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Subjective Measures of Risk Aversion and Portfolio Choice

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Abstract

The paper investigates risk attitudes among different types of individuals. We use several different measures of risk atttitudes, including questions on choices between uncertain income streams suggested by Barsky et al. (1997) and a number of ad hoc measures. As in Barsky et al. (1997) and Arrondel (2002), we first analyse individual variation in the risk aversion measures and explain them by background characteristics (both "objective" characteristics and other subjective measures of risk preference). Next we incorporate the measured risk attitudes into a household portfolio allocation model, which explains portfolio shares, while accounting for incomplete portfolios. Our results show that the Barsky et al. (1997) measure has little explanatory power, whereas ad hoc measures do a considerably better job. We provide a discussion of the reasons for this finding.

Jel–Classification: C5; C9; D12; G11
Keywords: Risk Aversion; Portfolio Choice; Subjective Measures; Econometric Models

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1 Introduction

This paper exploits direct measures of risk preferences in a model of household portfolio allocation. There are two main motivations for this. The first one is that if heterogeneity in risk preferences is important then empirical portfolio models should take this into account. The second motivation is that economic theory has a fair amount to say about how risk preferences should influence portfolio allocation. Having direct measures of risk preferences should therefore help us in better testing the validity or predictive power of economic theories of portfolio allocation.

Empirical analyses of portfolio choice of households or individuals appear to indicate that observed choices are often inconsistent with standard asset allocation models. As a consequence, several studies have focused on empirical failures of portfolio theory. The greatest failure is perhaps the fact that the majority of individuals do not hold fully diversified portfolios, although the percentage of households holding risky assets has increased over the last decade (Haliassos and Hassapis, 2000). A potential explanation for the fact that many households do not hold stocks may be lie in the costs of stock market participation (Vissing-Jorgensen, 2000).

The sub-optimal degree of international diversification known as "home asset bias" is potentially another empirical failure. It has been analyzed, among others, by French and Poterba (1990, 1991), Tesar and Werner (1992, 1994, 1995), Cooper and Kaplanis (1994), Glassman and Riddick (2001), and Jermann(2002). Possible reasons for the over-investment in domestic assets have been identified in different transaction costs between countries, additional sources of risk for foreign investments and explicit omission of assets from the investor's opportunity set.

A more fundamental piece of evidence against the rational model of portfolio allocation is provided by Benartzi and Thaler (2001) who find that the allocation of investors is heavily dependent upon the choices offered to them. Roughly speaking, if they are offered n choices they tend to allocate $\frac{1}{n}$ of their investment to each of the choices offered, independent of the risk characteristics of the investment opportunities.

Although these findings suggest that the rational model of choice is unable to explain several empirical phenomena, it is often hard to determine in more detail what the underlying cause of disparities between theory and empirical facts may be. The connection between theory and empirical evidence is often tenuous,

because too many intervening factors may explain why theoretical predictions are not borne out by data. For this reason some authors have turned to more direct, subjective evidence on preferencess to reduce the distance between theory and empirical facts. A prominent example is the paper by Barsky et al. (1997) who elicit several pieces of subjective information to improve our understanding of intertemporal choice and portfolio allocation.

In this paper we also aim to exploit subjective information to construct empirical micro-models of portfolio choice. In contrast with the work by Barsky et al. (1997) and Arrondel (2000), our model will be a formal structural model of portfolio choice, in which we consider several different measures of risk attitude. One measure is based on hypothetical choices between uncertain income streams in a household surve, and closely related to the aforementioned work by Barsky et al. (1997) and Arrondel (2000). The Barsky et al. measure has a nice direct interpretation if individuals have CRRA preferences. We will find however, that the measure also has theoretical and empirical problems. Hence we also consider alternative measures of risk attitude. We relate the different measured risk attitudes to observed portfolio choices of households. To deal with incomplete portfolios, we set up a simple rationing model that can endogenously generate corner solutions in portfolio allocation. Thus, we formulate and estimate a complete system of portfolio demand equations incorporating subjective measures of risk aversion. The model is closely related to rational portfolio theory and seems to do a reasonable job in describing differences in allocation across individuals who differ in socio-economic characteristics, wealth, and risk attitudes.

