulness in controlled experimental situations—and I have called such activity "experimental games." One result of such an activity should be a clarification of the need and direction for further modifications of the axiomatic systems. In this memorandum, I report the results of a few experimental games that I have conducted over the past few years.* Although this is the first time that I have written about these scattered experiments, I have used them in many meetings to illustrate one or another point about game theory. So much interest has been shown in them that I am encouraged to report them here, in spite of their limited extent and uncertain conclusions.

---

* Unfortunately, I usually kept no written record of the experiments so I reproduce them as faithfully here as I can from memory. Although the numerical and descriptive details are sometimes improvised, I believe that the main results and impressions are stated with sufficient accuracy to be useful.

** I take this opportunity to express my appreciation for many helpful discussions of game-theoretic points with John von Neumann, L. J. Savage, M. Dresher, and J. W. T. Youngs. I am also indebted to John Nash, L. S. Shapley, R. M. Thrall, M. Dresher, and D. R. Fulkerson for comments on a preliminary draft of this paper.

2. Buick (June 1949). A RAND employee (HK) was moving with his family to the East and decided to sell his Buick. MF was in the market for such a car so HK and MF had the usual bargaining problem on their hands. In its simplest form, the bargaining problem is essentially a non-constant sum two-person game. I shall not attempt to formulate this entire situation as a game but will now state an approximation to it.

The value to HK was the "best" price he could get. He could take the trouble to get offers from several used-car dealers, he could advertise in the local newspapers, and he could try various other means of disposing of his car. At the time he and I talked, he had somehow to make an estimate of the price he could get, discounted for each method by the nuisance value; the result would be some number of dollars $\bar{s}$. Actually, $\bar{s}$ is an estimate of some quantity s that might well be considered to have a probability distribution function $S(s)$. The value to MF was similarly the lowest price he would have to pay for such a car; there were exactly the same kind of difficulties in estimating this quantity $\bar{p}$ and its probability distribution function $P(p)$.

Neither party relished the task of estimating his price; both attached some additional value to completion of the deal without need for further searching. The matter was resolved by agreeing that a used-car dealer, who was well-known to both parties, would be asked in confidence to state his selling and buying price for the car in "as-is" condition, and that MF would then pay HK an amount intermediate between these extremes. This amounted to acceptance of the dealer's estimates for $\bar{p}$ and $\bar{s}$. It is assumed arbitrarily that the estimated nuisance value attached to further searching was the same quantity $\bar{d}$ for both parties.

If the deal were closed at an actual price a then the gain over both dealing with others would be:

$a-\bar{s}+\bar{d}$ for HK, and

$\bar{p}-a+\bar{d}$ for MF.

In the spirit of game theory, it would be accepted that no such transaction would occur if either party could do better by another action, whence:

$$a-\bar{s}+\bar{d} \geq 0, \quad \bar{p}-a+\bar{d} \geq 0.$$

The joint gain by dealing is $g = \bar{p}-\bar{s}+2\bar{d}$

In order to make this situation look like a formal game it is necessary to state the moves and payoffs. One way of doing this is to specify payoff matrices S and P defined for HK and MF as follows:

$$P = \left\| \begin{matrix} g & g & \cdots & g & g \\ 0 & g-1 & \cdots & g-1 & g-1 \\ 0 & 0 & \cdots & g-2 & g-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{matrix} \right\|, \quad S = \left\| \begin{matrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 2 & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & g-1 & g-1 \\ 0 & 0 & \cdots & 0 & g \end{matrix} \right\|$$

Rules of the game might be that HK and MF independently choose a non-negative integer not greater than g, that ownership of the car does not or does then pass from HK to MF according as the choice i (on rows) by HK is or is not greater than the choice j (on columns) by MF, and finally that MF pays to HK the amount $a = \bar{p}+\bar{d}+i-g$ if and only if $i \leq j$. There is nothing in the rules to prevent HK and MF from making a side agreement concerning their choices in the play of the formal game; they can also agree to make side payments if they wish.

It is not really necessary to state the formal game in finite form in terms of an indivisible unit of utility, as was done in writing S and P. Instead, the payoff functions could be written in terms of real variables x and y, as follows:

$$S(x,y) = \begin{cases} g-x & 0 \leq x \leq y \leq g \\ 0 & 0 \leq y < x \leq g \end{cases} ,$$

$$P(x,y) = \begin{cases} x & 0 \leq x \leq y \leq g \\ 0 & 0 \leq y < x \leq g \end{cases} .$$

If the solution requires a transaction involving some non-integral amount, when expressed in terms of the utility unit, this can be achieved by a probabilistic method yielding the correct expectation. For present purposes, either the finite or the continuous form is suitable.

Under game theory, the players would form a coalition and choose some pair of values for i and j that would maximize their joint return, and at the same time would agree on some side payment to be made from one to the other outside the formal rules of the game. Since all pairs with $i \leq j$ are equally satisfactory, we will suppose that $i = j = 0$ is chosen. But we are left unable to complete the transaction on the basis of game theory because it does not specify the amount of the side payment.*

I have resolved this dilemma in the two-person non-constant-sum case by adding what might be called "the split-the-difference principle." In general, if A^1 and A^2 are any two payoff matrices the joint payoff is the largest element in $J = A^1+A^2$, and if J_{rc} is such a maximal element then Player 1 chooses strategy r and Player 2 chooses strategy c under the coalition agreement. On the other hand, if Player 2 refuses the coalition he can guarantee himself a payoff $\bar{V}(A^2)$,

---

* The two-person market is discussed by von Neumann and Morgenstern in Section 61, pp. 555-566 [1]; they conclude only that the side payment will be some non-negative amount not greater than g.

