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Mathematical Proficiency for All Students: Toward a Strategic Research and Development Program in Mathematics Education

RAND Mathematics Study Panel
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PREFACE

Developing proficiency in mathematics is important for all students. However, when considered in light of current standards, or compared with performance in other countries, evidence on student achievement in mathematics makes clear the need for substantial improvement. U.S. students do not, as a group, achieve high levels of mathematical proficiency. The nation must seek to narrow the achievement gaps between white students and students of color, between middle-class students and poor students, gaps that have remained the same or widened over the past decade. To address these problems, the federal government and the nation's school systems have made and are continuing to make significant investments in the improvement of mathematics education. However, the knowledge base on which these efforts are founded has often been weak and speculative.

A strategic and coordinated program of research and development could contribute in significant ways to improving mathematics education by illuminating how to focus resources effectively and successfully. As part of a broader effort to inform the U.S. Department of Education's Office of Educational Research and Improvement (OERI) on ways to improve the quality and relevance of education research and development, RAND organized a panel of 18 experts to consider the problems and envision such a program. This Mathematics Study Panel, comprising education professionals, mathematicians, and researchers in mathematics education , represented a range of disciplinary and methodological perspectives. The Panel was charged with proposing a strategic agenda and guidelines for a long-term research and development program that could contribute to practice and policy in mathematics education.

This report presents a draft of the Mathematics Study Panel members' work on this agenda. The draft will be reviewed by other experts in the field. Widespread professional and public comment is also being solicited: The draft is posted on RAND's website (www.rand.org/multi/achievementforall) where visitors are welcome to contribute comments on the report. The Panel will reconvene in summer 2002 to use the comments and critiques to improve and develop the report.

The Mathematics Study Panel report is the second in a series of three RAND reports dealing with topic of education research and development. The first report, April 2002, proposes a program for research on reading comprehension education and the third report, scheduled for a draft publication in fall 2002, will address research and development management issues. These reports should be of interest to those involved with the planning of education research and development programs by public and private agencies. This particular report should also be of interest to researchers who study mathematics instruction and to practitioners who teach mathematics.

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INTRODUCTION AND OVERVIEW

Mathematics education has been a focus of public concern since the launch of Sputnik in 1957. The past 40 years have seen several waves of reform in school mathematics, each entailing serious efforts to improve mathematics learning. One well-known example was the "New Math." Another more recent example was the production of standards for curriculum and teaching by the National Council of Teachers of Mathematics and the ensuing development of new curriculum materials. Researchers have studied how students acquire and use various kinds of mathematical knowledge and skills, how teachers know and use mathematics in their teaching, and how schools can improve their mathematics instruction. Curriculum developers have produced new materials built on contemporary ideas about mathematics content and how it can be effectively represented and learned. These and other accomplishments are evidence of a continued concern for mathematics learning in this country.

Despite continuing concern, however, these efforts have not sufficed collectively to produce a high-quality system of mathematics education meeting today's needs. Some progress has been made, but the pace is slow, the gains are modest, and improvement is not at the scale now required. Some notable success stories do show that concerted, focused efforts can make a difference, but substantial effort is needed to make these successes the norm rather than the exception.

Mathematics education in the United States is plagued with serious problems. Most Americans graduate from high school without sufficient mathematical competence. In 2000, the National Assessment of Educational Progress (NAEP) found that only 16% of grade 12 students across the country were judged "basically proficient" in mathematics. The troubling achievement gaps between white students and students of color and between middle-class students and poor students have remained the same or widened. Despite extensive public investments in the improvement of mathematics education, many interventions have had disappointing impacts. And professional development has too often failed to provide teachers with useful knowledge and skills.

Tackling the problems of mathematics education depends on knowing where and how to invest efforts. It requires knowledge about problems of practice, and ways to address those problems. Where such knowledge exists and has been appropriately used, it has paid off. Examples of research that has made a difference in practice include studies of how teachers can use knowledge of students' arithmetic strategies to develop their problem-solving and computational skills, the characteristics of professional development that impacts teachers' practice and their students' learning.[1]

Yet, across the past four decades, coordinated and sustained investments in research and development have been inadequate. Federal agencies (primarily the National Science Foundation [NSF] and the U.S. Department of Education) have contributed funding for many of these efforts. But the investment has been small, and there has been no long-range programmatic effort devoted solely to funding research in mathematics education. Consequently, major gaps exist in the knowledge base. Problems have been unevenly identified and studied, and even where well-supported knowledge exists, it does not always reach the school classroom. Both the lack of a cumulative well-developed knowledge base and the fragile links between research and practice have been major impediments in creating a system of mathematics education that works.

The fact that we lack knowledge for the practice of teaching mathematics, and a system for its cumulative development, matters more than ever before. Although mathematics education has suffered problems for decades, the current goals present new challenges. The demands of life in the 21st century require high levels of mathematical proficiency. And although our system has been able to produce such proficiency in a small fraction of the population, *all* students now need competency in mathematics. The combined aim (mathematical proficiency) and its target (all students) are unprecedented.

To build the resources needed to reach these ambitious goals, this report maps out a long-term agenda of programmatic research and design in mathematics education that fundamentally connects theory and practice. If successful, such a program would produce resources for taking first steps, and, over the course of 10 to 15 years, build a solid base of knowledge useful for sustained support of effective instructional practice. The proposed agenda must take into account the reality that public investments in research are a fraction of those needed, given the scale and complexity of the problems. Hence, difficult choices and careful designs will be required, so as to gain maximum leverage and cumulative impact from available resources. In every aspect of the proposed agenda, central attention to the dual aims of mathematical proficiency and equity is vital.

Our argument is based on hypotheses about where and how to invest in improving American mathematics education. We begin by examining critical problems of mathematics teaching and learning. The program of research that we propose is aimed at solving those problems. Rooted in practice in both its inspiration and its application, this program seeks to coordinate progress in basic knowledge with multiple forms of empirical inquiry, interventions,

[1]Cohen & Hill, 2000; Saxe & Gearhart, 1998; Garet et al., in press; Silver & Lane, 1995; Silver & Stein, 1996.

and the wisdom of experience. Moreover, because solutions to these problems are not the province of any single community of expertise, a second goal is to build a multidisciplinary professional community of people who have experience and expertise in different parts of the enterprise. This community would work together to size up problems, set priorities, and plan useful programs of research. This panel's work represents one such effort to bring together some of the diverse groups who have a stake in the improvement of mathematics education—scholars, practitioners, and policymakers. What we envision is an approach that would coordinate the resources of research, development, and experience, and groups with a range of different skills and interests, to build systematic knowledge necessary for making mathematical proficiency an attainable goal for all students. Reaching these aims will require the creation of a research infrastructure to build the capacity for such work.

This report includes three main sections. We begin by reviewing why mathematics and its learning matter, and why now is a strategic moment for the improvement of mathematics education. Along with this rationale, we present an overview of the current state of mathematics education in the United States and make a case for a program designed to produce knowledge that is both useful to and usable for practice. The next section is the heart of our report. There we make the case for a set of specific priorities. We identify three focal areas in which a strategic investment in coordinated, cumulative research and development could improve the capacity of the system to help all students achieve mathematical proficiency, and we frame a set of key questions to focus programmatic research. In the last section, we turn our attention to what we mean by a *program* of work and consider the infrastructure that will be needed to develop capacity for this agenda.

WHY MATHEMATICS—AND THE IMPROVEMENT OF MATHEMATICS EDUCATION—MATTERS

The notion of *mathematical proficiency* that we use in this report offers a conception of what it means to be competent with mathematics.[2] It is represented as the intertwining of five strands:

- *Conceptual understanding*—comprehension of mathematical concepts, operations, and relations
- *Procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *Strategic competence*—ability to formulate, represent, and solve mathematical problems

[2]From *Adding It Up: How Children Learn Mathematics*, Kilpatrick, Swafford, & Findell, 2001.

- *Adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
- *Productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

The importance of mathematical proficiency can be seen from four perspectives. The first is social: Responsible and informed citizenship in a modern economic democracy depends on a host of quantitative understanding and skills for making personal and life-planning decisions. Such knowledge is important for making social and political judgments on many technical public issues and policies. Second, mathematics is crucial for the personal life and career options available to individuals. People's choices are shaped by whether or not they know and are able to use mathematics. A third foundation is cultural: Mathematics constitutes one of the most ancient and noble intellectual traditions of humanity. It is an enabling discipline for all of science and technology, providing powerful tools for analytical thought as well as the concepts and language for precise quantitative description of the world around us. It affords knowledge and reasoning of extraordinary subtlety and beauty, even at the most elementary levels. And a fourth basis for the importance of mathematics is economic: In the modern high-tech economy, the jobs that support a decent standard of living demand much stronger and more-flexible quantitative skills. These workplaces will constantly evolve, so a fixed knowledge base will not suffice to function effectively. Learning, communication, and analytical skills are necessary. If U.S. schools do not supply the skilled workforce for these jobs, the work may easily migrate abroad or may increasingly depend on skilled immigrants.

WHAT ARE THE FUNDAMENTAL PROBLEMS WE NEED TO SOLVE?

Consider for a moment how you would answer the following questions:

- Why does it work to "add a zero"—that is, to place a zero to the right of a number—when multiplying by ten, or two zeroes when multiplying by a hundred? And what if you want to multiply 1.5 by 10—then what? You *don't* add a zero. When the number includes a decimal, you "move the decimal point over" instead. Why?
- If the probability of rain is 50% on Saturday and 50% on Sunday, why isn't the probability of rain on the weekend also 50%?
- If you double the floor dimensions of a room, why does that double the amount of wall paint needed, but quadruple the amount of floor wax needed?

- If you receive a 10% salary cut, and later, a 10% raise, do you end up back at your original salary? Would you be better off if you received the salary increase of 10% first, and then, later, a 10% salary cut?
- What *is* pi (π)?
- For a given length of fencing, how would you create an enclosure with the maximum area inside?
- How would you prove to your brother-in-law that whenever you add two odd numbers the answer will always be even? What if he asked whether the same were true if you *multiply* two odd numbers? What would you say and how do you know that's true?
- You have turned in your math assignments on time 12 times, and turned them in late 6 times. How many days in a row will you now need to turn in your assignments on time in order to meet the teacher's requirement of an 80% on-time assignment rate?
- A state lottery prize of $1,000,000 must be paid out over ten years in $100, 000 annual installments. If investments earn at the rate of 10%, how much cash must the state put up to fund the prize?

As you respond to these questions, consider the nature of your own mathematical understanding, and how—and why—it developed as it did.

Most well-educated adult Americans cannot answer the above questions comfortably because their basic mathematical knowledge is thin. What they typically remember are rules, ungrounded by understanding. They may think pi (π) is 22/7 or 3.14. They may have no idea why the probability of weekend rain is *not* 50%. They cannot easily retrieve ideas or methods once forgotten. What they do know, they often cannot use effectively to solve real problems or understand quantitative information. They may struggle to figure their income taxes, to make sense of the test score data that represent their children's performance, to grasp the large numbers involved in policy questions on the economy or the environment, and to estimate the relative risk of life choices. In a host of everyday situations, they cannot estimate orders of magnitude, judge spatial arrangements, or reason effectively about quantitative relationships. They are unable to appreciate many aspects of the breathtaking array of natural wonders, scientific discoveries, and human creations and inventions. For example, the fact that scientists discover new stars that have already expired requires a grasp of what it means to *see*, what "light years" really mean, as well as just how very far away some parts of our universe are. Most adults do not make reasonable analyses of likelihood, and they are put off by symbolic notation. We cannot imagine that a similarly widespread lack of ability to read or write with understanding among educated adults would be viewed with complacency in the United States.

In the past, mathematical understanding and proficiency may have been seen as a luxury available to and expected of only a privileged elite. Some Americans might have reacted to this litany of problems with little more than a shrug. However, conditions have changed. For the first time, broad agreement exists that mathematical proficiency matters on a wide scale. That few people can answer the questions above is much more troubling today because most people agree that American adults are going to require substantial mathematical proficiency to participate fully and productively in the society and economy of the 21st century.

Defining the goal of mathematics education as providing *everyone* with the opportunity to gain mathematical proficiency brings the issue of equity front and center. The harsh reality is that our system produces starkly uneven results. Although a small fraction of students do develop mathematical proficiency in school, most do not. And those who do not are disproportionately represented among children of poverty, students of color, English language learners, and girls. Recent NAEP results show the gap in mathematics achievement holding steady or widening by social class and ethnicity. In 2000, over 34% of white students in grade 8 attained either "proficient" or "advanced" performance on the NAEP, up from 19% in 1990. Among African American students, results were dismal, with the proportion holding steady at 5% since 1990. Among Latino students, however, while the proportion of students who attained "proficient" or "advanced" status more than doubled since 1990, the absolute level remained low at 9%. Lack of success in mathematics has significant consequences: Algebra, for example, plays a significant gatekeeping role in determining who will have access to college and other career opportunities. Those gates tend to be closed to the less advantaged, by default (because the schools they attend simply do not offer advanced mathematics courses) and/or by discrimination (because tracking differentiates and further limits students' opportunities).

The idea of "mathematics for all," a slogan of the recent reform efforts in school mathematics, presents one of the most significant challenges to the U.S. education system. Taking as our goal that all students should now achieve ambitious levels in mathematics asks our schools to change practices that have effectively differentiated students' mathematics learning. In addition, schools and teachers are expected to teach new materials in new ways to attain high levels of mathematical proficiency. Meeting this challenge will require new practices to be developed, learned, and implemented on a broad scale, all in a society that has held mathematical proficiency to be more a matter of talent than effort, something that is more innate than teachable.

This report argues that these challenges—to achieve both mathematical proficiency and equity—demand the development of new knowledge and practice, rooted in systematic, coordinated, and cumulative research. We outline a strategic agenda for research and development aimed at solving core problems of mathematics instruction and its improvement.

WHERE DO THINGS STAND IN U.S. MATHEMATICS EDUCATION?

The waves of reform in school mathematics over the past forty years have represented serious efforts to improve mathematics learning. Each has attempted to upgrade what counts as "mathematics" in school, to alter students' mathematical experience, and to improve students' grasp of fundamental ideas and skills. Results of these efforts have produced some successes—the increase in the percentage of students taking the second year of algebra in high school, for example, and the steady improvement in students' performance at grades 4 and 8 since 1990 (NAEP). But the issues are more complicated than they appear. For example, according to data collected in NAEP 2000, 94% of grade 12 students report having taken Algebra I, 88% have taken Geometry, and 80% have taken Algebra II. This last number is astonishing because the percentage of students taking Algebra II was only 56% as recently as 1990. Thus, it seems that the access problem has diminished but the corresponding learning gains have been much more modest, which may suggest that there are problems with the quality of what is being offered, and that access is only part of the problem to be solved. Effective programs show that successful interventions are possible, but little is known about how to make sustained improvements at scale. The core problems—lack of mathematical proficiency and serious inequities—remain pervasive.

Efforts within mathematics education have escalated since 1989. The National Council of Teachers of Mathematics (NCTM) took a bold step as a professional organization by developing and disseminating a set of national standards for mathematics instruction. The NCTM Standards called for a reorientation and expansion of the content that students should be expected to learn. They emphasized understanding fundamental mathematical concepts and being able to apply mathematics to the solution of practical problems. The Standards were as notable for the changes they recommended in styles of instruction and forms of assessment as they were for the shifts in content expectations. Rooted in theories of learning and motivation that emphasized the human tendency to work actively in making sense of the world, the Standards' vision was one of learning through meaningful activity, and by building on what learners bring. Moreover, the Standards promoted, for the first time, an ideal of "mathematics for all." This ideal was as radical as any of the "teaching for understanding" imperatives. Taken together, these core ideas of understanding, meaning, and "mathematics for all" formed a set of goals that was unprecedented in American mathematics education.

The NCTM Standards quickly exerted broad influence. Many efforts were promoted as "based on the Standards." The Standards did in fact play a role in the redesign of state frameworks as well as the development of new assessments intended to measure achievement of state goals. A spate of teacher professional development efforts was based on elements of the Standards.

Perhaps most significant was the role played by the Standards in shaping the development of the new NSF-funded curriculum materials that were produced during the 1990s.For example, the Standards' emphasis on using mathematics in "real life" was reflected in an increase in application and contextualized problems.

Despite these efforts, serious reservations about the Standards were also voiced in some corners. Neither the NCTM nor the various state standards are universally endorsed. Critics in the mathematics, teaching, and policy communities are concerned about students' computational skills, worried about the use of calculators, and skeptical of the emphasis on applications, and of new teaching strategies and curricular orientations. Arguments flared over assessments, and teachers were unsure of how to measure students' learning toward new goals. The past decade has been a period of intense debate, not always constructive, for collective work on improvement.

During the same period, in district after district, and in many states, curriculum guidelines shifted toward requiring all students to take more demanding mathematics courses, particularly algebra and geometry. Policies also pressed content earlier in the school sequence, and urged the abandonment or de-emphasis of lower-track, repetitive arithmetic courses with euphemistic "business math" style titles. Many districts and states introduced high stakes tests with more demanding mathematical content, tests that must be passed for promotion at significant points in students' school careers, and for high school graduation. Still, many argue that we lack the necessary means to assess important aspects of students' mathematical development. Meanwhile, the number of schools offering—and the number of students taking—the College Board's Advanced Placement mathematics courses has risen. And efforts, such as the Algebra Project of Bob Moses and Equity 2000, have reflected the civil rights movement's recognition of the need to extend higher expectations and opportunity to its constituencies.

Federal funding for the pre- and in-service recruitment and training of the teachers needed to meet the demands generated by these new expectations has increased significantly. All of these developments have had strong support from the business and political communities, as expressed in an important series of National Educational Summits over the past fifteen years.

This is an important moment for mathematics education because, although the problems are far from new, the resources available for tackling them are substantially more developed than at any previous time in U.S. history. Supported by major federal and private investments, new curricula have been produced for pre-kindergarten through high school; professional development programs focused on mathematics abound; researchers have begun to amass knowledge about how students and teachers learn, and about approaches to specific mathematical topics; and teachers are eager, in many cases, to change their practice in ways that support student understanding and ability to use mathematics. Academic standards and curriculum frameworks have been developed to guide efforts of teachers, schools, and school systems to create and improve their mathematics programs, and new assessments have been designed. School mathematics is of interest to the public and the focus of discussion and debate at all levels. Never before has such an array of curricular and knowledge resources been available in such a remarkable context of public interest and concern. While these initiatives are not of an order of magnitude necessary to solve the problems, they are promising and stand as evidence that greater investment would yield greater results.

On one hand, then, a significant problem exists: Mathematics teaching and learning in the United States are not as effective as they need to be at the threshold of the 21st century. On the other hand, fortunately, some significant resources exist for the problem's solution: Much effort has already been invested in developing approaches, materials, and knowledge that can support the improvement of mathematics teaching and learning.

Closer scrutiny, however, reveals a pair of secondary problems that hamper efforts to improve mathematics education. Both involve resources, and both are problems that we seek to address with the work proposed here. First, despite the accumulation of potential resources for the improvement of teaching and learning, such resources do not consistently find their way into practice. Even when they do, their use is sometimes uneven, at variance with developers' intentions, or just disappointing. Certainly some of the explanation lies with the fact that some of what is produced is not useful to or usable for the improvement of practice. Other explanations rest with ineffective methods of representing and communicating knowledge. For example, because teachers often do not have systematic opportunities to learn these ideas and to link them to real classrooms, problems of access to and usability of knowledge persist. Often, there are simply no incentives for teachers to make use of research knowledge to improve their practice. In order to link the need for improved teaching and learning with existing resources and the design of new ones, questions of the usefulness and use of knowledge are crucial.