The paper is organized as follows. In the next section we describe the data we use in the analysis. In particular we present descriptive statistics on the various risk attitude measures and on the portfolio composition of households. Section 3 discusses some results from the literature regarding the classical theory of portfolio choice. Based on these results we formulate in Section 4 a simple static asset allocation model with rationing. The rationing emerges as the result of corner solutions (i.e. the existence of incomplete portfolios). We then derive an econometric model with switching regimes, where each regime is characterized by a particular asset ownership pattern. The model has two components: the first component determines the regime, and the second component describes portfolio shares of assets conditional on the regime. Section 5 presents empirical results. We find that the risk tolerance measure of Barsky et al. (1997) does a poor job in explaining choices between risky and less risky assets. Other, simpler, risk

attitudes measure do have a significant effect on the choice of risky assets. It thus appears that these simple attitude measures provide a better measure of risk tolerance.

2 Data on risk aversion and precaution and how they were collected

The data used in this paper have been collected from the households in the so-called CentERpanel. The CentERpanel is representative of the Dutch population, comprising some 2000 households in the Netherlands. The members of those households answer a questionnaire at their home computers every week. These computers may either be their own computer or a PC provided by CentERdata, the agency running the panel[1]. In the weekends of August 7-10 and August 14-17 of 1998 a questionnaire was fielded with a large number of subjective questions on hypothetical choices. The questionnaire was repeated in the weekends of November 20-23 and November 27-30 of 1998 for those panel members who had not responded yet. For this paper we exploit the section involving choices over uncertain lifetime incomes. We merge these data with data from the CentER Savings Survey (CSS). The CSS collects data on assets, liabilities, demographics, work, housing, mortgages, health and income, and many subjective variables (e.g. expectations, savings motives) from annual interviews with participants in the CentERpanel. Typically the questions for the CSS are asked in May of each year, during a number of consecutive weekends.

We discuss consecutively three measures of risk aversion elicited from the respondents in the sample.

2.1 Choices of uncertain lifetime income

Our first measure is based on a number of questions involving risky choices over lifetime incomes. This methodology, taken from Barsky et al. (1997) (BJKS, from now on), allows us to rank individuals with respect to their risk aversion without having to assume a particular functional form for the utility function.

[1]The description refers to the time of the survey. Nowadays, CentERdata does not provide a PC any longer but a set-top box.

In the BJKS experiment, questions are posed to all respondents, consisting of individuals aged over 50. Arrondel (2000) asked the questions to a representative sample of French households. In our case, the questions are asked only to people who have a job *and* who are the main breadwinner in a household (i.e. the person in the household who brings in the largest amount of money).

The structure of the questions is depicted in Figure 1. In the first round, respondents are asked the following question:

Imagine your doctor recommends that you move because of allergies. You follow his advice, and it turns out you have to choose between two possible jobs. Both jobs are about equally demanding (for example, both jobs involve the same number of working hours per week), but the income in one job is much more certain than the income in the other job.

The first job guarantees your current income for the rest of your life. In addition, we assume that income other members of your household may have, will also remain unchanged. In this situation, you know for sure that during the remainder of your life, the net income of your household will be equal to Dfl. Y.

The second job is possibly better paying, but the income is also less certain. In this job, there is a 50% chance that you will earn so much that the income of your household will be doubled for the rest of your life, that is, be equal to Dfl. Yx2.

There is, however, an equally big chance (50%) that you will earn substantially less in the second job. In the latter case, the net monthly income of your household will for the rest of your life be equal to Dfl. Yx07.

Which job would you take?

1 the job with the guaranteed fixed household income of Dfl. Y

2 the job that involves a 50% chance that the income of your household will for the rest of your life be equal to Dfl. Yx2, but also involves a 50% chance that the income of your household will for the rest of your life be equal to Yx07.