where $V(A^2)$ denotes the value to Player 2 (on columns) of the zero-sum two-person game with payoff matrix A^2; similarly Player 1 (on rows) can guarantee himself a payoff $V(A^1)$. The joint gain to the two players by forming a coalition is therefore $G = J_{rc} - V(A^1) - V(A^2) \geq 0$. The split-the-difference principle states that G will be divided equally between the two players; this is accomplished by cooperatively choosing strategy (r,c) and by a side payment from Player 2 to Player 1 in the amount*

$$1/2\left[A^2_{rc} - V(A^2) - A^1_{rc} + V(A^1)\right].$$

It is easily verified that this general solution reduces to that given for the Buick transaction between HK and MF, when S and P are identified with A^1 and A^2, respectively. In this case

$$r = c = 0,\ V(A^1) = V(A^2) = 0,\quad A^1_{rc} = 0,\quad A^2_{rc} = g,$$

and so the side payment from MF to HK is $g/2$. The net result is that HK receives:

a	$+\ g/2$	$-\ (\bar{s}-\bar{d})$	$=\ g/2.$
price	side payment	Buick	profit

Similarly, MF receives:

$(\bar{p}+\bar{d})$	$g/2$	$-\ a$	$=\ g/2.$
Buick	side payment	price	profit

---

* This is the same as the methods proposed by Shapley [2b] and by Nash [4a]. It is also the same as choosing the middle of the interval given by von Neumann-Morgenstern [1].

This illustrates the main point, namely an even division of the joint profit due to successful completion of the transaction.

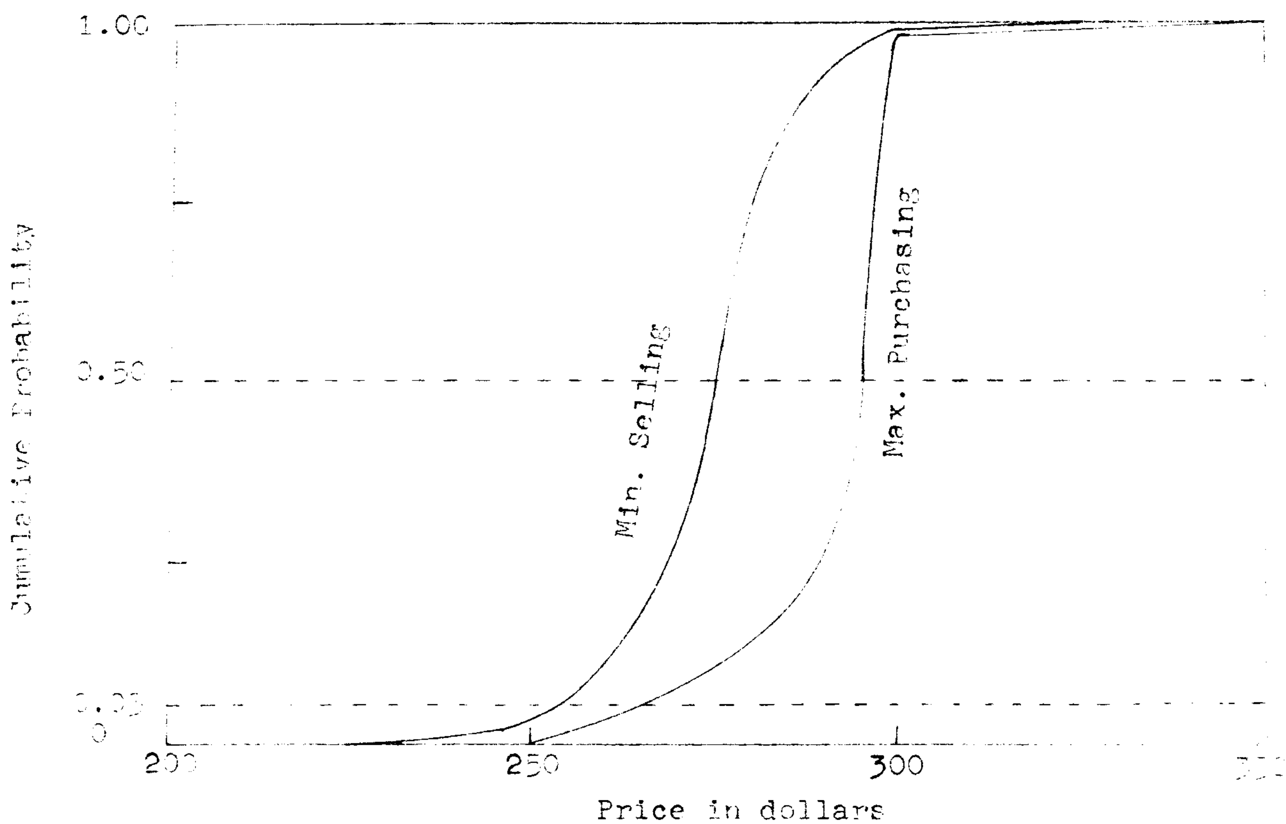
Unfortunately, the transaction was never actually completed because HK decided to drive the Buick East. It was quite apparent, nevertheless, that HK and MF were both content to accept the split-the-difference principle in this case where the personal utilities could be estimated so closely.

3. <u>Oldsmobile</u> (August 1949). A RAND Consultant (TA) was returning to the East after a summer in Santa Monica, and offered his Oldsmobile for sale. This was strictly a "transportation car" that he had purchased in California for summer use only. The game-theoretical problem is the same as that for the Buick, discussed in §2, but the emphasis here is on the estimation of the probability distributions $P(p)$ and $S(s)$ and on their rôle in the dynamic bargaining process. The entire theoretical problem was discussed frankly between MF and TA before the bargaining process began; it was agreed that haggling was in order.

MF and TA both knew that the dealer who sold the car to TA was ready to buy it back for at least $225, and that he had sold it to TA originally for $350. There was the usual discussion between TA and MF of the condition of the car; MF also had the dealer's opinion. The first stage in the bargaining was independent estimation by TA and MF of $P(p)$ and $S(s)$; in this case the nuisance value in failing to deal is taken to be zero, so $\bar{d} = 0$. The distribution functions chosen by MF are shown in Figure 1.