A second problem is one of squandered resources—the diversion of limited resources away from systematic knowledge building. Across the country, widespread concerns about school mathematics have not always proved a resource for collective effort. Instead, such concerns have too often spawned bitter debates about the best ways to remedy problems of mathematics instruction. In these debates, mathematicians, practitioners, and the university community have unfortunately been pitted against one another rather than working together to solve problems. Solutions are advanced with passion and often without evidence, which leaves schools, teachers, and students vulnerable to undisciplined skirmishes and multiple, conflicting signals. Public impatience for improvement has absorbed resources into one hotly argued program or another, and diverted scarce resources from much-needed research to build the evidence for effective practices. A vicious cycle is created in which opposing groups lobby for extreme and ungrounded approaches. When problems arise, the prevailing group is often replaced with another along with its supposedly more-effective plan. Not enough time is allowed for any approach to take hold, resources are drained away from research and toward direct action, and valuable professional attention is diverted into bitter arguments. Investments in research are crucial if much-needed evidence and knowledge are to be built.

The goal is clear: Schools must effectively help all students develop substantial mathematical proficiency. The problems are also clear: Neither do schools currently reach the goal of proficiency, except with a small fraction of students, nor are new resources sufficiently available or used effectively when available to reach the goal of proficiency for all. Achievement is all too often differentiated by class, ethnicity, and language. The improvement of mathematics teaching and learning is a problem still in need of substantial analysis and study, one for which new interventions should be designed and enacted, and their effects inspected.

This combination of needs, problems, and resources shapes a new agenda for research and development focused on the improvement of mathematics education. We seek to design a coordinated approach to research and development that would integrate attention to problems in and of practice with the resources of design and scholarship aimed at producing knowledge that is both useful to and usable for practice.

A COORDINATED AND PRODUCTIVE PROGRAM OF RESEARCH AND DEVELOPMENT

The proposed program of research and development is designed to produce continuous, evidence-based improvement in the teaching of mathematics in the United States. We seek to implement a new approach to coordinating research and development to ensure that data are collected in ways that allow knowledge to cumulate across projects, that results and products are useful for practice, and that efforts can scale up beyond the limits of individual studies. This is an ambitious task, but it is exactly what must be done if the federal investment in educational research is to have a positive impact on the education of American children.

An important impetus for our report lies with recurrent criticism of education research as fragmented, disconnected from problems of practice, and non-cumulative. In mathematics education, for example, where there is substantial concern about instruction and outcomes, improvement efforts often proceed without adequate empirical evidence and are independent of theory about promising courses of action. Likewise, researchers do not consistently take up the real problems that practitioners face. Why does this disconnection exist between research and practice?

One reason is that existing research does not provide adequate knowledge for well-conceived interventions. This shortfall is not surprising, considering the weak investment in educational research in comparison with the investment in innovation and development. But even when research has yielded relevant knowledge, it has not always been easily accessible or used effectively. On the one hand, some favored beliefs and strategies often remain popular despite contrary evidence. An example is the persistent notion that people from technical fields with majors in mathematics can step into classrooms and teach effectively with little additional professional development. On the other hand, some approaches for which empirical warrant exists command little support; for example, there is evidence that teachers can learn new instructional practices from highly scripted curriculum materials. That some see such learning as undesirable interferes with probing more carefully into what might distinguish helpful from unhelpful detail in teachers' guides. Our proposed plan has five key features intended to address these concerns.

First, we seek to better link both basic and applied research with development and practice. Our strategy is to focus on critical goals and to design programmatic research and development that can build knowledge for work toward those goals. Thus, a central feature of our plan is that it be both *grounded in practice* and strategically *goal-focused*.

Second, because problems of practice are *multidisciplinary*, so also are our plan and mode of working. The panel comprises people who have experience and expertise in different parts of the enterprise—education researchers, teachers, policymakers, and mathematicians. Engaging one another in examining collectively the state of the field provides important resources for this complex task of analysis and imagination. That is not incidental: Creating a research community that brings together this sort of experience and expertise is central to both the design and implementation of our agenda.

A third key feature of our proposed plan is its commitment to *multiple perspectives* and *methods*, organized for *collective* work. Rather than working separately on questions and issues that attract an individual's interest and attention, we seek to design the means to engage a diverse community of scholars, practitioners, designers, and policymakers in tackling some of the thorniest problems of mathematics teaching and learning. Some would do basic work, others would design interventions, and still others would conduct surveys or study effective cases.

A fourth feature of our proposed plan is that it be *strategically focused*. The problems we face cannot be comprehensively and concurrently treated in depth by research with present resources. We asked ourselves, "Where, with wise investments in the development of programs, practices, and knowledge, would we be able to say, after 10 to 15 years, that we are in a significantly new place, in terms of understanding and of knowledge-based design and implementation of interventions?"

Finally, a coordinated program of research should *build systematically on past work and established knowledge*. The questions we have are the products of both those we have been able to answer, and those we have not, and the issues we nominate to examine are ones that have emerged as past research and practice intertwine highlighting new problems that warrant systematic attention. Instead of working as though questions are all new—unlinked to the history of previous research, development, design, and policy efforts—we encourage the systematic use of such work, both across and outside mathematics education. This document does not provide a comprehensive view of the territory or of the critical connections that support and give body to the foci of our report. Developing such a view will be an important step in our collective effort toward coordinated work on the problems that focus the agenda we propose.

Toward that end, however, we have built this report on the foundation of two recently published National Research Council reports: *How People Learn*[3] and *Adding It Up: How Children Learn Mathematics.*[4] These two reports, written by interdisciplinary groups of scholars and practitioners similar to our panel, provide a comprehensive synopsis of what is known about student learning, and about the instructional interventions that seem to make important differences to that learning in particular areas. The work summarized is diverse, and spans multiple fields and periods. We saw our task as considering how a productive agenda of research and development could build on the record of such prior work.

FOCUSING A PRODUCTIVE AGENDA FOR RESEARCH AND PRACTICE

The overarching goal of our program of research is to achieve mathematical proficiency for all students. Because students' opportunities to develop mathematical proficiency are shaped within classrooms, through interaction with teachers and with specific content and materials, this agenda focuses on issues directly related to teaching and learning. We have therefore selected as initial areas of focus three domains that bear directly on instruction:

1. The teaching and learning of algebra for mathematical proficiency.
2. The teaching and learning of mathematical practices.
3. The knowledge of mathematics needed for teaching, the knowledge of students as learners of mathematics, and ways to deploy such knowledge in practice.

[3]Bransford, Brown, & Cocking, 1999.
[4]Kilpatrick et al., 2001.

Our aim is to map out a coordinated agenda of cumulative, programmatic research that, at the end of a decade or more, would provide us with the knowledge and practice needed to make this goal attainable. The program outlined below rests on the two foundations necessary for this goal: mathematical proficiency and equity.

Mathematical Proficiency

Because mathematical proficiency is the driving goal of this program, it is central to each of the areas selected for intensive, programmatic focus. The first area is **algebra**. Making algebra one of the three foci of this program calls for coordinated research and design that would probe the nature of mathematical proficiency in a major area of mathematics, and to investigate what it takes to develop it. Algebra is a strategic site for the program we envision for many reasons: some cognitive, some disciplinary, and others more social and cultural. Our second focus explores the mathematical skills, sensibilities, and habits of mind and action that are critical to doing, learning, and using mathematics proficiently but that often remain both unarticulated and untaught. Referring to these as **mathematical practices**, we argue that their identification, analysis, and development are a critical priority if we are to be able to help all students attain mathematical proficiency. Finally, because new knowledge about algebra and mathematical practices can only affect the quality of learning opportunities and outcomes if it can be used effectively for instructional practice, our third priority is the **knowledge needed for teaching**—both on what is needed and how and where it is used. Research and design to explore alternative means for helping teachers acquire and use such knowledge would be a central aim of this area of work.

Equity

Equity is central to this research and design program because attaining mathematical proficiency must be a goal for *all* students. This will require developing instructional practices that would enable all students to reach or exceed acceptable levels of mathematical proficiency, and end the relationship between student performance and race, class, gender, language, culture, or ethnicity. Achieving this goal requires concentrated research and development that focuses on equity issues within the teaching and learning of mathematics. Over the past 30 years, gender differences in mathematics achievement have been reduced, largely because of research knowledge that was accumulated over a number of years, pointing to sources of inequality within mathematics materials and classrooms. We need similar concentrated work to address current inequities for poor students, students of color, and students of diverse linguistic and cultural backgrounds. Examples exist of teachers who have countered patterns of inequality, achieving important successes with diverse groups of students despite poor resources and structural inequities. One important area of future research involves understanding what such teachers do in classrooms, in order to develop a broader knowledge base of equitable teaching practices. Equity issues figure importantly in each of our three proposed areas of focus—algebra, mathematical practices, and knowledge for teaching.

We turn now to discuss these three areas in the next sections. We examine the questions they raise, and sketch examples of the studies, interventions, experiments, and projects that would form the body of collective work that our agenda envisions.

2. FOCAL RESEARCH AREAS

FOCUS AREA #1: THE LEARNING AND TEACHING OF ALGEBRA

This panel believes that a research agenda aimed at developing mathematical proficiency must focus on important topics within the school mathematics curriculum as well as on broader ideas about what it takes to be mathematically proficient. Coordinated studies should investigate how students learn, in particular, mathematical domains; how understanding, skill, and the ability to use knowledge in those domains develop over time; and how such learning is shaped by variations in instruction or patterns of schooling. Where systematic inquiry, focused on specific areas of mathematics and their uses, has been conducted previously—for example, research on children's early learning about numbers, addition, and subtraction—the payoff has been substantial.[5]

To bring focus to the proposed research agenda, we recommend an initial focus on the development of proficiency in algebra. In doing so we define "algebra" broadly to include the mathematical ideas and tools that comprise this major branch of the discipline, including both classical topics and modern extensions of the subject. We include ideas that are found as early as kindergarten: When five-year-olds investigate relationships among colored wooden rods of different lengths, they are working with fundamental ideas of proportionality and measure. When, as six-year-olds, they represent these relationships symbolically, they are developing sensibilities and skills with notation. And when seven-year-olds "skip count"—for example, count by twos starting with 3 (3, 5, 7, . . .)—the basic ideas imply those of linear functions. Elementary school students work with symbolic representations of quantity using the number line, and they also develop fundamental properties of operations. The focus on algebra not only completes the study of arithmetic but also naturally extends to more-advanced, formal concepts and procedures, such as functions, solving higher-order equations, symmetry, the notion of isomorphism and other aspects of mathematical structure, and to connections between elementary and advanced ideas. The coordinatization of space brings algebra in close contact with geometry. In fact, algebra matters in all areas of mathematics because it provides fundamental tools for representing quantities and relationships, modeling situations, solving problems, and stating and proving generalizations. The proposed program of research and design would require mapping the terrain: What sort of mathematical knowledge, skill,

[5] Kilpatrick, Swafford, & Findell, 2001.

reasoning, tools, and dispositions comprise the mathematical domain of "algebra"? Moreover, what does it take to be "proficient" in algebra?

Why Focus on Algebra?

Algebra warrants the central focus that we are proposing for many reasons. One overrides all others: Despite the fundamental role of algebra in mathematical proficiency, an overwhelming proportion of students never become algebraically fluent or skilled. Without this crucial fluency and competence, students' access to many mathematical and other opportunities is gravely limited.

The need for focus led us to choose a single subject area—algebra – for the initial program of research and design. Algebra is an ambitious choice, and also a strategic one. Algebra, foundational to so much other mathematics, and so poorly learned in general, is an area in critical need of collective concentrated research and design.

Several reasons converge to support algebra—defined broadly—as a programmatic priority. First is that algebra functions within mathematics as the language system for ideas about quantity and space, and so serves as a foundation for all branches of the discipline. Algebra is a domain with wide range—its roots take hold before kindergarten and its branches extend through and beyond graduate school. It is also a domain in which core elements of mathematical practice can be seen and developed—symbolic fluency and naming, transforming and simplifying algebraic expressions, identification and use of mathematical structure, analyzing equivalence of representations, to name a few.

A second reason for a focus on algebra concerns its typical role in organized schooling in the United States. Algebra has traditionally been used as a gatekeeper in ways that have contributed to significant inequality of opportunity and outcomes, and, at the same time, it has long been a problem-infested area within the school curriculum. Algebra instruction has often been far from effective, and failure rates in first-level high school algebra courses remain high. For too many students, algebra is a terminal, and demoralizing mathematics course. Robert Moses[6] argues forcefully that algebra be regarded as "the new civil right," accessible to all citizens:

> . . . once solely in place as the gatekeeper for higher math and the priesthood who gained access to it, [algebra] now is the gatekeeper for citizenship, and people who don't have it are like the people who couldn't read and write in the industrial age . . . . it has become not a barrier to college entrance, but a barrier to citizenship. That's the importance of algebra that has emerged with the new higher technology. (Moses & Cobb, p. 14.)

[6]Moses & Cobb, 2001.

Because of algebra's centrality to mathematical literacy, and because of its position as the main access route through high school curriculum, it is an area critically in need of programmatic focus. Crucial is to build the means and tools so that all students are able to learn algebra successfully. Accomplishing this goal requires not only commitment, but also systematic knowledge about its teaching and learning.

Third, focusing on algebra would make it possible to explore its role in helping students structure and make sense of ideas that are often obscured. In elementary school, for example, students typically learn about numbers and the operations of arithmetic from the point of view of solving problems and making calculations. The simplicity, beauty, and power of arithmetic as a system do not readily emerge from these experiences. Algebra can provide structure and language for making the systemic structure of arithmetic visible to students, consolidating what they have learned into a flexible and coherent system. How algebra might play such a role in learning is an important area of inquiry.

A fourth reason to focus on algebra is that it is here that the disciplined use of abstract mathematical notation becomes central to the learning and doing of mathematics. Learning its syntax and grammar and becoming fluent with its meanings often presents students with some of the same cognitive obstacles encountered in early literacy learning. Constructing pathways from experience and natural language to the more structured and compressed forms of mathematical expression is often difficult. Without command of the powerful linguistic tools of mathematics, students' progress is often significantly hampered. A focus on algebra would spotlight this important problem of learning.

A fifth reason to focus on algebra is that this domain—especially when approached from the perspective of functions and modeling—also affords opportunities to study potentially powerful uses of technology in modeling and solving complex problems.[7] Thus, a focus on algebra allows an examination of conditions under which technological tools—such as graphing utilities, spreadsheets, or computer-based data collection devices—can support students' mathematical activity and learning. Moreover, a focus on algebra also opens the door to an examination of the application of school mathematics in modeling and solving contextualized problems.

Yet another reason to make algebra a central focus lies in current initiatives to improve mathematics education, initiatives in which algebra occupies a more central role than ever. As these initiatives are implemented, practitioners will be dealing with algebra more broadly and

[7] See, for example, Heid, 1996; Janvier, 1996.

across more of the school curriculum than previously. For example, NCTM's *Principles and Standards for School Mathematics* calls for algebra, including its more analytic elements such as functions, to be developed across the curriculum. The National Research Council (NRC) report, *Adding It Up: How Children Learn Mathematics*, also recommends that basic ideas of algebra as generalized arithmetic be introduced in the early elementary grades and learned by the end of middle school. The NRC report further recommends that strategies be developed and tested to help students move from arithmetic to algebraic thinking.[8] The intensified and extended emphasis on algebra will require new knowledge for instruction.

Finally, a focus on algebra offers great potential for the results of this research to inform both the framers and enactors of educational policies that affect mathematics teaching and learning. Algebra—perhaps more than any other topic—has become a deeply contested area of the curriculum in the past decade. Central to the debates has been the question: What should be the subject matter of "algebra" and what, and in what ways, should students learn? How prominent should applications be? Who should learn algebra, and when? Some critics decry the perceived decline in students' symbolic manipulation skills while others blast the dry and meaningless rendition of algebra that too often produces students who might be able to manipulate expressions but cannot do so sensibly in the course of solving problems. Although considerable research has been done in the past on the teaching and learning of algebra, the findings and insights gained from this work have made little penetration into the policy conversation. What is needed at this time is a systematic research and development agenda that is focused on algebra and that is intentionally designed to contribute information to the designers and enactors of policies that affect algebra teaching and learning. A successful endeavor of this kind could make significant contributions in moving the field from argument and critique-swapping to knowledge-based improvement efforts.

What Would a Focus on Algebra Afford?

Algebra, defined broadly, is fundamental to mathematics. Moreover, some significant research has already been conducted in algebra. Still, there is much that needs to be done to bring the work to full fruition. For example, we need a developmental perspective on the ideas of algebra and on how they may grow in students from the early grades through advanced study. We lack coordinated knowledge about instruction on specific central concepts and skills, and we do not understand sufficiently how different curricular choices and trajectories affect the development of algebraic proficiency. We understand too little about the ways in which

[8] Kilpatrick et al., 2001, pp. 11–14.

technological tools can be used to foster and enhance students' algebraic proficiency. These and many other related issues are ripe for systematic investigation at this time.

We continue by specifying three crucial areas of investigation that a focus on algebra would afford: (a) defining, investigating, and comparing alternative perspectives on algebra; (b) developing knowledge and practice centered on teaching and learning core ideas of algebra; and (c) using the focus on algebra to develop knowledge and practice related to broader aspects of the teaching and learning of mathematics.

1. Defining, investigating, and comparing alternative perspectives on algebra

Progress in our understanding of how students can develop proficiency in algebra will involve exploring the nature of sound understanding and competent performance in algebra. One foundation of a coordinated program of research and design would be to examine alternative perspectives on algebra, to consider what is meant by and included in "algebra," and to make visible and compare different implications for effective instruction.

Chazan[9] sketches three different views of algebra found across various curricula, studies, and other representations of the subject. One view sees algebra as a consolidation or generalization of ideas in arithmetic; a second represents algebra as the study of structures, patterns, and symbolic representation; and a third conceives algebra as the study of functions, covariation, and modeling. Other authors,[10] NCTM's Algebra Working Group, Lacampagne's *Algebra Initiative Colloquium*, the recent International Commission on Mathematical Instruction Algebra Study,[11] and framework documents such as_NCTM's *Principles and Standards* also provide ways of organizing the algebra domain. What do these different perspectives afford, how do they interact, and what are warrants for viewing algebra from any particular perspective? What kinds of student learning do they suggest and support? How have curriculum developers and teachers reconciled or integrated these different views of algebra? Investigating algebra as it is represented in curriculum materials, or in use, would be useful in contributing productive frameworks for inquiries into the teaching and learning of algebra.

Questions important to pursue include:

- What are the core ideas, skills, and practices encompassed within school algebra as it has been developed along different perspectives, and how are they related to one another and to other central mathematical ideas and practices? What are alternative

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[9]Chazan, 2000.
[10]See, for example, Bednarz, Kieran, & Lee, 1996.
[11]Chick et al., 2001.

maps of this content and its relationships and connections? In what ways are the answers to these questions different in technology-intensive instructional settings?

- What are the algebraic knowledge, skills, and practices needed for various later courses of mathematics instruction (geometry, discrete mathematics, statistics, calculus, etc.)?
- What is "algebraic thinking"? What is entailed when engaging with the concepts and processes of algebra?
- What are ways to bring empirical studies to bear on the contemporary debates about appropriate goals for the subject, and its placement in the curriculum?
- How can core ideas, skills, and practices of algebra be assessed? What tools can be developed in order to track more carefully students' difficulties and progress, understanding and misconception?