Various quantities in the question vary per respondent, exploiting the computerized nature of the interviews. The quantity Y is the respondent's selfreported after tax household income. Yx2 is twice the household income; Yx07 is household income times .7, etc. This is in contrast to the experiments by BJKS and Arrondel (2000), in which the incomes were the same for all individuals. Obviously, the question involves a choice between a certain and an uncertain outcome: the former is given by the actual income the respondent receives (Y), the latter is a 50-50 gamble over a good outcome (Yx2) and a bad outcome (Y x*07*).

In the second round each individual is asked a similar question. If she has chosen the certain outcome (Y) in the first round, she now faces another gamble where the risky outcome is more attractive. The 50-50 gamble now involves Yx2 and Yx08 (0.8 times income). If she has chosen the risky prospect in the first round, she is now asked to choose between her income for sure and a less attractive gamble, i.e. 50% chance of Yx2 and 50% chance of Yx05.

Similarly, in the third round the gamble becomes more attactive for those respondents who have once again chosen a certain income stream in the second round (the 50-50 gamble now involves Yx2 and Yx09), and less attractive for those respondents who preferred the risky choice (the 50-50 gamble now involves Yx2 and Yx025)

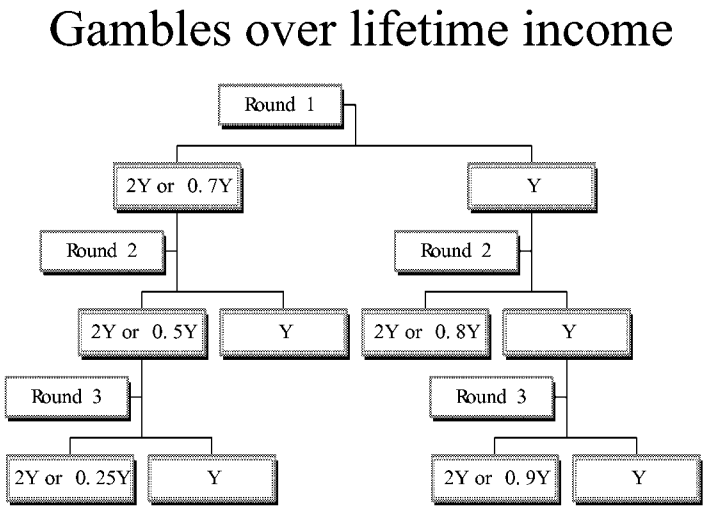


Figure 1: Choices of uncertain lifetime income

The answers to the questions allow us to identify six groups ranked from most risk averse to least risk averse (or equivalently from least risk tolerant to most risk tolerant; we will generally denote the variable defined by the six classes as "risk tolerance"). Both the BJKS study and Arrondel's involve only two rounds of questions rather than three as ours. For comparison we temporarily combine the two most extreme groups into one. Thus we have four categories of individuals,

from I to IV, where the I-group is the union of the 1 and the 2 groups and the IV-group is the union of the 5 and 6 groups. We can then compare the risk tolerance across the three studies. Table 1 gives the results. To facilitate a comparison with the BJKS study we split our sample in two age groups: 50 and younger and over 50[2].

An unfortunate aspect of the sample selection (respondents being employed and being the main breadwinner) is that it severely limits the number of observations. This clearly reduces the possibility of obtaining statistically significant results. Keeping this in mind, a comparison between France and The Netherlands on the basis of the complete age range suggests that there is a greater spread of risk aversion in The Netherlands than in France. The Dutch respondents are more heavily represented in the two extreme categories (almost 53% of the Dutch belong to the most risk averse group compared to 43% of the French, whereas 12% of the Dutch belong to the least risk averse group compared to 6% for the French). Summing the percentages of the first two groups and the percentages of the last two groups respectively, suggests that the Dutch are less risk averse than the French (only 69.5% of the Dutch belong to the first two groups compared to 82.5% of the French, whereas 30.5% of the Dutch belong to the last two groups compared to 17.5% of the French).