Figure 1

MF Estimates of Cumulative Distributions



The split-the-difference solution was estimated by MF as the abscissa of the point midway between the intersection of the cumulative distribution functions and the line denoting probability one-half, or approximately $285.

The first offer was made by MF for $225 with the argument that this would save TA the trouble of going to the dealer. TA countered by reminding MF that there were other ordinary citizens who might pay his original asking price of $325 but that he might take a little less to save trouble and expense of further searching. MF then said that it sounded like $300 was the real asking price, but that this was too much because anyone else would want a larger share of the dealer's margin and there was also some advantage in having the dealer handy if there was a complaint later about the car. MF then suggested that they close the deal at $250, since this was halfway between $225 and the fair asking price of $275. TA then proposed that the difference be split between the $250 offer by MF and the $300 offer by TA, at $275, with which MF agreed and the sale was made. In conversation after the deal was closed TA remarked that he had estimated the split-the-difference point at $275 but that he had not chosen complete distribution functions.

There is no striking conclusion to be drawn from this very ordinary bargaining example. The main point is that TA and MF were unable to formulate a systematic scheme that could be used to replace the awkward haggling process. A theory of haggling is needed for the bargaining problem if it is to be treated as a dynamic process in game-theoretic fashion. In the Oldsmobile transaction MF used the general haggling principle that the next offer should be no greater than the abscissa of the point on the selling curve (Fig. 1) whose ordinate when added to that from the purchasing curve (Fig. 1) for the last asking price was unity; this continues until some asking price is less than the next offer that would be made, but not greater than the split-the-difference price, and the deal is closed. This sequence was as follows:

Step	Next Offer		Asking Price	
	Abscissa	Ordinate	Abscissa	Ordinate
1	225	0.00	325	1.00
2	250	0.02	300	0.98
3	285	<0.50	275	0.50

Of course, the $285 offer was never made because the $275 asking price was less than $285 so it was accepted.

If TA, at Step 3, had asked $290 and stuck to it he could have forced an offer of $275 from MF and the final price would have been slightly higher.

4. Two Secretaries (October 1949). The central question in the two-person non-constant-sum case, when the available strategies and personal utilities are quite definite, is illustrated by the following situation:

> The Experimenter E offers to give Subject 1 an amount m but to give Subjects 1 and 2 together a greater amount m+g if they can agree on sharing the larger amount.

The payoff matrices for Subjects 1 and 2 are S and P, respectively, as defined in §2. The rules of the formal game are that Subjects 1 and 2 independently choose non-negative integers i and j not greater than g, that E pays Subject 1 the amount $m+S_{ij}$, and that E pays Subject 2 the amount P_{ij}; the subjects are permitted to come to a prior agreement before making their "independent" choices of i and j. The game-theoretic solution is substantially the same as for the general case discussed in §2. In both cases, actual side payments are unnecessary since the desired result can be achieved directly by choosing $i = j = g/2$.

This experiment was tried twice with two RAND secretaries, who shared the same office and who came to the experiment knowing in advance only that they were to serve as subjects in a game theory experiment. In the first trial, m = g = \$0.50; the result was the choice $i = j = g$, so that each secretary was paid \$0.50. In the second trial, m = \$0.50 and g = \$1.00; the result was the choice i = j = \$0.75, so again each secretary received the same amount (in this trial \$0.75). This is in contrast to the proposed theoretical solution in which the two secretaries would have shared the amount g only, with the first secretary receiving m in addition. Upon inquiry, it developed that they had entered into the experiment with a prior agreement to share any proceeds equally! It developed, after further inquiry, that neither of them would have felt bound by their prior agreement if the prizes had been very large but it was unfortunately not possible to test this question further.

The main lesson from this limited experiment is that the social relationship between the subjects can have a controlling influence on their choices. Often, as in this case, it is conceptually possible to allow for such an effect by appropriate corrections in the payoff matrices so that the assumed utilities are real and not just apparent. Usually, as in this case, it is practically impossible to take account of all the relevant factors—or to control them. Certainly, in this particular case, the experiment was invalidated as a test of the split-the-difference principle or any other ideal theory due to lack of control of the relevant experimental variables.

5. A Non-cooperative Pair (January 1950)*. There are now several theories for various special classes of games, some of which are not formally games in the von Neumann-Morgenstern sense. One theory that is of interest is that of Nash [4b] for games in which coalitions are prohibited, called non-cooperative games. We conducted one brief experiment with a two-person positive-sum non-cooperative game in order to find whether or not the subjects tended to behave as they should if the Nash theory were applicable,** or if their behavior tended more toward the von Neumann-Morgenstern solution, the split-the-difference principle, or some other yet-to-be-discovered principle.

The two subjects AA and JW were familiar with two-person zero-sum game theory. They also knew something of the von Neumann-Morgenstern theory for non-constant sum games, but were not familiar either with the Nash work or the split-the-difference principle. It was originally intended that non-cooperation be enforced by keeping each subject in ignorance about the identity of his opponent, but this was not done due to an accident at the outset; the experiment certainly seemed to be fully non-cooperative since there was no evidence of side payments, but there may well have been some implicit collusion within the rules of the game.

---

* This experiment was planned and conducted in collaboration with Dr. Melvin Dresher.

** Another series of experimental games was conducted by a group at The RAND Corporation, during the Summer of 1950, to test a theory of two-person cooperative games developed by Dr. John Nash [4c]. The results of this series were reported by Dr. Norman Dalkey (unpublished) as showing promising agreement between theory and experimental data.

The payoff matrices for AA and JW were:

$$A = \begin{Vmatrix} -1 & 1/2 \\ 0 & 1 \end{Vmatrix}, \qquad W = \begin{Vmatrix} 2 & 1 \\ 1/2 & -1 \end{Vmatrix}.$$

The elements of A and W are in pennies; a play of the game consisted in an independent choice by AA of row and by JW of column. There were 100 plays in all; the subjects were given ample time after the tenth play to make thorough (but independent) mathematical analyses of the experimental situation. Both subjects knew both payoff matrices and each kept a record of all earlier choices by both subjects prior to the current one. The subjects were asked also to make marginal notes after each play to indicate their reactions to the progress of the game as it went along. The actual choices are shown in Table 1. The running comments are given in the Appendix.