2. Investigating core algebraic ideas: Content, teaching, and learning

One thing that a coordinated program of research and design in algebra would do is to enable the probing of specific core ideas, analyzing their mathematical roots, meaning, and uses. Studies would examine how students apprehend, learn, and use these ideas, and would investigate the effect on students of different approaches to teaching these ideas. Examples of such core ideas include equality, variables, the field axioms, order, language and syntax, logic, linearity, functions, relationships among algebraic representations, modeling, and change or covariation.

Take the relationship of *equality*, for instance. Unquestionably a central mathematical idea, the meaning and use of the notion of "equal" is more complex than we often acknowledge. Moreover, the use of the equals sign (=) as a notation to indicate that two expressions are equivalent is central to progress in learning to solve algebraic equations in a meaningful way. Yet, seminal research studies of students' understandings and uses of equality and equation[12] have shown that many students come to their study of algebra with notions that may interfere with their acquisition of algebraic proficiency. For example, some students think of an equals sign not as a statement of equivalence, but rather as a command to perform an operation. That is, if they are asked 8 + 4 = ___ + 3, they fill the blank with 12. They have taken the **symbol** "=" as a command to perform the operation that precedes it: add 8 + 4 and then write down the answer. Researchers have suggested that this tendency derives from children's experience in executing arithmetic operations and writing down an answer immediately to the right of an equals sign.

[12] See, for example, Kieran, 1981; and Wagner, 1981.

Reminiscent of classical research on this topic,[13] recent work by Carpenter and Levi[14] has extended this earlier research to younger children in order to understand better the roots of this view of equality and possible instructional interventions that might lead to a more appropriate and productive view of equivalence.

What is it to understand the notion of equality in mathematics? What does it mean for two things to be equal, or equivalent? For example, when students work with fraction notation, they learn to determine whether two fractions are equivalent. What does this mean? How should students understand the idea of "equal" when we write that $3/6 = 1/2$, or when we write that $257 = 200 + 50 + 7$, or when we write $8 + 4 = 9 + 3$? Consider that, although these may look similar, each uses the notion of "equal" slightly differently as a function of the context. In the first example, we are saying that the symbolic expression "3/6" and the symbolic expression "1/2" represent the same numerical value. In the second, we are saying that the way we write "two hundred fifty seven" is composed of three values—two hundred, fifty, and seven, hence 257. And in the third, we are expressing the equivalence of two expressions: $8 + 4$ and $9 + 3$. One uses the equals sign here to indicate that the addition $8 + 4$ and the addition $9 + 3$ produce the same number. What is common to these examples is that, in each case, the expressions on each side of the equal sign are representations of a number, and the equation signifies that these two numbers (not their representations) are the same. As children encounter these different ideas in school, the equals sign reappears, each time to represent a slightly different idea about two things being "the same." Meanwhile, they also often see misuses of the equals sign—for example, as a running record of a series of operations: $2 + 4 = 6 + 3 = 9 + 1 = 10$, to record $2 + 4 + 3 + 1$. Yet what is actually written suggests not that $2 + 4 + 3 + 1 = 10$, but literally implies the absurdity that $6 = 9 = 10$. Other situations in which students might consider equality include such questions as whether or not two sets are equal: the set of even numbers and the set of all numbers divisible by 2, for example, or the set of prime numbers greater than two and the set of odd numbers. Later, in more-advanced algebraic work, students spend time solving equations, such as $3x + 8 = 9x - 7$, and at about the same time in their studies, encounter linear functions represented in the form $y = mx + b$. What are the implications of early understandings of the equals sign for students' proficiency in navigating among uses of these signs in expressions, equations to be solved, and functions?

Another major idea is the notion of variable. What are some different ways to think about the concept of a variable in mathematics? What difficulties arise for students as they learn to think about and use variables in mathematical expressions? How do early encounters with "missing addends,"

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[13] Behr, Erlwanger, & Nichols, 1976.

[14] Carpenter and Levi, 1999.

which signify specific unknown values, affect students' subsequent comprehension of variable? As with equality, there is a body of research on which to build, including several large-scale investigations of children's use of variables in mathematical expressions[15] and some clinical studies that have examined children's difficulties in more-individualized detail.[16]

Systematic, coordinated research is needed on the nature of these ideas, and what learning them involves. How do children think about and use equality or the idea of a variable? How might learning about more-advanced mathematical concepts, such as equivalence relations, give teachers a more robust understanding of the concept of equality in ways that make a difference for their instruction? How can early understanding of variables develop into mature understandings that support investigation of variation and change. What role might technology play in helping students and their teachers gain more-robust understandings of variables?

As our brief consideration of notions of equality and variable indicates, many important questions arise about the connections between arithmetic and algebra. What role is played by the fundamental properties of operations—for example, commutativity, associativity, distributivity, the neutral effect of adding 0 or multiplying by 1, the relation of subtraction to addition or of division to multiplication—in learning to work with numbers? How do students' explicit and implicit conceptions of these properties—the "field axioms"—affect learning to read, write, and manipulate algebraic expressions? How do students learn to think about these properties in general? Similar questions can be raised about the interplay between learning algebraic syntax and knowing the syntax of natural language. How can natural language be used as a scaffolding for the construction of algebraic knowledge? In what circumstances does natural language interfere with the acquisition of algebraic understanding? For example, when we say "two of the ten people in the room are vegetarians," we mean that two are vegetarians and eight are not: we have *subtracted* two from ten. But if we say, "Half of the people in the room are smokers," we are *multiplying* the number of people in the room by one-half. The preposition "of" describes both operations. What sorts of challenges to learning are presented by these different correspondences between natural and algebraic language?

The concept of function is viewed as central in most secondary school curricular treatments of algebra. Much has been demonstrated about students' fragile understanding of the function concept, and of the interrelationships among equations, graphs, and tabular or verbal representations. With a trend toward algebra as a strand across the grades, significant questions about how children's early experiences with patterns might develop into the basis for understanding relationships between two sets of numbers.

[15]Collis, 1975; Kuchemann, 1981.
[16]See, for example, Booth, 1984.

In making our call for fundamental research of this kind we are not ignorant of the considerable research knowledge base that already exists on many of the issues we have identified. The teaching and learning of algebra has been an active field of research in the United States and abroad for several decades, and there is a corpus of research on which to build.[17] Nevertheless, that research base is in need of further codification, clarification, and synthesis, and there is no doubt that further systematic inquiry is needed, especially in light of recent changes in curricular expectations and instructional methods across grades K–12. Questions that we view as important to pursue both in codifying existing research and pursuing new inquiry include:

- How do students develop an understanding of and facility for using various algebraic concepts?
- What, more explicitly, do students have to *do* to learn how to learn algebra? What are some of the key learning skills and practices needed to learn algebra?
- What are specific and predictable difficulties that students have with learning algebra, why do they experience those difficulties, and what kinds of instructional interventions might they call for?
- What are natural connections to other strengths and knowledge that students have that could be used as resources for learning algebra?
- How do the various representations of algebraic notions (symbols, words, graphs, charts) affect student learning of algebra? How do students learn to use those representations, and interrelate them, effectively?
- What is the interplay of contextualization and abstraction in student learning of algebra?
- How can and how should elementary algebra concepts and procedures connect to other areas of mathematics, such as data analysis, geometry, measurement, calculus, mathematical modeling, or discrete mathematics?
- In what ways can the algebra curriculum provide entry into the development of "mathematical practices"? What important mathematical practices figure prominently in algebra?
- What prior mathematical knowledge do students need to learn the core ideas, tools, and skills of algebra?
- What are useful roles for technology in the learning and teaching of algebra? In particular, how are graphing calculators and computer algebra systems used in the teaching of algebra, and what are their potential impacts on student learning of a

[17]See, for example, Bednarz, Kieran, & Lee, 1996; Sutherland et al., 2001; Wagner & Kieran, 1989; and Chick et al., 2001.

variety of algebraic ideas? How is the development of symbolic fluency affected in the computer algebra system environment?

- How does students' knowledge of algebra and their ability to use it develop over time? How can this be reliably assessed?

3. Using a focus on algebra to build knowledge about central elements of mathematics teaching and learning

A focus on algebra would also afford opportunities to make progress on several fronts strategic to the improvement of mathematics education. The codification of existing knowledge coupled with the systematic development of new knowledge about how students can be assisted to learn algebra well would provide resources to rationalize current debates that are often founded as much on ideology and belief as on evidence and logic. At the same time, underlying these issues lie elements of the teaching and learning of algebra that are also critical to the teaching and learning of mathematics more generally.

One fundamental element central to algebra, and also to other areas of mathematics, concerns relationships between abstraction and contextualization in learning. Algebra provides powerful abstract concepts and notation to express ideas and generalized relationships and a set of rules for manipulating them. These tools are invaluable for solving a wide range of problems. Learning to make sense of and operate meaningfully and effectively with these tools presents formidable challenges of learning and teaching. How do students develop these capacities with the power that abstraction can afford? This power involves both moving from contexts to abstract models and, conversely, interpreting abstract ideas skillfully in concrete situations.

Beliefs abound concerning the directionality of these connections in learning: Some argue that all meaningful learning must move from the "concrete" to the "abstract"; others insist that the power of the generalized, abstract forms affords learners greater insight and generalizable application. Notions of what counts as "concrete" or "abstract" remain vaguely and variously defined. Mixed into the debates are assumptions about what students find meaningful, as well as what affords access to the mathematical ideas. In fact, the notion of what is abstract or concrete may not be intrinsic to the knowledge but rather relative to the knowledge base of the learner. Ideas that, when first encountered, appear as general and abstract typically become familiar and experientially concrete once they are assimilated and used. And once they are familiar, they become, in turn, settings for further generalization and abstraction. An apparent weakness of so-called traditional school mathematics instruction that begins with abstract concepts and principles and moves eventually toward settings of application is that it is difficult to scaffold students' engagement with the abstractions without concrete referents. In an apparent effort to correct the tendency of school mathematics to overdo abstraction and formalism, some new curricula begin with and consistently engage students with richly contextualized examples and problems. Yet, a fundamental issue that confronts a more contextualized approach is how students can and do acquire appropriate generalizations from the particular examples they encounter. This issue is applicable to all school mathematics, not only algebra, but it can be productively engaged through research on algebra.

A second fundamental element of mathematics learning and teaching that is both central to algebra and applicable more broadly in mathematics concerns the development of symbol manipulation skills. Long a problem in mathematics instruction, arguments have centered on the nature and role of "practice" (or drills) in promoting skilled performance. Compelling arguments can be made that procedural fluency is enhanced by intense use. But such intense use might well be designed into conceptual explorations (for example, into the nature of algorithms and analysis of why they work); it does not have to be programmed as an isolated repetitive drill. A current issue is how different instructional uses of technology interact with the development of such skills. The increased availability of technology raises new questions about what is meant by "symbolic fluency." What role might be played by graphing calculators and computer algebra systems? Some studies have shown that computer-based visual and numerical supports for algebraic symbolic representation can help students develop meaningful understanding.[18] At the same time, important questions remain about the role of paper-pencil computation in developing understanding as well as skill. These are questions that appear at every level of school mathematics. Empirical investigation and evidence would be useful to debates that have been, until now, founded largely on belief and ideology and would enhance the existing base of research in this area.[19] What are students currently having opportunities to learn, and what knowledge, skills, and dispositions are they developing under different curricular and instructional regimes?

A third element is the development of students' mathematical knowledge over time. The context of the teaching and learning of algebra is particularly well suited to this because it is both conceptually rich and syntactically complex. How do mathematical ideas develop across time and with different instructional treatments? For example, as algebra shifts to being a K–12 enterprise, new questions emerge that invite systematic investigation. If students learn about variables and equations sooner and engage earlier in algebraic reasoning, how will these earlier experiences shape the development of students' algebraic proficiency over time? What precursor activities and experiences with pattern and relationships in the early grades will set the stage

[18]Kilpatrick et al., 2000.
[19]Heid, 1996.

effectively for understanding mathematical modeling in later grades? Addressing questions such as these will require significant attention to issues of assessment, both small and large scale.

Yet another fundamental element concerns how connections among areas of mathematics can be made to enhance students' learning. Too often students develop ideas in one mathematical domain but make no connections with fundamental ideas and problems in another. These connections are crucial within mathematics. Consider, for instance, the relationship of functions to the ideas of correlation and curve-fitting in data analysis. How can this relationship be best developed to help students understand and use the concept of function? The fact that space can be described with numbers (coordinates) is the foundation of analytic geometry, the bridge between algebra and geometry. How might students develop an understanding of this fundamental connection between geometry and number? Such understanding might help students with the notion that "absolute value" of a difference represents the distance between two points on the number line. Or take the Pythagorean Theorem ($a^2 + b^2 = c^2$), for example. Students usually learn this as a theorem about right triangles, but do not realize that it is the basis for the distance formula used for calculating distances between points in the plane. Similarly, they may learn to manipulate x's and y's and never realize that x^2 has a geometric representation as a square with side lengths of x. They do not recognize that they can visualize the difference between $x^2 + y^2$ and $(x + y)^2$ quite simply with a diagram. As these examples suggest, connections among areas of mathematics—algebra and arithmetic, or algebra and geometry—can be fruitfully examined using algebra as a focus domain. Yet, this focus also brings us squarely to issues related to the organization of the U.S. curriculum.

In the U.S. high school curriculum, different mathematical areas have been the basis for separate courses, and algebra is typically the first of these separate domains. In contrast, mathematics in the elementary and middle school is typically organized to include experiences with several different mathematical domains, but algebra has not traditionally been an explicit focus of attention in the elementary and middle grades. These traditions have recently been challenged by analyses showing that the secondary curriculum in most other countries do not isolate algebra within a course apart from other topic areas and the elementary and middle school curricula in most other countries treats algebra more extensively than has been the case in the United States. Many of the so-called "standards-based" mathematics curricula developed in the past decade include such features of integration of content areas at the secondary school level and attention to algebra at the elementary and middle school levels. Thus, a focus on algebra allows the impact of different curricular organization schemes to be an object of focus. How do these different curricular structures influence students' learning? As algebra begins to permeate the elementary curriculum in the coming decade, how will its curricular trajectory change? How will such changes affect students' proficiency with algebra? Algebra's curricular scope—whether located in the traditional high school course sequence or expanded across the grades—presents important questions about the mathematical opportunities available to diverse populations of students, with varied prior success.

One additional fundamental element deserves mention. Issues of equity in school mathematics are in need of considerably more attention than they have received in the past. Many of the most important issues are readily accessible for examination via analyses of access to and the teaching, learning, and assessment of algebra. Algebra is probably better suited than any other area of the mathematics curriculum as a focus for such investigations. Questions about school, district, and state policies on algebra requirements and assessments require serious research consideration. The role of algebra as a gatekeeper has divided students into classes with significantly different opportunities to learn. Currently, disproportionately high numbers of minority students are not adequately prepared in algebra and do not have access to serious mathematics beyond this level. What is the dynamic of this tracking process? What is it about this subject that has led to its critical political and social position in the curriculum? How could this be changed?

We need a close look at the issue of the learning of algebra in those segments of the population whose success rate in learning algebra has not been high. Are there specific routes to algebra proficiency that might be more effective within the social context of inner-city schools or minority cultures? How can mathematics instruction capitalize on the strengths that students from different cultural groups bring to the classroom in order to enhance the learning of algebra? We know that education is resource dependent, and that poorer communities often suffer from the lack of well-trained teachers, efficient administrators, and equipment that might support instruction. Are there strategies that might be used to ameliorate these problems so that their negative effect on students' learning can be reduced or eliminated?

Questions important to pursue include:

- How do different instructional arrangements, curricula, or approaches to instruction influence the development of students' usable knowledge of algebra?
- How do particular designs affect the difficulties that students often experience in learning algebra? How do they take advantage of students' strengths and resources?
- How is student understanding and continued use of algebra affected by algebra's being concentrated in one or more high school courses? By teaching algebra in conjunction with other subjects? By teaching algebra across the grades?

- How is students' learning influenced by particular interventions designed to affect their knowledge of and use of algebra? What are reasonable ways to assess students who are involved with different curricula, programs, and interventions?
- What are the effects of various curricula and teacher practices on student achievement in algebra? How do such instructional approaches impact issues of equity?
- What curricular and instructional approaches exhibit demonstrated potential to enhance the algebra proficiency of all students, especially those least well-served in current system?

FOCUS AREA #2: THE LEARNING, USE, AND TEACHING OF MATHEMATICAL PRACTICES

Competent learning and use of mathematics—whether in the context of algebraic, geometric, arithmetic, or probabilistic questions or problems—depend on the ways in which people approach, think about, and work with mathematical tools and ideas. Such competent use and learning is also influenced by habits of mind, orientations, and sensibilities. This insight has led us to recognize the pressing need for knowledge of what proficient users of mathematics *do*—what they attend to, care about, try, and pursue. The goals that drive this program—mathematical proficiency and equity—draw us to this focus: What *are* the tools, skills, and habits of mind and action that form the basis of learning, doing, and using mathematics proficiently, and how can they be developed more equitably? We refer to the collection of resources that constitutes skilled performance as mathematical *practices.*[20] The program of research and design that we envision would investigate and identify such practices and would explore opportunities to learn such practices. How are mathematical practices treated and used in different courses of instruction? In curricular programs? In teaching practices? Within experiences in and out of school? What affects their equitable acquisition?

What Do We Mean by "Mathematical Practices"?

Because this is not a familiar idea, some examples of what we mean by "mathematical practices" might be useful before proceeding. One domain of mathematical practice is the flexible and meaningful use of symbolic notation. Symbols and notation are important mathematical tools, and mathematics has a syntax and grammar ideally suited for expressing quantitative ideas and relationships. Whereas most Americans can read and interpret ideas expressed in written language, many do not read and make sense of symbolic notation fluently.

[20]Scribner & Cole, 1999.

People often report skipping over symbolic notation, and find it awkward when trying to write mathematical ideas. We would say that they do not have well-developed practices with symbolic notation. For example, in a now-classic study, college students were asked to express the relationship between the number of students and the number of professors.[21] Told that there were six times as many students as professors, more than one-half of social science majors and even more than one-third of engineering majors were unable to formulate this simple proportional relationship correctly, writing 6s = p instead of 6p = s. Likely they mapped the symbolic form onto the linguistic syntax directly—"*six* times as many *students*"—rather than considering the entire relationship that was to be represented. Using notation to write down similar kinds of mathematical statements has been shown to be difficult for high school students and adults, as well as for college students.[22] Being able to express ideas in clear and compact ways is often crucial to being able to figure out and solve problems, and being able to read mathematical notation is central to being able to make sense of many different forms of quantitative information, both within mathematics and in everyday life. The practice of using symbolic notation is one example of what we mean by the "tools, skills, habits of mind and action that form the basis of learning, doing, and using mathematics proficiently." Although reading and writing mathematical notation is crucial for mathematical reasoning and communication, it has rarely been taught with the explicitness given to learning to read and write natural language. Learning to use notation lies between the lines in most mathematics classes, and consequently far too few students develop adequate fluency.

Another example of practices are the habits and skills involved in sizing up problem solutions—the constellation of questions and self-checks that a mathematically proficient person uses to verify and evaluate the validity and completeness of a solution, whether in a school task, some problem in the workplace, or in everyday life. Closely related are the skills and tools involved in proving that an idea is true, a solution valid, or a method effective. Other practices useful to learning, doing, and using mathematics include practices of representation—making tables, lists, drawings, graphs, physical models—that make visible, accessible, and usable crucial features of a problem or an idea.