Considering the subsamples of respondents over 50, it appears that the Dutch have similar risk preferences to the Americans, although the Americans may be slightly more risk tolerant than the Dutch. Compared to the Dutch and the Americans, the French appear to be much more risk averse.

Table 1: Risk Tolerance in the USA, France and The Netherlands

	Total sample		Respondents over 50		
Group	**France**	**Neth.**	**USA**	**France**	**Neth.**
I	43.1	52.8	64.6	48.6	66.3
II	39.4	16.7	11.6	36.8	13.5
III	11.2	18.1	10.9	8.7	9.0
IV	6.3	12.4	12.8	5.9	11.2
Total	100	100	100	100	100
Obs.	2954	657	11707	2954	178

[2]The BJKS sample consists of respondents over 50.

Turning to a closer analysis of the Dutch data, we once again distinguish six classes of risk tolerance. Table 2 presents a number of descriptive statistics of the risk tolerance variable by several demographic and socio-economic characteristics. For the purpose of this table, the risk tolerance has been coded from 1 (least risk tolerant) to 6 (most risk tolerant). The p-values refer to one way analyses of variance of each of the risk attitude measures on the characteristics considered. Notice that there is a very uneven distribution of males and females in the sample. This is the result of the fact that only employed main breadwinners have been selected. The vast majority of the respondents fall in the most risk averse categories. Although the table might suggest that females are more risk averse than males, the difference in means is small and clearly not significant. Table 2 suggests that better educated individuals are generally less risk averse; the differences in risk tolerance between the three levels of education are statistically significant. Although the table suggests that the self-employed are substantially more risk tolerant than employees, the small number of observations of self-employed respondents leads to statistically insignificant differences.

Table 2: Risk attitude variable by background characteristics (means)

Characteristic	**Risk tolerance**	**N.obs.**
Male	2.724	567
Female	2.555	90
p-value	*.32*	
Low education	2.345	259
Middle education	2.882	212
High education	2.796	266
p-value	.001	
Employees	2.68	622
Self-employed	3.085	35
p-value	*.12*	
Whole sample	2.701	657

2.2 Risk attitude measures based on principal components

The CSS-questionnaire contains six direct questions about investment strategies. These are reproduced below. Respondents can express their agreement or disagreement with these statements on a seven point scale (1 means complete disagreement and 7 means complete agreement.

SPAAR1

I find it more important to invest safely and to get a guaranteed return than to take risks in order to possibly get a higher return.

SPAAR2

Investing in stocks is something I don't do, since it is too risky.

SPAAR3

If I believe an investment will carry a profit, I am willing to borrow money for it.

SPAAR4

I want to be sure my investments are safe

SPAAR5

I am increasingly convinced that I need to take more financial risks if I want to improve my financial position.

SPAAR6

I am willing to run the risk of loosing money if there is also a chance that I will make money.

The CSS also contains 13 questions about savings motives. Below we reproduce three of them that are related to precautionary motives and uncertainty. Answers can be given on a 1 to 7 scale, where 1 means "very unimportant" and 7 means "very important".

SPAARM03

To have some savings in case of unforeseen expenses due to illness or an accident.

SPAARM12

As a reserve for unforeseen events

SPAARM13

To have enough money in the bank, so that I can be sure to be able to meet my financial obligations.

Applying principal components analysis to these nine indicators of risk aversion and precaution, we find that three underlying factors can explain most of the variance in the answers. After applying varimax rotation we find the factor loadings presented in Table 3. The largest factor loadings in each column are given in bold face. We note that the factors which we have callled "riskat1" and "riskat3" mainly explain the six "spaar" variables, whereas "riskat2" mainly explains the "spaarm" variables. In view of the wordings of the nine questions, we interpret riskat1 and riskat3 as measures of risk aversion, whereas riskat2 is mainly a measure of prudence. Thus we would expect riskat2 to play a role in savings decisions whereas riskat1 and riskat3 should affect portfolio choice.