The frequencies with each of the four possible pairs were played are given in Table 2.

Table 2

Strategy Frequences

AA \ JW	1	2	Total
1	8	60	68
2	14	18	32
Total	22	78	100

Table 1

The Plays

Play No.	Strategies AA	Strategies JW	Payoffs to AA	Payoffs to JW
1	2	2	1	-1
2	2	2	1	-1
3	2	1	0	+
4	2	1	0	+
5	1	1	-1	2
6	2	2	1	-1
7	2	2	1	-1
8	2	1	0	+
9	2	1	0	+
10	2	1	0	+
11	2	2	1	-1
12	1	2	+	1
13	1	2	+	1
14	1	2	+	1
15	1	2	+	1
16	2	2	1	-1
17	1	1	-1	2
18	1	1	-1	2
19	2	1	0	+
20	2	1	0	+
21	2	2	1	-1
22	1	2	+	1
23	1	2	+	1
24	1	2	+	1
25	1	2	+	1
26	2	2	1	-1
27	1	1	-1	2
28	2	1	0	+
29	2	1	0	+
30	2	1	0	+
31	2	2	1	-1
32	1	2	+	1
33	1	2	+	1
34	1	2	+	1
35	1	2	+	1
36	1	2	+	1
37	1	2	+	1
38	2	2	1	-1
39	1	1	-1	2
40	2	1	0	+
41	2	2	1	-1
42	1	2	+	1
43	1	2	+	1
44	1	2	+	1
45	1	2	+	1
46	1	2	+	1
47	1	2	+	1
48	1	2	+	1
49	2	2	1	-1
50	1	1	-1	2

Play	Strategies AA	Strategies JW	Payoffs to AA	Payoffs to JW
51	2	2	1	-1
52	1	2	+	1
53	1	2	+	1
54	1	2	+	1
55	1	2	+	1
56	1	2	+	1
57	1	2	+	1
58	1	2	+	1
59	1	2	+	1
60	2	2	1	-1
61	1	2	+	1
62	1	2	+	1
63	1	2	+	1
64	1	2	+	1
65	1	2	+	1
66	1	2	+	1
67	2	2	1	-1
68	1	1	-1	2
69	2	1	0	+
70	2	1	0	+
71	2	2	1	-1
72	1	2	+	1
73	1	2	+	1
74	1	2	+	1
75	1	2	+	1
76	1	2	+	1
77	1	2	+	1
78	1	2	+	1
79	1	2	+	1
80	1	2	+	1
81	2	2	1	-1
82	1	1	-1	2
83	1	2	+	1
84	1	2	+	1
85	1	2	+	1
86	1	2	+	1
87	1	2	+	1
88	1	2	+	1
89	1	2	+	1
90	1	2	+	1
91	1	2	+	1
92	1	2	+	1
93	1	2	+	1
94	1	2	+	1
95	1	2	+	1
96	1	2	+	1
97	1	2	+	1
98	1	2	+	1
99	2	2	1	-1
100	2	1	0	+

+ denotes 1/2

The total payoffs were $0.40 to AA and $0.65 to JW.

The Nash [4b] equilibrium point is the strategy-pair (2,1). If this had been the invariable strategy of the two subjects their total payoffs would have been $0.00 to AA and $0.50 to JW. It seems unlikely that the Nash equilibrium point is in any realistic sense the correct solution; this is especially interesting since row 2 dominates row 1 for AA and column 1 dominates column 2 for JW. Here is an example in which a poor solution is chosen, for example, even if AA knows his own payoff matrix exactly but is misled into thinking that the game is zero-sum when actually the other player's payoff matrix is that of JW.

The von Neumann-Morgenstern solution, if side payments were allowed, would select the strategy-pair (1,2). Under this choice the payoffs would be \$0.50 to AA and \$1.00 to JW. The (1,2) choice would have been better for both players than was the result actually achieved. If side payments were allowed, the choice under the von Neumann-Morgenstern theory would be (1,2); the side payment from JW to AA would be not greater than \$0.50 nor less than -\$0.50, since $V(A) = 0$ and $V(W) = 1/2$.

The split-the-difference principle, with side payments allowed, leads to strategy-pair (1,2) with no side payments. Thus, even though the two subjects could have accepted the cooperative split-the-difference principle (in this particular game)* within the formal rules for the non-cooperative game they did not do so. There was a decided tendency to start with (2,1) and then to shift to (1,2) rather consistently after about thirty trials. It may be that players tend in real life situations to start near an equilibrium point and then progress toward a better solution if there is one; I hope to test this hypothesis in subsequent experimental games.

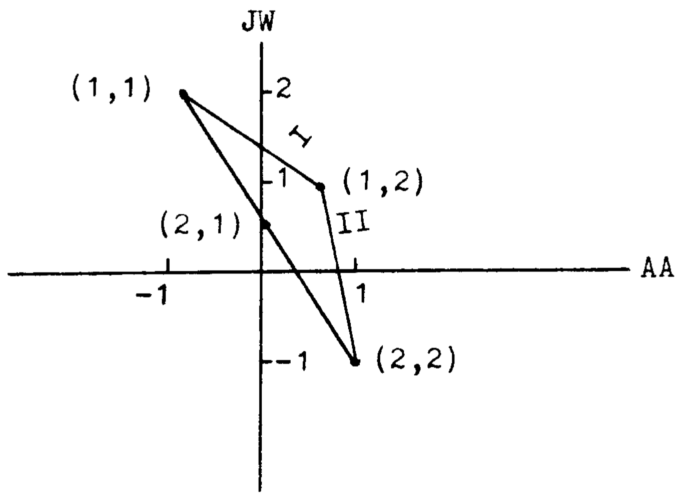
As an aid in the analysis of this game, the possible payoffs are shown graphically in Figure 2.