For example, consider the elementary and yet challenging puzzle of figuring out how many ways there are to use 8s strung together with plus (+) signs to equal 1,000.[23] One solution to this puzzle is 888 + 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8. But to determine all the possibilities, being able to set up and use a table strategically is important. What different sorts of

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[21]Kaput & Clement, 1979.
[22]Lochhead, 1980.
[23]Gelfand, 1993.

lists or tables might be made? Which would be most helpful? Solving this problem depends on more than the simple arithmetic that it seems to imply. It depends, crucially, also on skills and tools of organizing the collection of solutions. To talk about *practices* is to call attention to and elaborate what is involved in learning, doing, and using mathematics proficiently, with an eye to the tools, skills, and habits of mind and action used.

In order to elaborate the idea of mathematical practices, we turn next to an extended but simple example of mathematical problem solving. We use this example to make visible the practices entailed in even a rather simple bit of mathematical work.

Mathematical practices in use: An example of an elementary combinatorics problem. The problem: determining the fewest number of coins it takes to make 43¢ if you have available pennies, nickels, dimes, and quarters. How would you figure this out and know for sure that your answer is correct? One approach might be to find *all* the possible combinations and record them on a scrap of paper. Then you could count the number of coins in each combination, and pick out one with the fewest coins. Doing this would require being able to identify all the possible combinations, and to list them in a way that would enable you to be sure that you had found all the possibilities. Amazingly, 31 different combinations exist to make 43¢. This is a lot to deal with, but it does give assurance that you have found a minimum solution.

A second approach would be to size up the problem and realize that it is the quarters that get you the most value per coin, so there is an advantage to using them whenever possible. Given this initial insight, you might choose to proceed more directly, choosing the largest possible coin value at each stage, a process sometimes called a "greedy algorithm." With this approach, you would be able to determine the solution without listing all the possibilities. To do this, you would first choose a quarter. To make up the remaining 18¢, because quarters are no longer possible, you would next use a dime. Then you would make the remaining 8¢ with a nickel, and finish with three pennies. This is not unlike what a checkout counter clerk might do when making change. You might feel intuitively that this must give the minimal solution, and indeed it does. However, verifying this would require some further mathematical reasoning. You might appreciate that this is not transparent when you think of another case: Suppose that the only coins available were pennies, dimes, and quarters (no nickels), and you wanted to make 30¢. The "greedy algorithm" approach would lead one to choose a quarter, and then five pennies, six coins in all. But the minimal solution would be three dimes.

Both approaches work well to find the answer: Both improve upon guessing and are reliable and usable. But they draw on different tools, skills, and habits. For the first approach, a person obviously has to be able to calculate accurately. But this approach also depends on realizing that one needs to be sure that he or she has all the possibilities. Being able to make a list that makes plain that all combinations have been identified would be one way of doing this. For example, a list that is not structured in any way will not provide a convincing proof that all combinations have been identified. One list useful for this purpose could be made by dealing first with pennies-only solutions, followed by penny-and-nickel solutions only, followed by penny-dime and penny-nickel-dime solutions, and, finally, by solutions that use quarters. Other methods of organizing the list could be used; some might not provide as much help for establishing its completeness. Being able to evaluate alternative listing methods is yet another resource needed for this solution.

What practices are involved in the second approach? First it involves sizing up the problem. Using this method requires seeing that the first approach involves more work—that is, more calculation and more tabulation—than is necessary to solve the problem. This first approach furnishes more information, but the added information is not essential for solving the problem as posed. Approaching the problem in the second way involves using principles of "local optimization" to arrive at 43¢ as quickly as possible. This approach involves taking steps that are as big as possible toward this new goal—that is, starting with as many quarters as possible, then as many dimes as possible, then as many nickels as possible, and, finally, as many pennies as needed to arrive at 43¢ with the smallest number of coins possible. This approach is faster than the first, and requires far fewer calculations and virtually no skills of recording. Intuitively, this method seems to produce the combination with the fewest coins. However, proving this in general would require more work. A person who wanted to *justify* this method would have to use more tools and skills than mere implementation of the "greedy algorithm."

Examining two approaches to solving this simple problem provides a glimpse of some basic mathematical practices. On the surface, this seems to be a routine arithmetic problem, but when we look more closely at what it took to analyze this problem, we see more. We see notice different constellations of tools, skills, and habits that could be used in the course of its solution. The first approach involved conceiving the set of all coin combinations worth 43¢ in which the solution would be found, *a problem-solving strategy*, and then finding some way to *represent* and *mathematically organize* this set so as to exhibit its completeness. The second approach involved being able to *assess the computational work* and to *choose a method to maximize efficiency*. To prove that this method does indeed yield a minimal solution would further skills of *mathematical comparison and reasoning*.

We can identify some of the practices entailed in the two approaches and discern some of what each approach takes, but, because we lack detailed understanding of mathematical practices, much remains between the lines. Often only implicit, these practices shape the work. Someone who did not have these practices available would not be able to work in these ways.

Such a person might guess at a solution, or give up. In fact, in school, many students do not develop such practices, and although they may be able to add coins, they would not be able to size up the problem, work proficiently on its solution, or be able to justify its completeness (first approach) or validity (second approach).

Why Focus on Practices?

As with our focus on algebra, this proposal to focus an agenda for research and design on "mathematical practices" is a strategic one. Denning[24] writes, "It is not literacy, but practices, that create actions and constitute expertise." A focus on mathematical practices offers a different way to think about what it takes to do and use mathematics with the expertise needed to function in and out of school. This focus on practices calls attention to aspects of mathematical proficiency that are often left implicit—not just knowledge and skills, but habits, tools, and dispositions. It also calls attention to what it takes to be able to learn mathematics effectively and to the opportunities that are provided for such learning. Perhaps most important, a focus on practices affords a fresh and constructive perspective on understanding differences in mathematical proficiency: namely, the hypothesis that people who do well with mathematics tend to have developed these resources and to be able to use them flexibly and skillfully, while those who are less proficient often lack them. For these reasons, we argue for a systematic effort to understand these less-visible, often implicit, aspects of mathematical thinking, and the opportunities that might be designed for their learning. Such work could have high payoff for improving our ability to help all students develop mathematical proficiency.

One reason to propose a focus on practices for learning, doing, and using mathematics is to confront the pervasive—and damaging—cultural belief that only some people have what it takes to learn mathematics. If mathematics can only be learned well by a talented few, then not all students can really become mathematically proficient. This belief has consistently interfered with setting sufficiently high expectations for all students' mathematical achievement in this country. The U.S. curriculum has been characterized as "a mile wide and an inch deep," with few opportunities for sustained study or the development of stable and useful mathematical proficiency. As a society we do not believe that most people can be good at mathematics. Consequently, we accept the tracking of many students into self-fulfilling programs of low mathematical expectations, and invest far too little in understanding what it takes to teach all students to become proficient. Uncovering practices involved in learning, doing, and using mathematics can help demystify mathematical proficiency.

[24]Denning, 1974, p. 106.

A second reason to study the tools, skills, and habits crucial in learning, doing, and using mathematics is because they supply learning resources needed by teachers and students engaged in more-ambitious curricula and working toward more-complex goals. Without these resources, ambitious agendas for improvement are unlikely to succeed. When higher standards for students' performance are set, educators still know too little about what students and teachers would have to do and learn to attain these goals. What it would take for all students (and teachers) to reach such ambitious heights has not been adequately examined. For example, when a teacher whose students have never been asked to explain their thinking asks them to justify their solutions, she is likely to be greeted with silence. When she asks a student to explain his method, he will probably think he made an error. And when teachers assign more-challenging work, students, unsure of what to do, may ask for so much help that the tasks' cognitive demand declines.[25] Discouraged, teachers may conclude that these students cannot do more-complex work, and may return to simpler tasks. Observers may disparage such teachers for giving up. When students and teachers fail to reach high standards, the blame is often seen as the teacher's. Instead, we suggest that the reason teachers may fail is because they do not have ready the skills, tools, and habits required to do more-complex mathematical work because they have not had opportunities to learn them—nor do they even realize that such practices exist.

A third rationale for focusing on mathematical practices is that if the tools, skills, and habits involved in mathematics were made more visible, it would be possible to design and teach them directly, and, hence, for all students—not just a select few—to acquire them successfully. Like some of the groundbreaking work that Carol Lee[26] and her colleagues are doing in English, which is focused on literary interpretation, and on connecting students' prior skills and interests with literary practices, this focus could enable a serious challenge to pervasive inequalities seen in school mathematics. In their work, Lee and her colleagues work from two directions: on one hand, they seek to uncover and articulate closely the practices of literary interpretation involved in reading poetry or fiction; on the other hand, they study practices in which urban youth engage in other contexts—music and talking, for instance. They then build instructional correspondences between practices in which students already engage and structurally similar practices entailed by literary interpretation. Similar investigations of mathematically analogous practices in which students—especially those who have been disenfranchised from mathematics—engage in their own environments would be important to build parallel instructional mappings in mathematics.

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[25]Stein, Grover, & Henningsen, 1996.

[26]Carol Lee, 2000.

Our emphasis on investigating mathematical practices offers a means for uncovering what has to be learned in order to learn and do mathematics proficiently. Uncovering these practices can make it possible to design systematic opportunities for students (and teachers) to develop the learning resources needed to build a system in which all students can become mathematically proficient.

What Would a Focus on Mathematical Practices Afford?

We have provisionally defined mathematical practices as the tools, skills, and habits of mind and action that form the basis of learning, doing, and using mathematics proficiently, both in and out of school. We have offered a few examples and have illustrated what might be counted as practices. Our proposal for a coordinated research and design agenda focused on mathematical practices is based on the conviction that this focus would yield crucial resources for breaking down the broad gap between those few who become mathematically proficient and those many who do not. Building knowledge about what it takes to do mathematics well would help bring to the surface crucial elements of mathematical thinking and work whose acquisition has too often remained largely a matter of chance. It will help make visible the places in which mathematical practices may be learned, at school and at home, and the course and curricular arrangements that enhance or impede their learning. But so far, the notion of practices is an underdeveloped one, with connections to more-familiar ideas such as problem solving, communication, and reasoning. Where do we see critical investments needed if this focus is to have the payoff that we envision?

Work in cognitive psychology and in the study of learning make plain that there are skills, knowledge, and habits of mind related to learning in a particular field that are crucial to competence in that domain, both in using it and in learning it. Examining the work of scientists and mathematicians reveals "specific, repeatable sequences of activities on which scientists rely in their daily work."[27] This view of knowledge as practice permits a perspective on competence similar to what has enabled work on the teaching and learning of reading to make progress. Research in reading has highlighted the role of metacognition, self-regulation, and other skills of decoding text and interpreting its meanings. Researchers have investigated what competent readers do as they navigate and interpret different kinds of texts. Understanding what competent readers do has significantly contributed to our understanding of the goal, and of its elements. But what it takes to *move toward expertise* is not clear from studying expert performances. That is, the work required to develop expertise is not visible within portraits of

[27]Pickering, 1995, p. 4.

expertise itself. A discriminating appreciation of opera does not equip the music critic to learn to perform arias.

A rich body of work in mathematics education has sought to uncover and identify some of the characteristics of expertise in mathematical problem solving and reasoning. For example, experts notice patterns and are able to filter relevant from irrelevant features of a problem. Their knowledge is organized in ways that make it easy to retrieve for specific purposes. And they know things in ways that are "conditionalized."[28]

This focus on practices would build on this foundation, and would add to it by focusing on *activity*—on the work of *learning* and *doing* mathematics. We are proposing a line of work in mathematics education that would identify crucial aspects of what learning, doing, and using mathematics entails, and would seek to develop ways to help all students learn these tools and habits that we are calling *mathematical practices*.

In learning about the practices that constitute proficient mathematics, researchers may also become more knowledgeable about the ways we may educate citizens for the mathematical demands of adult life. The dawn of the 21st century has spurred a renewed concern for a kind of mathematical proficiency needed in a world flooded with quantitative information, decisionmaking, and the need for spatial reasoning.[29] Beyond everyday functioning, mathematics is needed for critical interpretation of information and analysis in domains as varied as politics, business, economics, social, and science policy. Knowing and using mathematics can be viewed as crucial for functional citizenship and empowerment of all members of a democracy. If the broad and effective development of mathematical proficiency is the fundamental goal of school mathematics education, we would then argue that a focus on the mathematical practices required outside of school is a critical component of a constructive research and development agenda.

Because the notion of "mathematical practices" is not common, we turn next to consider two specific domains of practice: *representation and justification*. We aim to make more plain the promise of these domains and to illustrate first steps in a trajectory of inquiry.

We chose to illustrate these particular practices because they are important to learning and using mathematics, but they are only illustrations drawn from a larger terrain yet to be explored and mapped. Other important practices include: producing and examining examples; making and testing conjectures; extending and generalizing; seeking and investigating similarities; reconciling differences; and sizing up problems and assessing solutions.

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[28]Bransford, Brown, & Cocking, 1999.

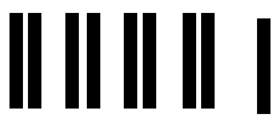
[29]See, for example, Banchoff, 1988; Devlin, 1999; National Research Council, 1989; Paulos, 1988, 1991, 1996; Rothstein, 1995; and Steen 2001.

The domain of *representational practices* includes the choices we make for expressing and depicting mathematical ideas and the ways in which we put them to use. The decimal representation of numbers, using place value, for example, is one of the most important historical examples of representation. Consider the difference between the Roman numeral representation of 1776 (MDCCLXXVI) and its base-ten representation. Representation also includes traditional modeling, in which a physical situation is described in mathematical terms, such as Newton's formulation of gravitational attraction. Representing ideas in different ways is fundamental to mathematical work. No ideal representation exists, for the quality of a representation depends on the purpose for which the representation is marshalled. For example, two "parametric" representations for the unit circle are: (1) $x = \cos(t)$ and $y = \sin(t)$, on one hand, and (2) $x = (1 - t^2)/(1 + t^2)$, and $y = 2t/(1 + t^2)$, on the other hand. The first way of representing the circle is well suited to describing circular motion and periodic phenomena; while the second is well suited to identifying the points on the unit circle with rational coordinates, which allows one to identify all "Pythagorean triples."

Crucial to representing ideas, then, is a sensitivity to the salient features of different representations, the things each makes most visible and accessible, and the ability to exploit these characteristics for particular purposes. What this sensitivity requires is fundamental awareness of the elements of an idea and its structure and the ability to re-present it in ways that highlight crucial aspects for a given purpose. The connections among representation, perception, and understanding are intuitively clear, but also inadequately understood.

For example, a rational number can be represented as many different fractions and also as a repeating decimal. Three-fourths can be represented as 3/4, but also as 6/8, 9/12, or 273/364. It is easier to compare 4/5 with 13/16 if the numbers are represented in decimal form (0.8 and 0.8125, respectively). What is less apparent from the decimal form is information about the commensurability of two numbers—that 3/4 and 2/3 and their difference can be composed of twelfths. Likewise, whole numbers—take 60, for instance—can be represented in base-ten place value notation (i.e., 60), but the prime factorization (i.e., $2^2 \times 3 \times 5$) is more informative for calculations—such as finding greatest common divisors—because it makes the multiplicative structure of a number visible in a way that the place value representation does not. Similarly, given a decimal representation of, say, the square root of 5 (i.e., approximately 2.2360679), one cannot see the fundamental property that the square of the number is a whole number (i.e., it is hard to see that 2.2360679 squared equals approximately 5). Such considerations bear on decisions about when it is appropriate to use calculators, which often prefer decimal representation and round-off. Choosing representations to use depends on the work one wants to do with the mathematical objects in question. This is a crucial part of the practice of representation.

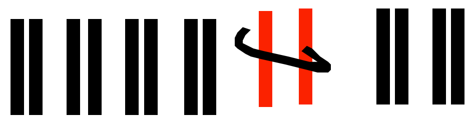
Using representations entails more than skill, however. It involves comprehension of the ideas, and appreciating and being able to use what particular representations do. For example, being able to represent an odd number as (n x 2) + 1 makes it easy to prove that adding two odd numbers will always produce an even number because it makes plain that all odd numbers have one left over after grouping by twos. Therefore, putting two odds together will produce an even by combining the ones left over to form another group of two. But being able to take advantage of this way of representing odd numbers requires a leap from thinking of odd numbers by example (e.g., 5, 13, 29, 45, 71) to thinking of them structurally. Using this representation also requires generalization, realizing that *all* odd numbers have this structure. Rather than using symbols, the same structural feature of odd numbers can be represented graphically:



This representation can also be used to prove that two odd numbers will combine to produce an even number by arraying:

and

and showing that the two "extra" ones can be grouped together to make a new group of 2, which means that the sum is even.[30]



As in this example, mathematical settings often present a whole class of equivalent representations of objects. Consider representing rational numbers as fractions as an example. One might seek a standardized representation—for example, the reduced form—because it is simplest or most revealing in some sense, and it resolves the question of equivalence. A historic example can be found in the description of planetary motion. The geocentric (earth-centered) view led to a complicated—but, in principle, correct—approximate description in terms of epicycles, whereas the solar-centered description led to the much simpler and more precise elliptical orbits around the sun. Similarly, in many geometric situations, the representation can often be greatly simplified by a strategic choice of the coordinate system used.

A fundamental practice of representation, then, is describing a situation mathematically. In a given situation, questions to ask include: What are the critical objects or variables that should be highlighted and named, and with what kinds of designations? What kind of image or drawing makes the geometric structure plain? How can the relations between these objects be represented? If you want to prove that an odd number plus an odd number always equals an even number, which of the above representations—the algebraic or the graphical—is better, or is there no difference between them? What about representing odd numbers not as (n x 2) + 1 but as (2 x n) + 1? These two algebraic representations differ in some important ways. The first describes odd numbers as structured in groups of two with one left over, while the second represents odd numbers as divided into two equal groups, with one left over. Does this make a difference here?

Knowing what to consider in answering these kinds of questions is also one of the practices important to the representation of mathematical ideas. For example, one consideration is that the algebraic representation is completely general: that is, it is clearly descriptive of *all* odd numbers. The graphical representation represents particular numbers (in the above case, 9 and 5), but its form provides an *image* of the general case. Although it is less rigorously general, the

[30] Example taken from Ball (1993) and Ball & Bass (in press).

graphical representation may make the principal concept more visible. Such considerations in using representations are crucial both to basic problem solving and to modeling (as well as to teaching and learning).

Included in this domain of representational practice is another critical area of practice—the fluent use of symbolic notation. Mathematics employs a unique and highly developed symbolic language on which many forms of mathematical work and thinking depend. It allows for precision in expression. In natural language, such a sentence as "This is a book about scientific conventions," can be interpreted in more than one way—namely, that it is a book about conventions used by scientists, or that it is a book about the professional meetings that scientists attend. Sometimes linguistic ambiguity is used purposefully, and some writers have exploited this feature to their advantage. Mathematical language, however, unlike natural language, is generally structured to eliminate ambiguity.

Symbolic notation is also efficient, compressing complex ideas into a form that makes them easier to apprehend and manipulate. Critical to mathematics learning and use is being able to fluently and flexibly encode ideas and relationships. Equally important is the ability to decode accurately what others have written.

Additionally useful is being able to compare two expressions and determine whether or not they are equivalent. Is $3 \times 5 + 7$ equivalent to $15 + 7$ or 3×12, and on what does this judgment depend? What is an equivalent, but more compact, representation of the same amount as that represented by $5 \times 10^2 + 4 \times 10^1 + 3 \times 10^0$? Does $(a^2 + b^2)$ express the same quantity as $(a + b)^2$? Is $2(b^2)$ the same as $2b \times b$ or as $(2b)^2$?

