Appendix D: Derivation of the Spectral Efficiency Form of Shannon's Capacity Formula

Our starting point is the standard form of Claude Shannon's formula for the capacity of a band-limited channel with additive white Gaussian noise (AWGN):

$$C = W \log_2\left(1 + \frac{P}{N}\right), \tag{1}$$

where C is the capacity, or maximum average rate at which information can be transmitted over the channel, and has units of bits per second; W is the bandwidth of the channel in Hertz; and P/N is the ratio of the signal power divided by the noise power passed by the receiver front-end filtering (a dimensionless quantity).

In order to get a capacity equation involving spectral efficiency in terms of E_b/N_0, start by making the substitution $N = W \cdot N_0$ in (1). Manipulating, we get

$$\frac{P}{N_0} = W\left[2^{C/W} - 1\right]. \tag{2}$$

Dividing both sides of (2) by C gives

$$\frac{P}{N_0 C} = \frac{W}{C}\left[2^{C/W} - 1\right]. \tag{3}$$

To introduce E_b/N_0, we now reason as follows. When operating at capacity, the average energy per information bit equals the average signal power divided by the average information rate in bits per second, i.e.,

$$E_b = P/C\,. \tag{4}$$

Substituting in (3) using (4) gives a useful formula relating the achievable spectral efficiency C/W to the E_b/N_0 signal-to-noise ratio:

$$\frac{E_b}{N_0} = \frac{W}{C}\left[2^{C/W} - 1\right]. \tag{5}$$

Suppose we want to find the minimum E_b/N_0 required to achieve a spectral efficiency, C/W, of 6 bits/sec/Hertz. Substituting in (5), we find that the minimum $E_b/N_0 = 10.5 = 10.2$ dB. (To obtain the maximum achievable spectral efficiency for given E_b/N_0, one must solve numerically.)