---

* The payoff matrices were purposely selected so as to make the von Neumann-Morgenstern cooperative solution available to the players in this case, with the thought that they might well choose it even though the formal mechanism for cooperative side-payments was not included in this non-cooperative game.

Figure 2

Payoff Values



The possible average payoffs to AA and JW are represented by points in the large quadrilateral of Figure 2. Any of these could be achieved in a sequence of plays within the formal rules of the game if the two players somehow independently determined to do so; this will be called "collaboration" when no side payments are allowed but the players agree on a pattern of play in advance of the experiment.

It is immediately plain that the only points that need to be considered are those (the "efficient" points) on the two edges I and II meeting at (1,2), since for every other point at least one player can improve his payoff without causing loss to his opponent. It is easily seen, also, that the two players can collaborate so as to achieve any payoff expectancy within the quadrilateral even though they make their choices independently at each formal play. Thus, if a denotes the probability that AA chooses row 1 and b the probability that JW chooses column 1 then points (x,y) within the quadrilateral are determined by the following relations:

$$x = (1-b) - \frac{a}{2}(1+b),$$

$$y = (2a-1) + \frac{b}{2}(3-a).$$

In particular, the edges I and II are as follows:

$$\text{I:}\quad x = \frac{1}{2}(1-3b),\quad y = 1+b,\quad \text{with } a = 1,$$

$$\text{II:}\quad x = 1-\frac{a}{2},\quad y = 2a-1,\ \text{with } b = 0.$$

In the collaborative game, the two players would agree on some point (x,y) on I or II and then compute (and use)the a and b values determined from one of the two sets of relations for I and II. The 100 plays, as they actually occurred, gave:

$$a_{100} = 0.68,\quad b_{100} = 0.22,\quad x_{100} = 0.3652,\ \text{and}\ y_{100} = 0.6152.$$

These expected values for x and y, based on independent choices with probabilities a_{100} and b_{100}, compare with \$0.40 and \$0.65 actually received by AA and JW. If the last fifty plays are used to estimate a, b, x, and y then:

$$a_{50} = 0.82,\quad b_{50} = 0.10,\quad x_{50} = 0.449,\ \text{and}\ y_{50} = 0.749.$$

From the last twenty-five plays:

$$a_{25} = 0.88, \quad b_{25} = 0.08, \quad x_{25} = 0.4448, \text{ and } y_{25} = 0.8448.$$

And from the first twenty-five plays:

$$\bar{a}_{25} = 0.44, \quad \bar{b}_{25} = 0.40, \quad \bar{x}_{25} = 0.292, \text{ and } \bar{y}_{25} = 0.392.$$

From this analysis it seems reasonable to infer that AA and JW were learning rapidly, and were converging to the split-the-difference strategy-pair (1,2). In particular, there was no tendency to seek as the final solution either the Nash equilibrium point or the point $x = y = 0.60$ which was available to them and might have been chosen by the two secretaries, for example.*

---

* Dr. Nash makes the following comment (private communication) on this experiment:

"The flaw in this experiment as a test of equilibrium point theory is that the experiment really amounts to having the players play one large multimove game. One cannot just as well think of the thing as a sequence of independent games as one can in zero-sum cases. There is much too much interaction, which is obvious in the results of the experiment.

"Viewing it as a multimove game a strategy is a complete program of action, including reactions to what the other player has done. In this view it is still true the only real absolute equilibrium point is for A always to play 2, B always 1.

"However, the strategies:
A plays 1 'till B plays 1, then 2 ever after,
B plays 2 'till A plays 2, then 1 ever after,

are very nearly at equilibrium and in a game with an indeterminate stop point or an infinite game with interest on utility it _is_ an equilibrium point.

"Since 100 trials are so long that the Hangman's paradox cannot possibly be well reasoned through on it, it's fairly clear that one should expect an approximation to this behavior which is most appropriate for indeterminate end games with a little flurry of aggressiveness at the end and perhaps a few sallies, to trst the opponent's mettle during the game.

"It is really striking, however, how inefficient AA and JW were in obtaining the rewards. One would have thought them more rational.

"If this experiment were conducted with various different players rotating the competition and with no information given to a player of what choices the others have been making until the end of all the trials, then the experimental results would have been quite different, for this modification of procedure would remove the interaction between the trials."

Dr. Dresher and I were glad to receive these comments, and to include them here, even though we would not change our interpretation of the experiment along the lines indicated by Dr. Nash.

6. An Estate (September 1949)* A RAND consultant, who knew of our experimental interests, gave five of us the left-overs from his summer's stay in Santa Monica. This estate consisted of such items as: a fifth of a fifth of Scotch whiskey, half a box of prunes, seven eggs, a dilapidated suitcase, some kitchen utensils, etc. The five of us agreed to dispose of the estate in the manner suggested by Steinhaus-Banach-Knaster [6].

The items were separated arbitrarily into five parcels in order to simplify the division process. Each of the five of us then gave an umpire a list of five values (in cents) for the parcels, as follows:**

Table 2

Estate Values and Division

(No coalitions)

Parcel	Heir				
	A	B	C	D	E
I	10	(55)	35	40	15
II	-10	(95)	50	30	30
III	50	40	35	(60)	35
IV	20	60	25	45	(110)
V	5	80	45	(95)	20
Values	0	150	0	155	110
Claims	15	66	38	54	42
Differences	-15	84	-38	101	68
Net Payments	-55	44	-78	61	28

* This experiment was planned and conducted in collaboration with Dr. J. W. T. Youngs.
** These data are fictitious because the original data were lost.

Each parcel was then given to the heir assigning it the highest value and the side payments were computed, with the results shown in the final row of Table 2. For example, Heir D received Parcels III and V and made a cash payment of $0.61.