And, in a still more complex example, are the following two expressions equivalent? First, as formal algebraic expressions? Second, as real-valued functions of the real variable *t*?

$$\frac{\sqrt{t+9}-3}{t} \qquad \text{(a)}$$

$$\frac{1}{\sqrt{t+9}+3} \qquad \text{(b)}$$

This pair is interesting because the answers to these two questions are different. First, (a) and (b) are equivalent as algebraic expressions. For example, if you multiply the top and bottom of (b) by

$$\sqrt{t+9}-3$$

you will be able to transform (b) into (a). This is analogous to multiplying 1/2 by 3/3 to show that 1/2 is equivalent to 3/6. However, although the two expressions are algebraically equivalent, notice that, for (a), t cannot equal 0, because the expression would then be undefined, whereas any value greater than or equal to –9 can replace t in (b). Thus, the functions defined by (a) and (b) have different domains of definition, but they agree where they are both defined. Expression (b) is the most convenient representation for evaluating the limit of (a) as t approaches 0.

Being able to read carefully, to consider the range of situations to which different expressions apply, and to appreciate the value of different representations is crucial to the skillful interpretation of symbolic expressions.

We turn now to a briefer exploration of another domain of mathematical practice: the work of justifying claims, solutions, and methods. *Practices of justification* center on how mathematical knowledge is certified and established as "knowledge." Understanding a mathematical idea means both knowing it and also knowing why it is true. For example, knowing that the probability of rolling a 7 with two dice is more likely than rolling a 12, is different from being able to explain why this is so. Although "understanding" is part of the contemporary reform rhetoric, the reasoning of justification, on which understanding crucially depends, is largely missing in much teaching and learning. Many students, even at university level, lack not only the capacity to construct proofs, but even the appreciation of what a mathematical proof is. That mathematical justification involving reasoning that is more general than what we typically call "proof" is often overlooked and underappreciated.

That practices of justification are fundamental to mathematical proficiency can be seen all too often by these lacks. Too many students learn mathematics based on what their books and teachers say, and remain unable to explain the bases for what they "know." Such knowledge, unanchored by reason, too often floats free and is forgotten. Further, knowing particular mathematical ideas and procedures as mere facts or routines is insufficient for using them flexibly in diverse cases.

Justification is a practice supported by both intellectual tools and habits. These are grounded on valuing a cluster of questions about knowing something and about what it takes to be sure: Why does this work? Is this true? How do I know? Can I convince other people? Such questions apply not only to sophisticated mathematical claims, but also to the results of the most-elementary observations and procedures. Determining why something is true depends on norms for what constitutes an adequate justification. It depends also on a body of publicly established and accepted knowledge that thus requires no further warrant. For mathematicians, this may derive from some system of axioms and definitions, plus established and publicly accessible theorems. But this notion of a body of publicly shared knowledge has a functional meaning also in a classroom that supports a culture of mathematical reasoning. Practices of justification depend on the habit of building deductive bridges from what is known to what is conjectured.

Another practice central to justification is the careful development and use of mathematical language, notably definitions of terms. Being able to prove that a claim is true depends on the possibility of refuting it. But there is no way to reconcile challenges if people use terms with divergent meanings. Take a simple example: Suppose two people disagree about whether the number 1,421 is even or not. One says, "It's odd, because it is not divisible by two." The other argues, "No, it's even because 1,421 divided by 2 equals 710 1/2." In fact, both are correct, given the meaning each assigns to divisibility by 2 and, consequently, to the definition of an even number. It is important to discriminate conflicts over substance from those that stem from incompatible use of terms. Reconciling conflicting arguments relies on shared meanings for terms and definitions. These two people cannot reach agreement about the parity of 1,421 without first reconciling their different meanings for "divisible by two."

Still other practices center on the critical abilities of inspecting and evaluating mathematical arguments, building alternative methods of proof, and comparing the efficiency and effectiveness of different arguments.

Representation and justification comprise two domains of mathematical practice, and our foray offers little more than a glimpse of what exists to be explored. We continue here by specifying three crucial areas of investigation that a focus on mathematical practices would afford: (a) defining, investigating, and elaborating the domain of mathematical practices; (b) developing knowledge about learning, use, and teaching within core domains of mathematical practice; and (c) using the focus on mathematical practices to develop knowledge and practice related to broader aspects of the proficient uses of mathematics in a variety of work and life settings.

1. Defining, investigating, and elaborating the domain of "mathematical practices"

Because this focus breaks new ground, it will require exploration. Our discussion of representation and justification offer a view of both the intricacy and the potential of a close focus on mathematical practices. The proposed program of research and design would seek to uncover what people *do* as they learn and use mathematics. Programmatic work would identify those aspects of learning and using mathematics that are crucial to proficiency. Our examples include tools—ideas in use—such as optimization; models, such as number lines; and sensibilities, such as appreciating precision or seeing an efficient path to a solution. Skills of certain kinds seem part of this domain, such as those with writing and reading mathematical notation, drawing, and organizing lists. And habits such as asking particular kinds of questions, checking particular kinds of cases or numbers, and being concerned with certain kinds of issues—these too have emerged in our initial forays in this domain. Systematic effort would be needed to identify some of the "habitual patterns of action . . . standard patterns, routines, procedures, patterns, and habits."[31] Precisely because they are internalized and tacit, these practices are often invisible even to proficient mathematical users, and are hence not easy to identify.[32]

No detailed map exists for the domain of mathematical tools, skills, and habits of mind and action. Developing a taxonomy of mathematical practices will require systematic inquiry. What mathematical practices are crucial to proficient mathematical thinking, to learning, doing, and using mathematics? People who use or do mathematics skillfully may be successful precisely because they are fluent in mathematical practices. But because we have lacked a focus on such habits and tools, we may not even notice what such individuals are doing or understand how they are thinking. Further, we do not know how the different activities in which students engage—in and outside of courses, at home and at school—impact students' development of mathematical practices.

Questions important to pursue include:

- What tools, skills, or habits of mind seem to underlie particular forms of mathematical reasoning, work, or thinking?
- What is it that competent learners and users of mathematics seem to do, consider, and be sensitive to that appears to influence their success?
- In different contexts, with different sorts of problems or users, do different sorts of mathematical practices appear, or are there common practices across a wide range of contexts?
- Is there a difference between identified and familiar practices, such as "problem solving" and less-familiar, or less-visible, practices? What does it take to locate, identify, and make visible such practices?
- What sorts of things comprise practices? We hypothesized "tools, skills, and habits of mind." Are there other categories of activity or resources that constitute what we are calling "practices"?

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[31]Denning, 1974, p. 108.
[32]Polanyi, 1953.

2. Developing knowledge about learning, using, and teaching core mathematical practices

In order to build the sort of knowledge needed, research and design projects must do more than unearth and make visible the practices needed and used in a variety of mathematical, everyday, and work environments. The program must delve into specific core practices, and how they might be taught and learned. We will need to understand much more about what particular practices involve, what they require, and how they can be learned and taught. Important practices include: producing and examining examples; organizing examples; investigating pattern and structure; making and testing conjectures; extending and generalizing; seeking and investigating similarities; reconciling differences; representing ideas; justifying claims; and sizing up problems and assessing solutions. Projects might design ways to help students develop particular mathematical practices, and study what it takes for teachers and students to make this focus both explicit and effective. Others might analyze how students make use of the practices that they are learning, and how those practices impact their learning. Still other investigators might examine different opportunities to learn mathematical practices and compare their outcomes and impact.

Questions important to pursue include:

- In what ways do students' opportunities to learn, or not learn, important mathematical practices relate to the different resources and curricular opportunities available?
- How can instruction be designed to help students develop "mathematical practices"? Projects would design and study alternative instructional treatments and their impact on students' development and use of such practices as they work on particular mathematical topics.
- How are differences among students' mathematical proficiency affected by efforts to develop particular mathematical practices? Do some practices have higher leverage than others in impacting students' proficiency?
- How does knowledge of particular mathematical ideas and procedures interact with mathematical practices in a setting? For example, how does one's understanding of an idea relate to the ability to represent it? How does the effort to represent an idea influence understanding?
- What patterns exist in students' ideas about and learning of particular mathematical ideas, procedures, and practices? Where is there substantial individual variation?

- Who are the students who are successful with these practices? How do they learn them? Does this relate to gender, culture, language, or class? Projects would design and study interventions in any such differential patterns in attainment.
- Can achievement measures be constructed that enable assessment of students' development of these "mathematical practices"?

3. Exploring and developing mathematical practices in work and life settings

Examining a range of domains of mathematics use is a third line of work that would build knowledge about mathematical practices, and how they might be learned and taught. Understanding better the ways in which adults use (or could use) mathematics in a variety of settings—both in work and in the course of everyday adult activity—would extend knowledge about practices important for mathematical proficiency.

The situations that call for mathematical reasoning arise in domains as various as personal health (e.g., weighing costs and benefits of new drug treatments), citizenship (e.g., understanding the effect of different voting procedures on election outcomes), personal finance, culture, professional practices, and work. Consider the recent need to consider the effect of different voting procedures on election outcomes: Such analysis involves substantial mathematical reasoning and use of mathematical procedures. Newer voting machines, for example, reject certain improperly completed ballots and allow voters to correct their errors. If districts with these machines count a greater percentage of their voters' ballots, how might the election be affected?

Examples of mathematics as it arises in everyday situations are fascinating and remind us of the many things we do and face that invisibly demand mathematical practices. The most familiar examples come up in domains involving money (e.g., calculating tips), cooking (e.g., measuring out amounts for recipes), games of chance (e.g., estimating probabilities), and newspaper tables and graphs (e.g., interpreting data). One obvious example is the mathematical reasoning inherent in managing one's financial situation. Mathematical decisions regarding money have profound implications and the many Americans living in poverty are vulnerable to exploitation without a usable knowledge of mathematics. For example, the popularity of state lotteries—a fundraiser for state coffers—is based, in part, on people's misunderstanding of probability. Other examples of mathematics use are less familiar. Consider, for example, the task of packing a car trunk with many irregularly shaped parcels, pieces of luggage, and loose objects. Many people do not even recognize this as a task to which mathematical knowledge and reasoning can contribute. If they do, they may see it as a context for calculating measurements. They are less likely to appreciate that knowledge of geometry and spatial reasoning might contribute to finding workable arrangements in such a constrained and complicated solution space. How differently shaped pieces compete with one another to form a hierarchy of what to place in the trunk first, second, and third is a kind of strategic knowledge important to solving such problems. Other examples include those of managing a car (e.g., calculating the amount of gas needed to complete a journey) or programming a video recorder. Even working out the number of paving stones needed to cover a yard, and the orientation of the stones, requires the enactment of mathematical practices. These are all interesting examples of mathematically entailed tasks of everyday life that many people perform with variable skill and often without benefit of the mathematical practices that could help them solve such everyday problems.

Many adults' jobs entail the use of mathematics. These jobs include such mathematically intensive professions as engineering, nursing, banking, and teaching,[33] as well as a host of occupations in which workers employ a range of mathematical skills and practices (e.g., waiting tables, carpentry, tailoring, operating a sandwich cart). This program of work should investigate the practices used in different work environments in order to build a broad perspective on mathematical practices important to learning and using mathematics.

Systematic empirical studies of mathematics as it is used in everyday life and in work settings will inform the development of a taxonomy of mathematical practices and considerations of what schools teach, and with what connections. In addition, such work could provide resources for improving students' interest in and motivation for learning mathematics. In an age where mathematics is fundamental to the working of society, still many students believe it is a subject with no practical relevance. This perception stops many students from engaging in their studies or taking the subject at higher levels.[34] The numbers of students choosing mathematics as a college major are diminishing to dangerously low proportions, as more and more students turn to subjects for which they see practical relevance. Further, those who do choose to study mathematics beyond high school are disproportionately white, Asian, and male.[35] A crucial aspect of success for many students, particularly those who are first-generation college students, is to develop a positive sense of themselves as mathematics learners and to see value and purpose in the study of mathematics. This is particularly true for girls and students of minority ethnic and cultural groups.[36] Studies of mathematics in use can therefore help to raise awareness of the importance of mathematics as a field of study.

[33] See, for example, Noss & Hoyles, 1996; and Hoyles, Noss, & Pozzi, 2001.
[34] Boaler & Greeno, 2000.
[35] Tate, 1997.
[36] Martin, 1989; Boaler, 2000.

Prior studies of the range of mathematical demands that citizens face in life and work have revealed some surprising results and had significant impact upon curriculum ideas.[37] Building a continuing systematic knowledge base of the mathematical practices entailed in everyday life and work settings is foundational to making the learning and teaching of such practices a more explicit part of students' education in mathematics.

Questions important to pursue include:

- What are some of the important arenas of everyday life that entail mathematics? What mathematical practices play a role in everyday settings? How are these used?
- What mathematical practices are demanded by different occupations—teaching mathematics, working as an engineer or a pharmacist, or even serving coffee?
- What are some of the important arenas of everyday life that entail mathematics? What mathematical practices play a role in everyday settings? How are these used?
- How do people function in these everyday and work arenas? What practices are widely used, and with what levels of success?
- What are some of the mathematically-entailed problems that people have to solve in everyday or work contexts, and what are some of the different, including more-or-less successful, ways in which they try to solve them?
- What situations do people confront in everyday life or at work for which they seem to lack effective approaches, sensibilities, or tools?

FOCUS AREA #3: KNOWLEDGE OF MATHEMATICS FOR TEACHING

Our proposed program centers on building the resources needed for high-quality mathematics instruction. Given that the quality of instruction depends fundamentally on what teachers *do* with students with particular mathematics, we propose a third programmatic strand, focusing on what has been an almost intractable problem: the content-specific knowledge, skills, and sensibilities needed to teach mathematics effectively to a wide range of students. In order to teach mathematics well—to make wise decisions, skilled judgments, and make effective moves—teachers need to understand what they are teaching and be able to connect students with that content. This decision to focus on *mathematical knowledge for teaching* rests on our collective judgment that what teachers know and believe, and the tools and arrangements that equip or constrain their work, mediates what students can learn, the contribution made by curriculum materials, and a host of other elements of instruction. As with all other domains of complex human performance, the quality of instruction depends on the skilled use of knowledge in

[37]See, for example, Cockcroft, 1982.

context. What teachers notice, how they interpret what they see and hear, and the ways in which they marshal mathematical knowledge in response to particular students and situations are all central to the effectiveness of what they do.[38]

By focusing on knowledge and its use in teaching students, and by coordinating work across a variety of studies, interventions, and projects, this strand of the program would aim to build and contribute to the improvement of initial and continuing professional education, and of other resources designed to support the quality of teaching (e.g., teachers' guides, certification requirements, teacher assessments).

Although many kinds of knowledge may play a role in teaching, we propose a central focus on the mathematical knowledge needed for teaching, and its interplay with the knowledge of students' mathematical development and ways of thinking needed to build effective connections between students and the content. This choice reflects the overarching goal on which our program of research is based: to achieve mathematical proficiency for all students. Crucial therefore is better insight into how mathematical knowledge—of what kinds—plays a role in teachers' ability to address inequalities in opportunities to learn. And intertwined with this is a second issue: How does knowledge of students' mathematical thinking interact with knowledge of the content to shape teachers' decisions about the presentation and representation of content, their use of materials, and their ability to understand their students?

What Would a Focus on Mathematics Knowledge for Teaching Involve and Afford?

Over the past 15 years, two refrains have echoed through the discourse about teachers' knowledge of mathematics: (1) that U.S. teachers' mathematical knowledge is weak and (2) that the mathematical knowledge needed for teaching is different from that needed by mathematicians. Clearly, these two refrains are importantly related, both central to the problem of developing teachers' knowledge for teaching. Still, efforts to develop teachers' mathematical knowledge continue to lack an adequate theoretical or empirical basis for what to work on, and how to connect it to the work that teachers do.

This has created impediments to improvement. Since the late 1980s, new programs, materials, curricular frameworks, standards, and assessments have been developed, all aimed at providing resources for the improvement of mathematics education. Still, teachers are the crucial link. For teachers to use new curriculum materials that emphasize understanding as well as skill, to open their classrooms to wider mathematical participation by students, to challenge widespread inequalities in students' learning opportunities, and to help students succeed on more-challenging assessments require substantial mathematical insight and skill.

[38]Bransford, Brown, & Cocking, 1999.

In light of this, many efforts have been mobilized to help teachers develop more-robust mathematical understanding for teaching: Courses and workshops offer teachers opportunities to revisit and relearn the content of the school curriculum, states have raised the content knowledge requirements for teacher certification, and programs have been developed to attract mathematically experienced and skilled people into teaching. Herein lies the practical problem: Because of weaknesses in the research base, these initiatives have been designed and implemented with scant evidence about their efficacy. Despite some successful efforts to develop teachers' mathematical knowledge through professional development, participating teachers are not necessarily able to put new knowledge to use in their classrooms. Even when they do gain important mathematical insights, they may be no better able to understand their students' ideas, to ask strategic questions, or to analyze the mathematical tasks contained in their textbooks.[39]

Although widespread agreement exists that effective mathematics teaching depends on teachers' knowledge of the content, the nature of the knowledge required for teaching is underspecified. On one hand, the need seems obvious. Who can imagine teachers being able to explain how to find equivalent fractions, answer student questions about why 3 + 5 equals 5 + 3, or represent place value without understanding the mathematical content? On the other hand, less obvious is what "understanding the mathematical content" *for teaching* entails: *How* do teachers need to know such mathematics? What *else* do teachers need to know of and about mathematics? How and where might teachers *use* such mathematical knowledge in practice? And what does it take for teachers to use such knowledge of mathematics effectively as they make instructional decisions and moves with particular students, in specific settings?

In 1985, Lee Shulman and colleagues introduced the term "pedagogical content knowledge" to the lexicon of research on teaching and teacher education.[40] This term called attention to a special kind of teacher knowledge that links content and pedagogy. In addition to general pedagogical knowledge and knowledge of the content, Shulman and his students argued,[41] teachers needed to know things like what topics children find interesting or difficult or the representations most useful for teaching a specific content idea.[42] This notion of "pedagogical content knowledge" not only underscored the importance of understanding subject matter in teaching, but it also made plain that one's personal knowledge of a subject could often be inadequate for teaching.

[39] See, for example, Borko et al., 1993; and Wilcox et al., 1992.
[40] Shulman, 1986.
[41] Wilson et al., 1987.
[42] Shulman, 1986; and Shulman, 1987.

Research to date paints a sobering view both of what teachers know about mathematics and of what we know about high-quality teacher preparation and professional development. Although common sense suggests that the best preparation for teaching K–12 mathematics might be an undergraduate degree in mathematics, a few complications are evident.