The point of interest here is not especially the fair division process itself, based on honest evaluation by each heir of the parcels, but is rather this process thought of as a 5-person non-zero-sum game. First of all, if the evaluations are honest, the allocation of the parcels to heirs attaching highest values is equivalent to the von Neumann-Morgenstern [1] requirement that the joint payoff be maximized. But if the heirs form coalitions, and then choose the values given to the umpire in a manner intended to maximize their individual gain, the fair division process may no longer be fair.

As an example, suppose that three heirs assign values honestly and the other two (say A and B) form a coalition. A and B agree to split the joint gain resulting from the coalition equally between them. Their procedure is to first make up their honest value list, as in Table 2, and then to give to the umpire identical lists in which each entry is the larger of their two honest values. The result is shown in Table 3.

Table 3

Estate Values and Division

(AB coalition)

Parcel	Heir A	Heir B	Heir C	Heir D	Heir E
I	55	(55)	35	40	15
II	95	(95)	50	30	30
III	50	50	35	(60)	35
IV	60	60	25	45	(110)
V	80	80	45	(95)	20
Values	0	150	0	155	110
Claims	68	68	38	54	42
Differences	-68	82	-38	101	68
Net Payments	-97	53	-67	72	39

The joint gain to A and B from this coalition is $0.33.

As a still more extreme coalition example, suppose that Heirs A, C, D, and E form a coalition against B and employ the same principle as that just computed for A and B. The result is shown in Table 4.

Table 4

Estate Values and Division

(ACDE coalition)

Parcel	Heir				
	A	B	C	D	E
I	40	55	40	40	40
II	50	95	50	50	50
III	60	40	60	60	60
IV	110	60	110	110	110
V	95	80	95	95	95
Values	0	150	0	155	110
Claims	71	66	71	71	71
Differences	-71	84	-71	84	39
Net Payments	-84	71	-84	71	26

The joint gain to ACDE from this coalition is $0.27.

In the most extreme case, namely that in which ACDE form a coalition and also have full knowledge concerning the values chosen by B, the joint gain to ACDE is $0.40. It is interesting to see that A and B alone in coalition can increase their take by $0.33, as compared with only $0.40 if all of A, C, D, and E join in a coalition. On the other hand, as Steinhaus [6] has noted, there is no way in which any coalition can prevent one heir from receiving at least his fair share as determined by his own private evaluation if he states it honestly to the umpire.

7. Three Baby-sitters (August 1949). Non-zero-sum games with more than two players are fundamentally more complicated than those with only one or two players [1]. Shapley [2b] has strengthened the axiomatic framework so that a satisfying solution is determined for some essential games, with more than three players, but the general case is still open. I shall now describe one very limited experiment with a three-person non-zero-sum game.

One of my three teen-age children (S, M, W) was needed to serve as a baby-sitter for three hours one Saturday afternoon. In order to settle which one, I offered at dinner a week in advance to conduct a "reverse auction" after the fashion popularized by the Gilbreths [3]. The children liked the idea.

W first opened with a bid of $4.00, then quickly remarked that the three of them could do better if they teamed up against me, and they all listened inquiringly to hear me say that collusion was not allowed. I agreed instead, and to their surprise said that coalition was all right provided: a) the final bid was not greater than $4.00, and b) they would tell me how they reached their agreement. After a short but unsuccessful attempt to come to some agreement then and there, including a discussion of possible sidepayments, they asked for and were allowed more time. Each evening at dinner I asked for their decision until finally on Friday, a week later, they all asked to go ahead with the reverse auction because they had been unable to agree among themselves. We did and the end result was that W won the job with a bid of $0.90!

This is probably an extreme example, although not really so extreme when you compare the magnitude of the children's error with that made by mature nations at war because of inability to split-the-difference. I have noticed very similar "irrational" behavior in many other real life situations since August 1949 and find it to be commonplace rather than rare. The reasons for such non-optimal behavior seem to turn about inadequate administrative and enforcement mechanisms, and about difficulties in making the difficult and obscure value judgments necessary for estimation of the relevant payoff matrices.*

It may be of some interest to compare the solutions that the children would use if they followed the type of rationale proposed by Steinhaus [6] or Nash [4a]. For this purpose, the "estate" consists of $4.00 cash and a baby-sitting obligation. Also, the game could be represented by the payoff function

$$P^{\ell}_{ijk} = \delta_{ijk\ell} s_i + 400\, \delta_{ijk}\, \delta_{\ell 3} \quad \text{for} \quad i,j,k,\ell = 1,2,3,$$

where:

a δ is 1 or 0 according as all its subscripts are or not equal, $\ell = 1,2,3$ denote the children in the order W, M, S, $s_1 = -90$, $s_2 = -100$, $s_3 = -125$, and a play consists of a

* Dr. Howard Raiffa [5], in his very interesting paper, has made some game-theoretic suggestions for arbitration schemes that could be used to resolve some of these difficulties.

choice of values for i, j, k independently by W, M, S respectively. It follows easily that the estate would be settled by W paying M and S the amounts \$1.05 and \$0.967, respectively, and that W would acquire the \$4.00 and the baby-sitting obligation. It also follows easily that the game could be settled, under the assumption that the product of utilities is maximized as a non-cooperative game with side payments, by W accepting the baby-sitter obligation and by side payments from S to W and M in the amounts \$1.933 and \$1.033 respectively. In both cases W serves as baby-sitter, but the \$3.10 profit is divided equally among the players under the Nash method whereas the division depends on the different values assigned by the players in the Steinhaus method.

This case illustrates a central question. In many positive-sum situations the players know in advance that the value may differ between them after they know the details of the game, but beforehand they agree that they are entitled to equal shares. One solution that always works in such a case is for the players to settle by lot which one shall receive all the payoffs; a simple variant when side payments are possible is to agree in advance to split all the payoffs equally, as was done by the two secretaries. Under this solution, or its variant, the actual payoffs can be disposed of so as to maximize the total payoff—as is done when an estate is auctioned off to a market including the heirs. The virtue of such a method is that it makes the situation symmetric as between the players and leads to an easy solution. Of course, this method cannot be used in all cases; for example, it could not have been used by AA and JW in playing the non-cooperative game discussed in §5 if neither player had any way of communicating with the other.

8. House of Delegates (January 1939). The West Virginia Legislature includes a Senate and a House of Delegates. The Delegates in the House are elected representatives of the fifty-five counties. The usual majority voting procedure is in effect. Here is a social process that could be approached from a game-theoretic standpoint, in principle at least, even though the actual situation is too complicated for detailed analysis in our present state of knowledge.

The situation of interest now is the problem faced by the House in 1939 in determining the distribution of state aid for schools among the counties for the next biennium. A study was made [7] of the effects on the school system of various alternative distributions; a few preferred ones were presented to the Legislature for consideration and final determination. The alternatives were compared two at a time; there was an evident tendency for the House to choose the one that captured the majority vote when all representatives from each county voted as a block for the distribution that gave their county the larger amount. It became more and more obvious, as the informal voting went on, that the alternative that would win would be the one that prevailed over all others under the majority voting procedure—so a computation was attempted to determine this "best" distribution. Unfortunately, or fortunately, there was no such "best" distribution because the majority preference ordering was intransitive.* This type of situation will now be illustrated by some examples but no attempt will be made to analyze the actual situation before the 1939 House of Delegates.

Suppose first that the House had three Delegates, one from each of three counties, and had four identical but indivisible items to allocate. The possible allocations are shown in Table 5.

Table 5

Allocation Example

County	Allocation			
	1	2	3	4
1	4	1	2	0
2	0	0	1	2
3	0	3	1	2

It is to be understood that each column represents all distinct permutations of its rows, thus:

0		2	2
2	includes	0	2
2		2	0

* A full discussion of the problem of social choice may be found in a recent monograph by Arrow [8]. The present author has reported a series of preference experiments relevant to this discussion [9].

Under majority voting, it is immediately obvious that for each allocation there is at least one other that is better— for example: 3 > 1, 3 > 2, 4 > 3, 2 > 4. By simply interchanging county designations these four preference relations will provide a better allocation in every case. This intransitivity is the rule rather than the exception; it always occurs if the number of indivisible items is large enough, unless some one county has enough votes to control the result entirely on its own.*

We are left with the important question: how does the democratic majority voting process yield decisions? I believe that the answer starts with the observation that the real-life cases are not constant-sum, as was assumed explicitly in the example just given, and that the decision is essentially to choose the course of action that maximizes some joint utility measure—such as the sum of utilities used by von Neumann-Morgenstern [1], the product suggested by Nash [4a], or some other.** The von Neumann-Morgenstern solution would not specify the side payments, but an easy generalization of the Nash solution to cover the case of more than two players in this kind of bargaining situation does yield a complete formal solution. An example will be given now

---

* This is essentially the dilemma confronting the three children in the situation discussed in §7, if they tried to resolve their problem by selfish majority voting.
** This general question has been treated in some detail by Raiffa [5].

to illustrate this point.

We start with the completely symmetric case in which the payoff functions for the constant-sum game are taken from Table 5, starting with a complete enumeration of strategies. To accomplish this, first define the allocation matrix A_{ij} as follows.

$$A_{ij} = \begin{pmatrix} 4 & 0 & 0 & 1 & 1 & 3 & 3 & 0 & 0 & 2 & 1 & 1 & 0 & 2 & 2 \\ 0 & 4 & 0 & 0 & 3 & 1 & 0 & 1 & 3 & 1 & 2 & 1 & 2 & 0 & 2 \\ 0 & 0 & 4 & 3 & 0 & 0 & 1 & 3 & 1 & 1 & 1 & 2 & 2 & 2 & 0 \end{pmatrix}$$

Then the payoff function for player ℓ, if strategies i, j, and k are chosen by players 1,2, and 3, respectively, is defined as follows:

$$P^{\ell}_{ijk} = 0 \quad \text{if } i,j,k \text{ are distinct,}$$

$$P^{\ell}_{ijk} = A_{i\ell} \quad \text{if } i = j = k,\ \ i = j \neq k, \text{ or } i = k \neq j,$$

$$P^{\ell}_{ijk} = A_{j\ell} \quad \text{if } j = k \neq i.$$

These relations simply express the fact that two or more players who agree on a strategy can insure that this determines the allocation. This game is completely symmetric, and is represented by the convex in the first octant defined by the condition:

$$u_1+u_2+u_3 \leq 4.$$

The point $u_1 = u_2 = u_3 = 4/3$ is the Nash [4a] solution; this can be achieved by mixing strategies 10, 11, 12, for example, but it can also be achieved by mixing various other sets of strategies. Of course, if side payments are allowed any strategy may be selected and this same solution then be achieved.

One way of changing this game into a non-constant-sum game, that is also a little more realistic for this section, is to replace the elements of A_{ij} by their positive square roots; we denote this new allocation matrix and consequent payoff function by a_{ij} and p^{ℓ}_{ijk}. This game is still symmetric but is not constant-sum. Diminishing marginal utility is perhaps a bit more realistic in the school aid case since extra money for schools may be a good deal more important relatively in a poor program than in a rich one. At any rate we shall consider this game as an example. It is easily seen that the Nash solution, namely the point where the product of the utilities is maximum, is the point $u_1 = u_2 = u_3 = \frac{1}{3}(2+\sqrt{2})$; this solution can be achieved only by mixing strategies 10, 11, and 12. Again, if side payments were allowed, the same result could be achieved by choosing any one of the pure strategies 10, 11, or 12.