First, elementary school teachers are responsible for teaching all subjects, not simply mathematics, and so they seldom major in mathematics as undergraduates. Instead, they typically are required to take a few mathematics courses in mathematics departments. Second, the mathematics of the K–12 curriculum does not map well onto the curriculum of an undergraduate mathematics degree. That is, even if prospective teachers majored in mathematics as undergraduates, they might not have studied the mathematics entailed in the K–12 curriculum since they were K–12 students themselves. So, although it is often overlooked, the problem of mathematics knowledge for teaching matters for the preparation and professional learning of secondary as well as elementary and middle school teachers.[43]

Research sheds little light on these issues. There have been a few studies that investigated the relationship between teachers' majors (or course-taking patterns) in mathematics and gains in student achievement. Several studies have found a positive (slight) correlation between teachers' majors in a subject matter and student achievement gains.[44] But in another study, the researchers found a "ceiling effect"—that is, increases in student achievement were positively correlated with teachers' courses in mathematics up to about five courses.[45] After that, the benefits of taking mathematics courses appeared to be negligible. To complicate matters further, some positive correlations have been found between mathematics-specific education coursework and student achievement, but not between mathematics courses and student achievement.[46]

A major problem is that proxies for teachers' knowledge—such as undergraduate degrees or number of courses taken—are poor indicators of what teachers actually know, and how they use that knowledge in teaching. Complicating things further is the variation across institutions of higher education in what constitutes a major (e.g., at some colleges and universities, one does not "major" at all, but fulfills a "concentration"). This variability makes it difficult to know exactly what teachers have had the opportunity to learn in particular masters or undergraduate courses. In sum, the research suggests that teachers' knowledge of mathematics and mathematics pedagogy can be positively correlated with student achievement. However, the lack of sophisticated, robust, valid, and reliable measures of teachers' knowledge means that we know

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[43] See Conference Board of the Mathematical Sciences (2001) for a thorough examination and recommendations for the mathematical preparation of teachers at all levels.

[44] Ferguson, 1991.

[45] Monk, 1994.

[46] Begle, 1979; Monk, 1994.

little empirically about what teachers need to know about mathematics and mathematics pedagogy, and how such knowledge impacts students' learning opportunities and their development of mathematical proficiency over time.

Meanwhile, existing investigations of teacher knowledge have opened up a vast and distressing portrait of teachers' mathematical knowledge. In the late 1980s, researchers at the National Center for Research on Teacher Education developed new and better methods of assessing teacher content knowledge—for example, interviews designed around tasks of teaching that involved mathematics, observations, and discussion of classroom teaching. Once researchers began to look closely, their analyses revealed how thin most teachers' understandings were. Both elementary and secondary teachers, whether they went through traditional or alternate routes, appeared to have sound mechanical knowledge, that is, they were able to solve straightforward, simple problems. When asked to explain their reasoning, however, or why certain algorithms work, neither elementary nor secondary teachers displayed conceptual understanding. Secondary teachers who had majored in mathematics were unable to explain why division by zero was undefined, or, thinking of slope formulaically as "rise-over-run," were unable to connect the concept of slope to other important mathematical ideas. Other researchers, using the same instruments or similar ones, have found similar results.[47] Although they could get "right answers," they lacked understanding of the meanings of the computational procedures or of the solutions. Their knowledge was often fragmented, and ideas that might have been connected (e.g., whole number division, fractions, decimals, division in algebraic expressions) were held separately. Other researchers probed other specific areas of teachers' knowledge, each revealing the thin and inexplicit nature of elementary teachers' mathematical understanding. This should not have been a surprise, given that teachers have learned mathematics in the same system that so many seek to improve. That their understanding is more rule-bound than conceptual, and more fragmented than connected, merely reflects the nature of the teaching and curriculum that they, like other American adults, experienced. However, if teachers are to lead the improvement of mathematics teaching and learning, opportunities to revise and develop their own mathematical knowledge are crucial. And for this, we need better insight into the nature of the mathematics used for the work of teaching.

Recently, Liping Ma's work[48] has added to work on knowledge of mathematics for teaching with an important idea that she calls "profound understanding of fundamental mathematics." Ma describes the "knowledge packages" that were part of the knowledge of the

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[47]See, for example, Borko & Eisenhart, 1992; Even, 1990; Simon, 1993; Ma, 1999; Wheeler & Feghali, 1993; and Graeber & Tirosh, 1991.

[48]Ma, 1999.

72 Chinese elementary teachers she interviewed. These knowledge packages constitute a refined sense of the organization and development of a set of related ideas in an arithmetic domain. The teachers Ma studied clearly articulated ideas about “the longitudinal process of opening up and cultivating such a field in students’ minds.”[49] Their knowledge packages consisted of key ideas that “weigh more” than other ideas in the package, sequences for developing the ideas, and “concept knots” that link crucially related ideas. Ma’s notion of “knowledge packages” represents a particularly generative form of and structure for pedagogical content knowledge. Key in her notion of mathematical knowledge for teaching is a kind of culturally situated and curricular structuring of the content that readies it for teaching, by identifying central ideas and their connections, in addition to the longitudinal trajectories along which ideas can be effectively developed.

What has all this research on teachers’ mathematical knowledge produced? First, it has helped define the problem of subject-matter knowledge for teaching by bringing it into the center of research on teaching and teacher learning, and by distinguishing *teachers’ knowledge* from the knowledge held and used by others who draw from that same discipline in their work. For example, teachers must use and know mathematics differently from ways in which engineers or physicists know and use it. Second, studies have demonstrated repeatedly how weak many teachers’ knowledge is. Still, the advance of the concept of *pedagogical content knowledge* makes plain that the sort of knowledge required for teaching may be developed in practice.

Considering research on mathematical knowledge for teaching, there remains much that we do not know. First, studies of teacher knowledge have left a gap, having only mined a few key topics. Division has garnered enormous attention, followed by fractions, rational numbers, and multiplication.[50] Obviously, many other key topics warrant attention—discrete mathematics, number systems, infinity, integers, geometry, place value, probability—to name a few. We know little about the ways in which teachers need to understand these topics, the kinds of unpacking of ideas that they need to be able to do, the connections they need to be able to see—in short, both what they would need to know of these other areas and how they might need to use such knowledge.

Work on teachers’ mathematical knowledge has been dominated by attention to substantive knowledge—or *topics*. As Kennedy[51] points out,

> Because the main goal of [current] reformers is to instill a deeper understanding in students of the central ideas and issues in various subjects and to enable students to

[49]Ma, 1999, p. 114.
[50]See, for example, Post et al., 1988; and Simon & Blume, 1994a, 1994b, 1996.
[51]Kennedy, 1997.

> see how these ideas connect to, and can be applied in, real world situations, it therefore makes sense to require that teachers themselves also understand the central ideas of their subjects, see these relationships, and so forth.

But herein lies a second crucial gap. Less work has probed teachers' use of or knowledge about *mathematical practices* as a component of mathematical knowledge. Why does this matter? First, as students learn mathematics, they are engaged in using and doing mathematics, as are their teachers. They are representing ideas, developing and using definitions, interpreting and introducing notation, figuring out whether a solution is valid, and noticing patterns. They are engaged in mathematical practices as they engage in learning mathematics. Inevitable as this is, teachers and curricula vary enormously in the explicit attention they give to this component of mathematical knowledge. Second, we propose in this report that facility with mathematical practices is crucial for success with mathematics. Being able to represent one's ideas (to oneself as well as to a teacher and peer) is a critical part of doing mathematics. Limited ability to interpret and use symbolic notation, or other forms of representation, can impede students' work and their learning. Similarly, being able to inspect, investigate, and determine whether two solutions, two representations, or two definitions are similar, or equivalent is fundamental to many arenas of school mathematics. Students and teachers are constantly engaged in situations in which mathematical practices are salient. Yet conceptions of teacher knowledge have barely probed the surface of what of mathematical practices teachers would need to know and how they would use such knowledge.

Another tack-on question about knowledge for teaching is to focus on teachers' capacity to identify, interpret, and respond to students' mathematical ideas, difficulties, and ways of thinking. A research focus on knowledge for teaching would involve close examination of what it takes for teachers to hear, understand, and work effectively with the widest possible range of students in mathematics.

Researchers have examined teachers' knowledge of and beliefs about students' mathematical thinking, abilities, and learning, and explored whether and how such mathematically oriented knowledge of students affects the course and quality of instruction. Some projects have investigated how teachers' familiarity with research-based knowledge of students' mathematical thinking affects teachers' instructional effectiveness[52]. Evidence exists that such knowledge can make a significant impact on instruction. However, the research base on students' mathematical thinking is uneven—strong for certain mathematical topics, and for some kinds and ages of students, and lacking for others. As such, more remains to be learned

[52] See, for example, Carpenter, Fennema, & Franke, 1996.

about the ways in which such knowledge might contribute to the quality of instruction. Other research has investigated the role played by teachers' beliefs and expectations about different kinds of students.[53] Much of this work demonstrates how teachers' knowledge and expectations about students impact the students' opportunities to learn as well as their learning. However, here, too, more remains to be uncovered about how such expectations and beliefs play out for particular topics or mathematical practices. Understanding how teachers' beliefs and expectations of particular students' mathematical knowledge, or of the mathematical capabilities of certain groups of students, interact with their knowledge of mathematics offers a crucial perspective on instructional practice and its influences and demands.

In summary, because the investigation of teachers' subject-matter knowledge has been unevenly intense, collective knowledge about the mathematics that teachers need to know is underdeveloped. Aspects of mathematical knowing, appreciation, and practice go unnoticed and unarticulated. The conceptual scheme for what it means to think mathematically as a teacher remains inadequate, often falling by sheer habit into all-too-familiar lists of topics. Moreover, there is a great deal more to understand about what any of these aspects of mathematical knowledge might look like or mean as they are used in teaching to work responsively and effectively with diverse students.

What Do We Need to Know About Mathematics Knowledge for Teaching?

We identified three areas to frame and focus a fruitful line of work focused on knowledge for teaching: (1) developing better understanding of the knowledge of mathematics needed in practice for the actual work of teaching; (2) developing ways to deploy useful knowledge of mathematics and students in practice; and (3) developing valid and reliable measures of knowledge for teaching.

1. Learning about the mathematical knowledge and knowledge of students' mathematics needed for the work of teaching

One area that we target for programmatic work concerns the content-specific knowledge used for teaching mathematics, and how and where the use of such knowledge makes a difference for high-quality instruction. Much remains to be uncovered about the mathematical resources that can make a difference for the quality of teaching, and about the dynamics of their use in relation to other beliefs, dispositions, and orientations.

Questions important to pursue include:

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[53]See, for example, work by scholars such as Aguirre et al., Khisty et al., and Warren.

- What particular knowledge of mathematical *topics and practices* is needed for teaching specific mathematics to particular students?
- What are the mathematical practices needed for teaching? That is, what do teachers need to know about such practices, and how and where are such practices needed in the course of teaching?
- What is the scope and content of such knowledge—that is, what specific mathematical knowledge is critical to the decisions, moves, and judgments that teachers must make in the course of their work? How and where is such knowledge needed in the course of teaching?
- What knowledge and expectations of students' mathematical thinking and capabilities, and how they learn mathematical topics and practices, are needed for teaching specific mathematics to particular students?
- How does content-specific knowledge of students interact with mathematical knowledge to shape teachers' decisions about the presentation and representation of content, the use of materials, and the ability to hear and understand their students?
- How does teachers' knowledge of mathematics and of students' mathematics play a role in teachers' ability to address inequalities in opportunities to learn?
- What mathematically oriented dispositions are needed to enable teachers to use what they know effectively in practice?

2. Developing means for the deployment and use for instruction of mathematical knowledge for teaching

The second target of this line of work concerns the construction of a system in which the mathematical knowledge needed for teaching is deployed effectively to improve the quality of instruction. One obvious element of this problem concerns teachers' professional learning opportunities, while a second element concerns the arrangements for professional work that would support both learning and the use of what is learned. A third element might center on the design of useful tools to support mathematically knowledgeable practice. Issues important for development and research include:

- What opportunities for learning enable teachers to develop the mathematical knowledge, skills, and dispositions needed for teaching? How can teachers be helped to develop the requisite mathematical knowledge, skills, and dispositions in ways that enable them to teach each of their students effectively? What opportunities ensure teachers' learning to *use* such mathematical knowledge and skills and act on such dispositions that they develop?

- In what ways can teachers' work and roles be organized to support the use of such mathematical knowledge in teaching? For example, different arrangements for distributing professional practice might be investigated that permit specialization on the part of some teachers, while others would focus on instruction in other content areas. Another possibility is to organize teachers' grade level assignments in ways that maximize their collaboration and learning from their teaching: Teachers moving through the grades with students might, for instance, afford curricular perspectives difficult to attain when teachers remain at the same grade level. However, collaboration on lessons with others teaching the same level, with the same materials, as teachers engaged in practices of "lesson study" similar to those widely used in Japan strive to do, could also provide structure for the development of knowledge for teaching.

Not all questions center on professional learning, or on instructional organization. A third important arena ripe for exploration is the development of materials and tools designed to make possible and support knowledgeable practice.

- How can tools be developed in ways that support the effective use of mathematical knowledge in teaching? Such tools might include curriculum materials, technology, distance learning, and assessments. A raft of new curriculum materials, for instance, has been designed with substantially enhanced teachers' manuals. These manuals are intended to provide teachers with opportunities to learn about the mathematical ideas, about student learning of these ideas, and about ways to represent and teach these ideas. How might such materials be used by teachers, and with what effects on practice?

A third element of this agenda, critical for progress on these two problems, centers on the need for measures of the content knowledge needed in teaching. In order to uncover the mathematical knowledge needed in teaching, and to study the impact of different kinds of learning opportunities, the field needs reliable and valid measures of teachers' knowledge and of their use of such knowledge in teaching. Needed is a range of tools, from survey measures, to performance tasks, to written and interactive problems. This line of work should build constructively on the last 15 years of work on teacher assessment.[54] Such measures would

[54]For example, Shulman and colleagues at Stanford University with the Teacher Assessment Project (1985–1990); National Board for Professional Teaching Standards; INTASC; Praxis; the National Center for Research on Teacher Education and the Teacher Education and Learning to Teach study at Michigan State University (1986–1991); and the Study of Instructional Improvement currently under way at the University of Michigan.

permit teacher knowledge to be investigated as a variable in virtually all studies of mathematics teaching and learning

3. ELEMENTS OF A STRATEGIC PROGRAM OF MATHEMATICS EDUCATION RESEARCH AND DEVELOPMENT

DESIGNING FOR A STRATEGIC PROGRAM OF RESEARCH

The RAND Study Group on Mathematics Education has sought to envision a system leading to continuous, evidence-based improvement in mathematics teaching and learning. The overarching goal of such a system would be to produce the knowledge needed to develop high-quality instructional practice. Such practice would substantially enable all students to reach or exceed acceptable levels of mathematical proficiency. Moreover, differences in race, class, gender, language, culture, or ethnicity would no longer be predictive of differences in student performance. The study group believes that such a system will require a coordinated agenda of cumulative, programmatic research that, at the end of a decade or more, would provide us with the knowledge and practice needed to make this goal attainable.

At the same time, progress must be made now. The students of today cannot wait a decade for improved instruction. Thus the program should be designed to work at both long and short range, and goals that are within reach now should be part of a well-designed system. With appropriate design, we believe that it is possible for the short-range programs to contribute materially to the long-range goals.

This final section of our report thus addresses requirements for organizing and managing an effort that embodies these perspectives. It begins by laying out what might be termed "design imperatives" for such a system and the research and development program it requires. It illustrates these by sketching classes of projects related to the problem areas discussed in the previous section. It concludes with a discussion of the requirements for managing a strategically focused, problem-centered program of research and development.

This report has proposed three specific substantive problems that could be productive initial targets of such a systematic program—the learning and teaching of algebra; the learning, use, and teaching of mathematical practices; and the development of the knowledge needed for teaching mathematics. Each focuses on a strategic element of the equitable development of mathematical proficiency. However, to speak of a "system leading to continuous, evidence-based improvement in mathematics teaching and learning" is to signal that what is needed is a strategy for developing, studying, and improving instruction that yields both better practice and more-robust and useful knowledge for practice. This is not business as usual. We envision a new and ambitious approach to the construction of systematic knowledge and the development of practice. Doing this would require deliberate program design, imagination, and an emphasis on experimentation and evaluation for continued development and improvement.

Our proposals are intended to boost the quality and usefulness of research in mathematics education and require substantial revision in current assumptions and approaches to the planning and management of research. In this last section of the report, we turn our attention to what it would mean to design and build a system of programmatic research that would both draw from and aim to solve problems of practice.

The program we propose has five key features.

- First, its agenda is both grounded in practice and focused strategically on specific, important educational goals. The program is designed to build and discipline the connections between practice and basic inquiry in ways that strengthen both, and that maximize the ways that each can build on and contribute to the other.
- Second, it depends on and is committed to building a community of research and practice that brings together expertise from a variety of disciplines and professional experience.
- Third, our proposal deliberately engages multiple perspectives and methods and would be organized for collective work.
- Fourth, the program builds systematically on past work and established knowledge.
- Fifth, the program requires active design and management and explicit attention to the need to develop the new institutions and leadership that are required by its reliance on multiple perspectives, collective activities, and focus on practice.

Building on prior work and knowledge is a cornerstone of a systematic program. The questions we focus on derive both from those we have been able to answer and those we have not. The issues we nominate to examine are ones that have emerged as past research and practice intertwine, highlighting new problems that warrant systematic attention. Toward this end, we take a specific, and uncommon, view of what it might mean to tie research more tightly to practice.

Consider first a common view that relations between research and practice are linear. The first step is to develop basic theoretical knowledge; second, develop instructional applications; third, run trials of specific tools and interventions accompanied by careful documentation, evaluation, and analysis; and, finally, scale up tested models for practice.

Our program is based on an alternative conception of the fundamental relations of research and practice. In our view, these relations are more cyclical than linear. Practice always demands action, so interventions of myriad kinds are *always* at play. These interventions produce questions, problems, insights, and, sometimes, useful new knowledge. Research presses problems and questions into forms that can be studied, and seeks to produce theoretical knowledge, empirical results, concepts, and practices. Important to note is that sometimes one informs the other, but sometimes not. Our proposal is that program designs must be developed that strategically link improvement efforts with research. This implies a variety of relations between research and practice that together comprise a coordinated *program* of work. That program of work produces knowledge useful for practice, new practices, and improved outcomes for students. Moreover, it produces improved questions, new problems, and better formulations of old ones. It spawns new methods for studying practice and new ideas for analyzing it. Finally, it yields knowledge about what it takes to make evidence-based improvements in mathematics teaching and learning.