Another change will now be made in the payoff functions, to obtain a non-symmetric non-constant-sum game, by adding $(\ell-1)$ units to each element of p^{ℓ}_{ijk}. Our new payoff function is:

$$Q^{\ell}_{ijk} = \ell - 1 + \sqrt{P^{\ell}_{ijk}} \, .$$

This change simulates the difference in the effect of the allowance due to differences in wealth initially in the state aid case. I have not computed the Nash solution in this case, but this would be done by finding the point (u_1, u_2, u_3) on the convex set (determined by the Q^{ℓ}_{ijk}) where $u_1u_2u_3$ is maximum.

In the actual school aid situation there is no way now to convert the dollar allocation matrix into utilities. I can easily imagine the Delegates themselves being able and willing to provide utility estimates for each alternative distribution under consideration, and the theory could then be applied to determine the Nash solution. It would be entirely unreasonable to try to apply this method to help in making the decision, but it might be instructive in some instances to see how the computed solution compares with the one actually chosen by the regular voting process.

mhb

Appendix

Running Comments*

I. Subject AA

Play No.	Comment
1	JW will play 1—sure win. Hence if I play 1—I lose.
2	What is he doing?!!
3	Trying mixed?
4	Has he settled on 1?
5	Perverse!
6	I'm sticking to 2 since he will mix for at least 4 more times.
9	If I mix occasionally, he will switch—but why will he ever switch from 1.
10	Prediction. He will stick with 1 until I change from 2. I feel like DuPont.
19	I'm completely confused. Is he trying to convey information to me?
28	He wants more 1's by me than I'm giving.
31	Some start.
32 - 40	JW is bent on sticking to 1. He will not share at all as a price of getting me to stick to 1.
49	He will not share.
58	He will not share.
59	He does not want to trick me. He is satisfied. I must teach him to share.
67	He won't share.
68	He'll punish for trying!

* The two subjects are friends.

Play No.	Comment
70	I'll try once more to share—by taking.
91	When will he switch as a last minute grab of (2). Can I beat him to it as late as possible?

II. Subject JW

Play No.	Comment
1	Hope he's bright.
2	He isn't but maybe he'll wise up.
3	O.K., dope.
4	O.K., dope.
5	It isn't the best of all possible worlds.
6	Oh ho! Guess I'll have to give him another chance.
7	Cagey, ain't he? Well ...
8	In time he could learn, but not in ten moves so:
10	I can guarantee myself a gain of 5, and guarantee that Player AA breaks even (at best). On the other hand, with nominal assistance from AA, I can transfer the guarantee of 5 to Player AA and make 10 for myself too. This means I have control of the game to a large extent, so Player AA had better appreciate this and get on the bandwagon.
	With small amounts of money at stake, I would (as above) try (by using Col. 2) to coax AA into mutually profitable actions. With large amounts at stake I would play Col. 1 until AA displayed some initiative and a willingness to invest in his own future. One play of row 1 by AA would change me from Col. 1 to Col. 2, where I would remain until bitten.
	On the last play it would be conservative for me to switch to Col. 1, but I wouldn't do so if the evidence suggested that AA was a nice stable personality and not in critical need of just a little extra cash.
11	Probably learned by now.
12	I'll be damned! But I'll try again.
13	That's better.
14	Ha!
15	(bliss)

Play No.	Comment
17	The stinker
18	He's crazy. I'll teach him the hard way.
19	Let him suffer.
21	Maybe he'll be a good boy now.
22	Always takes time to learn.
23	Time.
27	Same old story.
28	To hell with him.
31	Once again.
32	---, he learns slow!
33	On the beam again.
39	The ---.
41	Always try to be virtuous.
42	Old stuff.
50	He's a shady character and doesn't realize we are playing a 3rd party, not each other.
52	He requires great virtue but doesn't have it himself.
60	A shiftless individual—opportunist, knave.
62	Goodness me! Friendly!
68	He can't stand success.
71	This is like toilet training a child—you have to be very patient.
80	Well.
82	He needs to be taught about that.
92	Good.

BIBLIOGRAPHY

[1] von Neumann, John, and Morgenstern, Oskar. Theory of Games and Economic Behavior. Princeton, Princeton University Press, 1944.

[2] a. Shapley, L. S. Quota Solutions of n-Person Games. P-297, The RAND Corporation, 7 April 1952. Also, to be published in Contributions to the Theory of Games—II, by the Princeton University Press.

b. ————————. A Value for n-Person Games. P-295, The RAND Corporation, 18 March 1952. Also, to be published in Contributions to the Theory of Games—II, by the Princeton University Press.

[3] Carey, Ernestine Gilbreth, and Gilbreth, Frank B., Jr. Cheaper by the Dozen, Grosset and Dunlap, Inc., New York.

[4] a. Nash, John F., Jr. "The Bargaining Problem," Econometrica, Vol. 18, 1950, pp. 155-162.

b. ————————. "Non-cooperative Games," Annals of Mathematics, Vol. 54, No. 2, September, 1951, pp. 286-295.

c. ————————. Two-person Cooperative Games. P-172, The RAND Corporation, 9 August 1950.

[5] Raiffa, Howard. Arbitration Schemes for Generalized Two-person Games. June 1951, 71 pp., Report No. M 720-1 R 30, under Contract N6ONR 232-1.

[6] Steinhaus, Hugo. "The Problem of Fair Division," Econometrica, Vol. 16, No. 1, January 1948, pp. 101-109.

[7] a. Flood, M. M. "The Foundation School Program," West Virginia School Journal, April 1939, p. 11.

b. ————————. "Sharing School Costs in West Virginia," West Virginia School Journal, April 1939, p. 22.

[8] Arrow, Kenneth, Jr. Social Choice and Individual Values. John Wiley and Sons, New York, 1951.

[9] a. Flood, M. M. A Preference Experiment. P-256, The RAND Corporation, 13 November 1951.

b. ————. A Preference Experiment (Series 2, Trial 1). P-258, The RAND Corporation, 5 December 1951.

c. ————. A Preference Experiment. (Series 2, Trials 2, 3, 4). P-263, The RAND Corporation, 25 January 1952.

mhb