Thus we see the production of knowledge and the improvement of practice as a cycle of study, development, improved knowledge and practice, and new development. This implies that problems can be entered and worked on at different points in that cycle, as shown in Figure 1:

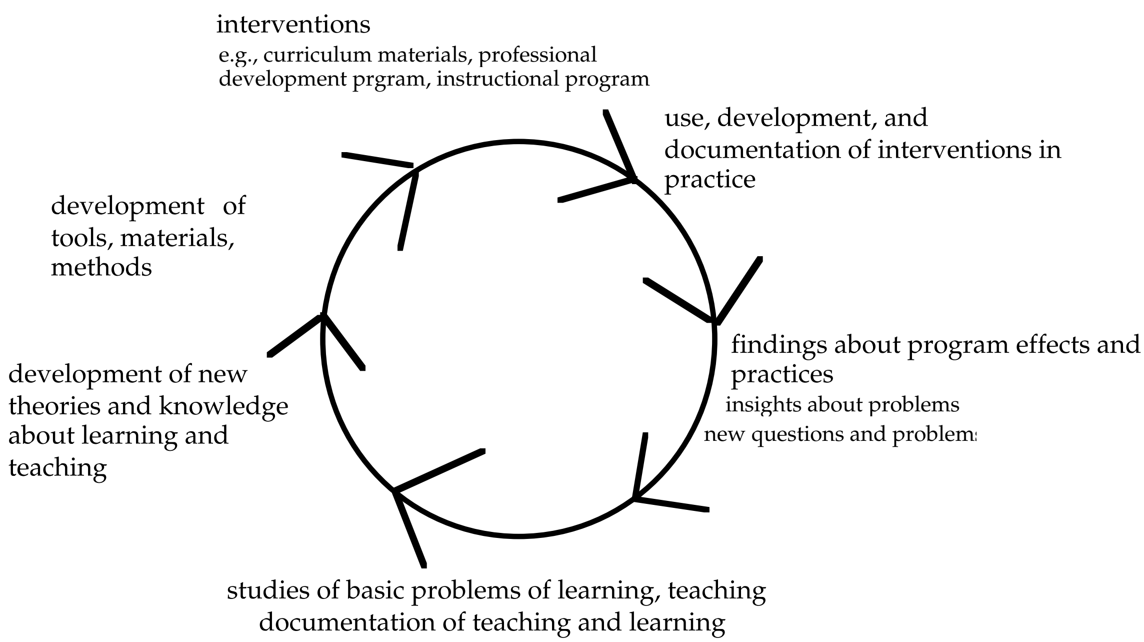

Figure 1: Cycle of Knowledge Production and the Improvement of Practice

Even this picture is a simplification because the points on the circle can often collapse on each other in constructive ways. For instance, the development of tools and materials, the design of interventions, and their use and documentation in practice can all go on together in a complex design process, or in smaller cycles between design and use. And there can be instances of quite basic or theoretical work that happen in the midst of design and practice. Furthermore, the circle is misleading for effective programmatic work is not repetitively cyclical. Instead, research and development would be progressive, such that work in one or more parts of the cycle would fruitfully shape advances in others. This simple model, however, makes visible a set of useful analytic distinctions and provides a sense of the complexity and iterative nature of the larger process.

A coordinated program of research and development would be designed to strengthen relations among these different efforts so that investments and yields are interrelated, that knowledge produced builds on what is being tried in practice, and that what is developed for practice draws on new insights from research. Individual projects might work at one or more points in the cycle. No one project by itself would be expected to yield the definitive answer to any significant problem. Instead, design and leadership would coordinate an evolving variety of projects, from interventions to studies of different kinds, in ways that built knowledge and practice. One corollary of this approach is that interventions, which have as their primary goal the improvement of practice, have also, by design, as a concomitant goal the testing of theoretical ideas and the generation of new theoretical insights and questions.

Our proposal, then, involves building a system that can reliably and programmatically produce knowledge that is usable for and useful to the improvement of practice, and is developed into tools, approaches, programs, practices, and organizational structures that improve student learning. These, in turn, would lead to more-robust and reliable knowledge of learning and teaching in general and in specific contexts.

Such a research and development system itself has to be designed, for without that attention these activities would proceed, as we have often seen, with little connection or coordination. Basic work in the disciplines is often remote from efforts to improve practice. And efforts to improve practice either remain unanalyzed, or they yield insights and ideas of practitioners that remain unexamined by researchers, and so are untested and unwarranted. This compartmentalization results in fragmentation and lack of progress in both the production of usable knowledge and the improvement of practice. Most important, little warranted knowledge or practice is produced in ways that can provide resources across contexts. Rarely are theories or programs accompanied with knowledge of what it takes to make them work with particular students in particular settings. To be most usable, the products of such programmatic work should be accompanied by knowledge of the chances that their use would succeed in meeting teachers' goals in their immediate situation, and the likely limits on or uncertainties in their use. Given the incentives that drive education research today and the limited resources available to support such work, this kind of knowledge is unlikely to be produced without encouragement from a strategically organized program.

Such a program should be designed as a *collective* undertaking. Collective does not mean that all work would be guided by some rigid program design. But projects of different sorts would be linked in a system that coordinated different kinds of work, from basic inquiry to survey research to small-scale developments, to efforts to develop and use programs or approaches across contexts. Designing a collective undertaking to be programmatic requires directed investment and imagination. Significant responsibility for this design has to lie with the sources of funding of the work and the management authority that inheres in that function. But crucial to stimulating imagination will be to institutional structures to engage the communities of research and practice in collective and disciplined responsibility for this work.

If so, core questions would be articulated, rearticulated, and improved as a product of the collective enterprise of disciplined inquiry and development. Findings and results would be publicly shared, critiqued, and improved on the basis of cross-project scrutiny, comparison, and critique. Methods, instruments, tools, and resources would be shared and used in different projects, settings, and for different parts of the program's work. Investments would be made in the development of common instruments to permit comparisons across settings and investigations. Contrasts and differences in methods, instruments, tools, and resources would also be strategically cultivated, investigated, articulated, and used. Contrasting or conflicting theories or hypotheses would be confronted and resolved, or superseded. The design of a collective and coordinated program of work on fundamental problems of practice that values and uses difference and that seeks and supports connections, is at the heart of what it would take to improve the production of knowledge useful for the improvement of practice.

We turn now to some examples of what it might mean to design disciplined programs of coordinated work in the focal areas proposed in this report. These are intended as examples only, to illustrate what might constitute productive work, to show a range of such work, and to consider what it might mean to coordinate such work.

1. The learning and teaching of algebra

One strand of work might investigate student learning of ideas and practices central to proficiency in algebra. This strand could be built from work at all points on the cycle in Figure 1. For example, some work could be designed to investigate students' learning of central algebraic proficiencies with different materials and instruction.[55] A plethora of new material already exists, and little work has been done to capitalize on the enormous investment that has already

[55]Boaler, 2002.

been made in curriculum development. These curricula differ in content emphasis, instructional approach, scope, and sequence. Close, comparative analyses of the materials could set up systematic studies of students learning in classes using particular materials. This set of work would begin at the "use of interventions" point in the cycle, and be designed to identify what is currently promising while at the same time suggesting new insights into what makes the programs work and what might make them work better in improving students' learning of algebra.

But the same strand of work might also include the development of new tools or programs aimed to help students learn specific ideas or skills already known to be difficult to learn. For example, studies of young children show that they often have difficulty learning to understand algebraic equality and using the syntax of equations correctly.[56] A series of instructional experiences might be developed to address this difficulty explicitly. Work of this kind would produce resources for other interventions and studies, and would begin at the "development of tools, methods, and materials" part of the cycle.

This strand of work would also need to include surveys of current practice in the teaching and learning of algebra. What is being taught, to whom, and with what materials and outcomes? Common measures could be designed, and researchers, working in different environments, could collect data on large numbers of classrooms. Data would be collected on key elements of the instructional experiences, curriculum, and environments of students' opportunities to learn; these might be log or survey data. Work of this third kind would be located at the "documentation of teaching and learning" part of the cycle.

Some studies would be designed comparatively and cross-sectionally, others would be longitudinal. Longitudinal studies would make it possible to analyze instruction as a *stream,* which is important to learning about how children's mathematical proficiency develops over time.

Some projects would work around major sectors of the circle, beginning, for instance, with an intervention, and using knowledge from close examination of its use, to remodel and retest it in a wider range of settings. Some projects might be developed to study closely some crucial elements of what it takes to be proficient in algebra and, over time, to develop alternative tools and approaches that might support more robust learning.

Important to the strategy of making the work programmatic is the careful design of the strand as a collective undertaking that has conceptual coherence and that is structured to make the sharing of progress usable to the development of work at all points on the cycle.

[56]See, for example, Carpenter & Franke, 2001.

For example, carefully designed working conferences could be held to examine primary records of student performances, the development of particular mathematical ideas, and practices among students. These conferences would be designed to bring together people doing similar work, but also people doing work based on different hypotheses, methods, or with different approaches or tools. Data sets that could be examined by different research groups would be developed with the technological supports that could make these data accessible and usable by research groups with different constellations of expertise.

The cumulative result of this collective process could be the development of a clearer understanding of the nature of proficiency in algebra and the practices that undergird it; the proper relationship of that to the rest of mathematics; and an understanding of what is required for teachers to teach it and students to learn it, across the grades and diversity of settings in American schools. This work would be connected with projects that seek to develop instructional approaches to the teaching of algebra. The latter would draw on evolving knowledge about proficiency in algebra and would also contribute to that knowledge.

2. The learning, use, and teaching of mathematical practices

The notion of "mathematical practices" is newer and less articulated and thus in need of some kinds of work that differ from work on the learning (and teaching) of algebra. Basic investigations are needed to uncover and articulate core practices of learning, doing, and using mathematics. What do people who use or learn mathematics proficiently *do*?

Some studies could profitably be developed to probe the nature of such practices, following and observing closely the less visible processes of learning, doing, and using mathematics. For example, what is it that mathematicians do when they are puzzling about an idea, stuck on a problem, or unsure of a solution? What do other skilled users of mathematics do to solve similar difficulties? How do mathematicians develop ideas, and how do they manage discrepancies in concepts, representations, definitions, or solutions? What role does language—written and oral—play in their learning and their work? Because so much of what it takes to be proficient in mathematics has remained invisible, much basic work will be needed to unearth, identify, test, and articulate elements of proficiency that constitute such mathematical practices.

But not all work on mathematical practices would start with basic inquiry. We already know a great deal about some practices, and about what makes them difficult for students to learn. For example, one core practice of mathematics is the use of notation and graphical representations to encode and represent ideas. Far too many students never develop fluent command of the linguistic and representational tools of mathematical expression. They have difficult writing their ideas and solutions, and are too often stymied in interpreting mathematically encoded statements. Supportive tools or approaches could plausibly be designed and developed into small or large interventions to be tested. What sorts of approaches might improve students' development of representational skills? Efforts to intervene in students' learning could be documented and analyzed, developing both knowledge for instruction and improved insights about the nature of practices of representation and what is entailed in acquiring and developing them.

Existing curricula do, of course, involve students in writing, representing, and reading mathematical notation and other representations. Although they are not identified as "practices," they are nonetheless part of what proficient students and teachers are using and doing. But these practices are often ancillary to other mathematical tasks, are not explicitly articulated, and are not a dedicated component of instruction. Some projects might analyze and compare different curricula, and study teachers' attention to practices of representation as they use particular curricula. Analyses could provide insight into how teachers develop these practices, and what students learn, and could yield new ideas about what might need to be supplemented, augmented, or altered so as to make more reliable the development of students' fluency with representation and notation. Experiments with different approaches might be designed and conducted longitudinally, or at different points in students' mathematical development, to develop more-robust knowledge of what it takes to produce, over time, flexible and usable fluency with mathematical notation and representation.

Verification and justification constitute another crucial arena of mathematical practice. Students often remain at a loss about whether their answers are right, their ideas plausible, or their solution methods valid. Although they have learned particular procedures and ideas, they are not sure of their underlying reasons. They are asked to explain their thinking, or their solutions, but do not know what constitutes a mathematical explanation. Projects might be designed to support the development of students' capacity to produce mathematical explanations, to engage in practices of mathematical justification. Some might work from inside existing curricula in which students are asked for their reasoning or to show why their answer is right or their solution makes sense. Other projects might probe the patterns of reasoning that characterize students' practices at different ages, and based on that, might develop scaffolds or other tools to support the development of students' practices of justification.

Connections could be built between work on practices of representation and practices of justification, for language and representation can play an important role in the construction of mathematical explanations and proofs.

Coordinated work on mathematical practices, then, would strategically intertwine efforts at different points in the cycle: basic research into the nature and scope of fundamental practices and how they are deployed in the course of learning, doing, and using mathematics; investigations of students' capacities and skills with such practices, obstacles faced by students; and interventions designed to influence the development of such practices.

New work would be required to develop tools for probing and assessing proficiency with particular mathematical practices. This program would be strengthened if common measures were used across a variety of settings and kinds of projects. Because mathematical practices constitute a new focus in the field, investments in coordinating approaches, findings, insights, and problems across projects will also be important for making progress in this arena. Systematic efforts to both spread and focus the work—for example, with different ages of students, with different characteristics, in different settings, or in the context of different areas of mathematics—would also be crucial to make productive progress, refining core questions, formulating emerging theories, and articulating instructional approaches.

Well-coordinated work in this domain could contribute in the short run to progress in developing students' skills in learning and doing mathematics if investments are made in areas about which we already have knowledge (e.g., representation and the use of notation, explanation and justification). At the same time, progress over a longer period of time will be facilitated if early basic work is possible to extend and develop collective ideas about the nature of mathematical practices that are currently less visible, less well-identified, and about which less is known with respect to students' capabilities and development. And most important in such a new, but promising, area is to ensure adequate and flexible opportunities for projects of different kinds to form a community of practice and scholarship with productive means of interacting, disagreeing, sharing findings, and developing with greater precision ideas, questions, tools, and knowledge useful for both research and instruction.

3. Developing knowledge for teaching

Work in our third focal area is at a point that differs from work either in algebra or in practices. Substantial evidence exists about lacks in teachers' knowledge, and inadequacies in the supports provided for them to acquire mathematical knowledge for teaching. Programmatic work in this area must be strategically designed to move forward on this large problem of capacity for high-quality instruction.

This is an area ripe for carefully planned and varied interventions. Alternative approaches to helping teachers develop mathematical knowledge for teaching could be designed and tried. With common instruments to track teachers' learning, these different approaches could be compared over time, as teachers teach particular mathematical ideas or skills (and develop experience with these in the context of their work) and also engage with particular opportunities to learn. One such example would be to investigate teachers' learning from new, highly-detailed teachers' guides and from teaching, with and without a particular supplementary support. Another would be teachers' learning from well-designed mathematics sessions or close study of students' mathematical work, especially error patterns and conceptual difficulties. Needed here to advance the work are well-articulated hypotheses about teachers' learning, designs that can test those hypotheses, and shared tools for studying teachers' learning from different opportunities. Progress would require design and coordination to build cumulative knowledge.

At the same time, more needs to be learned about the mathematical resources needed for teaching. Measures developed can only be as good as the conceptions of what needs to be measured. We propose systematic analytic and empirical work on what mathematical knowledge is needed in order to teach, and how it is deployed in the course of teaching for what sorts of tasks.

Moreover, the ongoing work on algebra and mathematical practices would also provide new insights about the mathematical knowledge and awareness that might be required to foster students' learning in these areas. For example, to develop students' fluency in using and interpreting mathematical notation, teachers would obviously need to be fluent users of symbolic notation themselves. But, in addition, they might need to be able to think conceptually about issues of precision and representation that would enable them to work with students' emerging skills in using notation to record ideas, not unlike working with young children's early efforts to write in natural language. What might be the core elements of recording an idea in mathematical notation, and how might teachers elicit and build those elements with students over time? For example, one principle of mature mathematical notation is that it is unambiguous and, unlike natural language, not connotative or subject to multiple interpretations. A teacher who had a sense for this principle would be able to ask questions of students to help them discover ambiguities in their use of symbols, terms, or representations, even at the early stages. Similarly, teachers might need understandings of the concept of a variable that would guide their work with students as they begin to work with ideas such as unknown and variable quantities in solving equations. Being able to unpack explicitly and conceptually the central elements of the relations among variable, constants, equations, and solutions would likely be important for teachers in ways that matter for teaching about these fundamental elements of algebra.

Programmatic work, then, would link not only different kinds of work—projects of different kinds, and studies of multiple methods and designs—*within* a strand of work such as algebra, practices, or knowledge for teaching, but also to do so *across* strands. This again would require careful design.

WHAT BUILDING A STRATEGIC RESEARCH PROGRAM REQUIRES

In this section we have described some of the essential design features of a "system of continuous, evidence-based, improvement in mathematics teaching and learning." We have offered examples of what programs of work might involve in the areas we propose for priority attention: algebra learning and teaching; the learning, use, and teaching of mathematical practices; and understanding the nature of the mathematical knowledge and knowledge of students required for effective teaching.

We have argued for a coordinated program of research that derives its agenda from a focus on the goals and problems of practice. Over time, such a program would set out a strategic menu of research and design that would orchestrate basic work that could produce new insights about effective learning and teaching; it would also contribute to the solution of intractable problems of practice, along with the applied design and development of approaches to practice, and the knowledge and materials needed to support it. Such a program should also include the testing of these designs, both as a way of determining whether and over what range of situations they work and as a way of identifying new problems and new insights that might inform further basic work. It would involve balancing efforts to make important improvements in the shorter run with investment in work that might offer the chance of much greater improvement in the longer run. An important source of stimulus for the latter would come from the lessons learned about the limitations of current improvements drawn directly from the experience of implementing them.

We suggest that the strategic orchestration of work taking place at the various points in this "Cycle of Knowledge Production and the Improvement of Practice" (see Figure 1, p. 59) requires a conscious and intelligent appreciation of how developments at those points are relevant to and should take account of each other. This would be facilitated by the development of a multidisciplinary community of researchers and practitioners who accept the goal of ensuring students' mathematical proficiency and who are committed to developing the necessary knowledge for meeting that goal. They would be similarly committed to developing knowledge and practice through a cumulative, scientific process of building on what is known as well as testing new theories and designing new approaches based on those theories while addressing and resolving conflicts. Different perspectives and claims would be explored using empirical evidence, sufficiently varied but rigorous and publicly accessible methods for gathering that evidence, and, where possible and appropriate, shared measures and agreed constructs. The examples we give of possible work on algebra, practices, and teaching suggest more specifically what this collective attention to building the knowledge needed for success in mathematics instruction might look like.

To achieve this sort of work will require major renovation and redesign of our current approach to planning, managing, and building useful research and interventions. The communities of research and practice that now focus on mathematics learning and instruction are neither equipped nor organized with the leadership, resources, and full set of institutional structures that would be required to carry out systematic, strategic work of this sort. Significant pockets of constructive work exist, but they tend not to include the full range of disciplines and personnel or kinds of work that need to be included; and they are not sustained coherently over the time spans that would be required to cycle between knowledge development and practice long enough to ensure widely applicable, trustworthy improvement.

Recognizing this, we conclude that an additional level of design is required that is the design of the a research and development system that could produce such knowledge and support. Because the education research and practice communities are fragmented, diffused, and do not interact in a collective enterprise, responsibility for initiating this level of research system design must rest with the institutions that fund research. Given the resources likely to be required, the federal government, especially the Office of Education Research and Improvement (or successor agency, if one is created during the current reauthorization process), should play a key role.

The design of such a system involves many tasks. One of the first will be to find a way to engage the full range of relevant research fields and the fields of practice in a commitment to working cumulatively toward the realization of the program's goals. This will require investment in developing the community we have described, one that can take advantage of the resources the agency has to offer. This should include developing a commitment to shared norms of evidence, method, and ways of resolving points of theoretical or practical contention.

To accomplish this, the agency's program leadership must be able to earn the respect of the relevant fields of scholarship and practice. Crucial will be a knack for combining the modeling of scientific norms and values with institutional creativity. Important also will be the ability to identify those in the field who can develop the new kinds of organizations, programs, and structures that will be needed to do this work. This leadership would take advantage of efforts such as ours as it develops ways of engaging the wisdom of the field in shaping its planning and funding decisions and arguing for public support for its initiatives.

While the study group emphasizes the importance and potential of the U.S. Department of Education's Office of Educational Research and Improvement (OERI) or its successor in providing this leadership, an important reality is that several federal agencies make significant investments in activities intended to improve mathematics education through research and development or other activities. Of particular importance is NSF, which we estimate supported 75 percent of the research and development in mathematics education in fiscal year 2001. OERI, NSF, and the National Institute for Child Health and Human Development also jointly manage the Interagency Education Research Initiative that provides significant funding for scaling up and testing education interventions that have been found promising in laboratories and limited school settings. Some of the initiative's projects deal with mathematics.

Our argument is that a fully strategic approach would try to keep all of these issues—fundamental work, design and application, and scaling up—continually in focus. To facilitate this, we think that attention could be maintained more effectively and constructively if the program we envision took more narrowly defined goals of practice (such as the aspects of mathematics proficiency we have identified) as its strategic focus rather than spreading its attention over all student achievement. This last point about the most promising problem focus is quite debatable, however, and probably could be resolved only by practical experience. We do, though, think that a strong case can be made for the virtue of starting with the goals and problems of practice and working back strategically, if the ultimate hope is to have an impact on practice.

What we propose may sound strongly directive. In fact, we propose a role for the government that is more akin to orchestration. It is important for the program's management, and for the general strategic planning processes from the field that the management organizes, to support the program and maintain an overview of the component parts of the strategy and their potential interactions. The government's program leadership and its advisors may make choices concerning disciplinary or subdisciplinary foci that are rooted in the larger program's goal- or problem-oriented strategic judgment, and the appropriate balance between work with a basic emphasis and programs focused on design and application based on that work. However, it is not crucial and need not always be a requirement for the scholars or practitioners working in any of the subparts of the problem, and certainly not for each particular study, to take account of all of the other elements of the strategy. Subprograms funding basic and disciplinary work should appear to the participants to be just like any basic research funding program, and be judged by the same criteria.

To support this orchestration, our study panel believes that one or more standing study panels, such as this one, may be needed. Such panels or groups would be a major agent for fostering the collective approach to addressing the challenges related to improving the mathematical proficiency of all students. They should engage the multiple practice and disciplinary perspectives we argue are needed for this work. Periodically, they would assist leaders of the funding agencies in assessing and synthesizing the results of the program activities and reformulating program strategies. They might suggest new institutions and research management procedures needed to support evidence-based improvement of mathematical education practice.

The RAND Study Group on Mathematics Education hopes that this report will provide a useful base on which such groups can build.

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RAND MATHEMATICS STUDY GROUP

Deborah Loewenberg Ball is Arthur F. Thurnau Professor of mathematics education and teacher education at the University of Michigan. Her work as a researcher and teacher educator is rooted in practice, drawing directly and indirectly on her many years of experience as a classroom teacher. Ball's work focuses on studies of instruction and of the processes of learning to teach. She also investigates efforts to improve teaching through policy, reform initiatives, and teacher education. Two research projects are the site for her current work.

With David Cohen and Brian Rowan, she is currently co-directing a large longitudinal study of whole-school reforms designed to improve instruction and learning in reading/language arts and mathematics in high-poverty elementary schools. The Study of Instructional Improvement seeks to develop a theory of instruction and of intervention that will develop knowledge of the processes and outcomes of instructional improvement. The research team is studying the course of four major whole-school reform programs in 100 schools over five years, tracing the interventions' efforts to make change, and the responses of school personnel, teachers, parents, and students. Teachers' learning, as well as students' opportunities to learn and their performance, will be followed across the entire period of the study.

Ball also studies the practice of elementary mathematics teaching. The Mathematics Teaching and Learning to Teach Project focuses on the work of teaching and seeks to uncover the mathematics that teachers need to know in order to teach mathematics well. Ball's principal collaborator on this work is Hyman Bass, a research mathematician. The research team studies classroom teaching and analyzes the mathematical entailments of the work, a sort of job analysis. This project also explores how records of practice (e.g., videotapes of classrooms, student work) can be used to support communication about teaching and learning among a wide range of stakeholders.

Ball's publications include articles on teacher learning and teacher education; the role of subject-matter knowledge in teaching and learning to teach; endemic challenges of teaching; and the relations of policy and practice in instructional improvement.

Hyman Bass is the Roger Lyndon Collegiate Professor of Mathematics and Professor of Mathematics Education at the University of Michigan. His mathematical research publications cover broad areas of algebra, with connections to geometry, topology and number theory. He has received the Cole Prize in Algebra from the American Mathematical Society, and the Van Amringe Book Award from Columbia University for a book that helped found the subject of algebraic K-theory. He has held visiting research and faculty positions at mathematical centers around the world, including Paris, Bombay, Rio, Cambridge, Stockholm, Mexico, Rome, Trieste, Hong Kong, Berkeley, and Jerusalem. He has lectured widely, in particular as a Phi Beta Kappa National Visiting Scholar. He is a member of the National Academy of Sciences and the American Academy of Arts and Sciences. Bass is president of the American Mathematical Society. He recently chaired the Mathematical Sciences Education Board at the NRC, and the Committee on Education of the American Mathematical Society, and he is President of the International Commission on Mathematics Instruction. During the past six years, he has been collaborating with Deborah Ball and her research group at the University of Michigan on the mathematical knowledge and resources entailed in the teaching of mathematics at the elementary level. In all of this work, a major challenge has been to build bridges between diverse professional communities and stakeholders involved in mathematics education, both here and abroad.

Jo Boaler is an Associate Professor of mathematics education at Stanford University. She has been teaching and conducting research in mathematics education for the last eight years: the first five years at King's College, London University, and the latter three at Stanford University. Jo is a former secondary school teacher of mathematics. She taught in diverse, inner-London comprehensive schools, across the 11–18 age range. Her interests include teaching and learning through different mathematics teaching approaches, equity, and teacher education. She is author of the book *Experiencing School Mathematics: Teaching Styles, Sex and Setting* that was published by Open University Press in 1997 and won the Outstanding Book of the Year award for education in Great Britain. She won the Best Ph.D. in Education Award in the United Kingdom and is author of the book *Multiple Perspectives on Mathematics Education*. She is currently president of IOWME—international organization for women and mathematics education, a subgroup of ICME.

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Thomas Carpenter is Professor of Curriculum and Instruction (Mathematics Education) at the University of Wisconsin–Madison and is Director of the National Center for Improving Student Learning and Achievement in Mathematics and Science, funded by the Office of Educational Research and Improvement of the Department of Education. His research integrates the study of the development of children's mathematical thinking, instruction that supports that development, and professional development that fosters instruction that leads to learning with understanding. His current research focuses on the development of algebraic thinking in the elementary school, in particular the development of generalization, justification, and proof. He is former editor of the *Journal for Research in Mathematics Education* and served on the National Research Council study panel The Mathematics Learning Study.

Phil Daro is the Executive Director of the California Institutes for Professional Development and the Director of Research and Development for the National Center on Education and the Economy. Mr. Daro's career has included his tenure as the Director of the Office of Project Development with the California Department of Education, Executive Director of the American Mathematics Project, and Executive Director of the California Mathematics Project. He received his B.A. in English from the University of California–Berkeley, with a minor in mathematics. A former high school math teacher, he received his teacher training at the State University of New Jersey, Trenton.

Joan Ferrini-Mundy is Associate Dean for Science and Mathematics Education in the College of Natural Science at Michigan State University, where she is a Professor of Mathematics and of Teacher Education. She holds a Ph.D. in mathematics education from the University of New Hampshire and was a faculty member in mathematics there from 1983–1999. Dr. Ferrini-Mundy taught mathematics at Mount Holyoke College in 1982–1983 where she co-founded the Summer Math for Teachers program. She served as a visiting scientist at NSF, 1989–1991. She has chaired the Research Advisory Committee, and been a member of the Board of Directors of the National Council of Teachers of Mathematics. She served on the Mathematical Sciences Education Board, and was its director, 1995–1999. Dr. Ferrini-Mundy's research interests are in calculus learning, mathematics education reform, K–14, and secondary mathematics teachers' learning. She served as chair of the Writing Group for NCTM's *Principles and Standards for School Mathematics*.

Ramesh Gangolli was born in India, and was educated in India, England, and the United States. He served as a member of the mathematics department at the University of Washington from 1962 to 1997. His mathematical research has been in the areas of probability theory and harmonic analysis. He has also maintained an active interest and participation in mathematics education in schools for over 30 years. He has served in various capacities within and outside the University of Washington: as chair of the mathematics department, as a member of NSF's Advisory Committee for the Division of Mathematical Sciences, as member of MSEB, as a trustee of the American Mathematical Society, and so on. At present he is partially retired from his university position and spends his time pursuing his interests in mathematics, music, and mathematics education. He is the Principal Investigator of a local systemic change project funded by NSF, involving nearly 600 mathematics teachers drawn from six school districts in the Seattle metropolitan area.

Rochelle Gutiérrez is Assistant Professor in the Department of Curriculum and Instruction, College of Education and in the Latina/Latino Studies Program, College of Liberal Arts and Sciences at the University of Illinois at Urbana-Champaign. She received her bachelor's degree in human biology from Stanford University, and her master's and Ph.D. in curriculum and instruction from the University of Chicago. She has been a summer fellow at the Center for Advanced Study in the Behavioral Sciences at Stanford University, a Dissertation Fellow with the Spencer Foundation, a Post-Doctoral Fellow with the National Academy of Education/Spencer Foundation, and currently is a Faculty Fellow in the Bureau of Educational Research at the University of Illinois. Her research interests center on issues of equity for marginalized students, especially those living in the inner city. She is specifically concerned with the sociocultural and organizational factors that play out in the teaching and learning of mathematics for Latina/Latino and African American students.

Roger Howe has been teaching and doing mathematics at Yale University for over 25 years. He has been concerned with issues of mathematics education since 1990. He has served on MSEB and on the board of directors of the Connecticut Academy for Education in Mathematics, Science and Technology. He served on the Mathematics Learning Panel of the National Research Council, and on the Steering Committee of the CBMS Mathematics Education of Teachers project. He was chair of the American Mathematical Society's committee to provide input to the NCTM Standards 2000 project. For the past two years, he has been chair of the AMS Committee on Education. He has been a consultant to commercial mathematics textbook publishers, and has published articles on mathematics and education in several journals.

Jeremy Kilpatrick is Regents Professor of Mathematics Education at the University of Georgia. After receiving an A.B. and M.A. from the University of California–Berkeley, he went to

Stanford University where he earned an M.S. and a Ph.D. (in mathematics education under E. G. Begle). Before joining the faculty at Georgia in 1975, he taught at Teachers College, Columbia University. He co-edited the series *Soviet Studies in the Psychology of Learning and Teaching Mathematics* from 1969 to 1975 and was editor of the *Journal for Research in Mathematics Education* from 1982 to 1988. Among his other editorial work, he edited the chapters on curriculum, goals, content, and resources for the 1996 *International Handbook of Mathematics Education* and co-edited the 1998 volume *Mathematics Education as a Research Domain: A Search for Identity.* His publications include a chapter on the history of research in mathematics education in the 1992 *Handbook of Research on Mathematics Teaching and Learning* and coauthorship of a research report on an innovative precalculus course in the 1996 Volume 3 of *Bold Ventures: Case Studies of U.S. Innovations in Mathematics Education.* He has taught courses in mathematics education at several European and Latin American universities and has received Fulbright awards for his work in New Zealand, Spain, Colombia, and Sweden. From 1991 to 1998, he was Vice President of the International Commission on Mathematical Instruction. He chaired the Committee on Mathematics Learning of the National Research Council; the committee report *Adding It Up* was published by the National Academy Press in 2001. His present research interests include mathematics curricula, research in mathematics education, and the history of both.

Karen D. King, Ph.D., is Assistant Professor of Mathematics Education in the Department of Mathematics, Michigan State University. While studying at the University of Maryland, she conducted research on undergraduate teacher thinking. Her recent research has focused on undergraduate student learning of differential equations, with a particular focus on the role of technology in supporting students' learning. Her current research merges these two lines of inquiry, to focus on undergraduate teaching of preservice secondary teachers, with attempts to coordinate the experiences of the different members of the classroom community, students and teacher. Karen's work in teacher preparation focuses on the content development of future teachers, with particular attention to the classroom experience of the prospective teacher as a model of standards-based teaching.

W. J. "Jim" Lewis is Professor and chair of the Department of Mathematics and Statistics at the University of Nebraska–Lincoln (UNL). Under his leadership the department won the University of Nebraska's 1998 University-wide Department Teaching Award as the outstanding teaching department in the four-campus university system. His department also won a 1998 Presidential Award for Excellence in Science, Mathematics, and Engineering Mentoring. Lewis is also an award-winning teacher having received teaching awards from UNL and from the MAA. He was a principal investigator for the Nebraska Math and Science Initiative, Nebraska's NSF-funded SSI. Currently he is a co-PI for a NSF grant to revise the mathematics education of future elementary school teachers at UNL. He is a past chair of the American Mathematical Society's Committee on Science Policy and currently serves on the AMS Committee on Education. Lewis was co-chair of the NRC Committee on Science and Mathematics Teacher Preparation that produced the report *Educating Teachers of Science, Mathematics, and Technology: New Practices for the New Millennium*. Currently he is chair of the Steering Committee for the U.S. Department of Education–funded Conference Board on the Mathematical Sciences (CBMS) project that recently released the report *The Mathematical Education of Teachers*. He received his Ph.D. in mathematics from Louisiana State University.

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With Michelle Perry, James Stigler, and David Brady, he is currently running a large-scale study on how cross-national video-based records of classroom practices can be used in improving mathematics education. This project, Representing and Learning from Classroom Processes, aims at providing a research base for designing and implementing systems that use video of classroom interactions as part of teacher education and educational research.

Marge Petit is Senior Associate at the National Center for Improving Educational Assessment (NCIEA) (Center for Assessment). Prior to assuming the position of Senior Associate of NCIEA, Ms. Petit served as Deputy Commissioner of Education, Vermont Department of Education. From September 1999 to February 2000, she served as Acting Commissioner of Education. Ms. Petit provided statewide leadership with Commissioner Marc Hull in the implementation of the quality aspects of the Vermont Equal Educational Opportunity Act. Previous to being named Deputy Commissioner, Ms. Petit was the Assessment Specialist with the Vermont Institute for Science, Mathematics, and Technology, a position she held from 1993 to 1996. She has been a Vermont educator since 1968. Her experience includes working with students in the classroom in mathematics and science, statewide and national development in assessment, mathematics and science materials, state and national policy development, as well as working with teachers and administrators around Vermont. Ms. Petit was a summer writer and assessment consultant to the STEM (Mathematics) Project at the University of Montana. She was a member of the national advisory board for the National Test in Mathematics and Mathematics Advisory Board for the Achieve Middle School Mathematics Project. She is presently a member of the NAEP Planning Committee and the MSEB Board.

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Mark Saul is a classroom teacher at the Bronxville Schools, a suburban district just north of New York City. Throughout his 30-year teaching career, he has taught students from a wide variety of backgrounds, in inner-city schools, as well as upscale suburban environments. He has worked with students on a variety of levels from third grade through high school, from remedial students to the most advanced. He has also worked extensively with pre- and in-service teachers. In addition to his classroom work, he has published numerous books and articles, including translations from French and Russian. Internationally, he has served as a consultant and led exchange programs to Taiwan, Russia, Bulgaria, South Africa, and Romania. He has served as Chief Guide for the 2001 International Olympiad, a Director of the Research Science Institute, a program for high-ability high school students at the Massachusetts Institute of Technology, as president of the American Regions Mathematics League, and as an executive board member of the Mathematical Sciences Education Board. Currently, he serves on the Board of Directors of the National Council of Teachers of Mathematics, and as Associate Editor of both *Notices of the American Mathematical Society* and of *The Mathematics Teacher*, the NCTM journal for high school teachers. In 1984, he received the Presidential Award for Excellence in the Teaching of Mathematics from NSF, and, in 1998, he received the Paul Erdos Award from the World Federation of National Mathematics Competitions. In 1997, he was elected as a Fellow of the American Association for the Advancement of Science.

Geoffrey Saxe is a Professor in the Graduate School of Education at the University of California–Berkeley. He is known internationally for his empirical and theoretical contributions in areas of culture and cognitive development with a focus on children's mathematics. He has served as PI and Co-PI on federal and private foundation grants concerned with children's cognitive development and processes of teaching and learning. Sites for his research have included remote areas of Papua, New Guinea, urban and rural Brazil, and urban schools and home settings in the United States. His current work focuses on the interplay between teaching and learning in the domain of fractions in the upper elementary grades. He has served as a member of various standing committees and task forces and review panels for private and public foundations, including the MacArthur Foundation, the Spencer Foundation, NSF, and the National Institutes of Mental Health. He currently serves on numerous editorial boards for scholarly journals, and he is the Incoming Editor of the journal, *Human Development*.

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Thomas K. Glennan, Jr. (Ph.D., Economics, 1968, Stanford University) is a Senior Advisor for Education Policy in the Washington Office of RAND. His research at RAND has spanned a wide variety of policy planning issues in such diverse areas as education, manpower training, energy, environmental enforcement, demonstration program management in health and human services, and military research and development. Through 1997, he led RAND's analytic effort in support of the New American Schools Development Corporation. He has also examined potential national and federal policies in support of the use of technology in elementary and secondary education. Currently, he is leading an effort to develop plans for coherent, long-term programs of R&D in Reading and Mathematics Education for the Office of Education Research and Improvement in the federal Department of Education. He is a coauthor of books on the management of research and development and the use of social experiments in policy planning. Dr. Glennan served as Director of Research and Acting Assistant Director of the Office of Economic Opportunity for Planning, Research and Evaluation before becoming the first Director of the National Institute of Education in 1972.

Carole Lacampagne (Ed.D., Teachers College, Columbia University) is currently Director of the Mathematical Sciences Education Board, National Academy of Sciences. During the work of the RAND Mathematics Panel, she served as its senior research advisor. Before coming to

RAND, Dr. Lacampagne served as Director of OERI's National Institute on Postsecondary Improvement, Libraries, and Lifelong Learning, as a visiting scientist for NSF, and as Associate Professor of Mathematics, Northern Illinois University. Dr. Lacampagne has publications in number theory and in mathematics education, has served on articulation committees for mathematics in the states of New Jersey and Illinois, and has chaired several committees of the Mathematical Association of America.

Frederic A. (Fritz) Mosher is an independent consultant on education policy and research planning, management, and funding. He is Senior Advisor to the Spencer Foundation and a RAND Adjunct Staff member, working with RAND's project, supported by the Office of Educational Research and Improvement, to examine ways in which OERI might improve the quality and relevance of the education research it funds. He has been an advisor to the Assistant Secretary for Research and Improvement in the U.S. Department of Education and to also Achieve, Inc. In 1998, he retired from Carnegie Corporation of New York (a philanthropic foundation) after 36 years as a program specialist and policy analyst. Over that time he worked in the full range of the Corporation's programs, including international affairs; U.S. governmental reform; education at all levels; and the role of universities in the planning and development of national education systems in Anglophone Africa. In the 1970s, along with Vivien Stewart, he developed Carnegie's initial program in the reform of public education; and in the 1980s and early 1990s, under the leadership of David Hamburg, he chaired the Corporation's program on Avoiding Nuclear War (later Cooperative Security), which dealt extensively with U.S.-Soviet relations. In recent years he returned to a focus on the policy issues involved in transforming the U.S. public education system into one that would enable substantially all students to reach high standards of achievement. He is a cognitive/social psychologist by training, with a Ph.D. from Harvard University.

Gina Schuyler received an M.A. in teaching from Trinity College, a B.S. in policy and management from Carnegie Mellon University, and a B.S. in history and policy from Carnegie Mellon University. She is an Education Research Analyst for RAND Education in the Washington office. Her primary interests lie in K–12 education reform, at-risk students, and teacher quality. Her current projects include an evaluation of the Ford Foundation's Collaborating for Educational Reform Initiative, a study of 10-year strategies for programs of research for the Department of Education's Office of Educational Research and Improvement, and continuing work on an evaluation of New American Schools. Ms. Schuyler has also taught kindergarten and first grade in Washington, D.C.