Table 3: Rotated factor loadings (varimax rotation)

Variables	**riskat1**	**riskat2**	**riskat3**	**uniqueness**
spaarm03	-0.038	**0.796**	-0.017	0.365
spaarm12	0.163	**0.838**	0.023	0.270
spaarm13	0.154	**0.785**	0.036	0.360
spaar1	**0.838**	0.111	0.058	0.282
spaar2	**0.632**	-0.007	0.370	0.464
spaar3	0.006	0.030	**0.724**	0.475
spaar4	**0.817**	0.151	0.046	0.307
spaar5	-0.015	0.037	**0.785**	0.382
spaar6	0.346	-0.027	**0.758**	0.304

Table 4 presents a number of descriptive statistics of the three risk attitude measures by several demographic and socio-economic characteristics, organized in a similar way as Table 2. The risk attitude measures are normalized such that they have zero mean and unit variance for the complete sample. The variance varies slightly across subgroups but usually very little. Table 4 shows that all three risk attitude measures are significantly less for males than for females, implying that males are less risk averse and have a less strong precautionary motive. Education only has a significant effect for riskat1, with a pattern that is hard to interpret. Despite the small number of self-employed in our sample we do find significant less risk aversion (or precaution) among the self-employed than among employees for riskat2 and riskat3. For riskat1, the difference is not significant.

Table 4: Risk attitude variables by background characteristics (means)

Characteristic	**riskat1**	**riskat2**	**riskat3**	**N.obs.**
Male	-.071	-.092	-.126	529
Female	.123	.159	.219	304
p-value	*.007*	*.0005*	*.0000*	
Low education	.032	.087	.060	259
Middle education	.075	-.007	-.048	255
High education	-.119	-.066	-.026	283
p-value	*.06*	*.21*	*.422*	
Employees	-.098	.039	-.092	407
Self-employed	.259	-.356	-.575	21
p-value	*.13*	*.07*	*.03*	
Whole sample	-1.85e-09	9.98e-10	4.42e-10	833

2.3 Direct questions on precaution and risk aversion

The third type of subjective measures are the answers to the following two questions, which were included in the same module of the CentERpanel as the BJKS risk aversion measure. The questions read as follows:

Would you rather describe yourself as a carefree person, or rather as a careful person?

When there is possible danger, do you take many precautions?

Responses in both cases could be given on a seven point scale. We will refer to the first variable as "careful" and to the second variable as "precaution". Table 5 presents a number of descriptive statistics of the two risk attitude measures by several demographic and socio-economic characteristics, analogous to Tables 2 and 4 above. The table contains the mean scores on the seven point scales for both variables. We observe that females are significantly more "careful" than males. Both careful and precaution appear negatively related with education.

Table 5: Risk attitude variables by background characteristics (means)

Characteristic	careful	precaution	N.obs.
Male	5.001	4.843	977
Female	5.110	4.896	734
p-value	*.04*	*.44*	
Low education	5.096	4.948	560
Middle education	5.048	4.877	531
High education	5.003	4.764	523
p-value	*.001*	*.04*	
Employees	4.916	4.711	786
Self-employed	4.914	4.851	47
p-value	*.90*	*.31*	
Whole sample	5.047	4.866	1711

2.4 How are the subjective measures related?

Before turning to the analysis of the relation between the various measures of risk tolerance and portfolio composition we present analyses of the interrelation between the various measures. First of all, comparing the outcomes presented in Tables 2, 4, and 5, we notice the following patterns. Splitting the sample by gender, risk tolerance does not reveal significant differences, but riskat1, riskat2, riskat3, and careful all indicate that males are more risk tolerant and less precautious than females. Distinguishing respondents by education shows that risk tolerance, riskat1, careful, and precaution all suggest that people with higher education are less risk averse and less precautious. Riskat2 and riskat3 do not differ significantly by education. Finally, only riskat2 and riskat3 differ significantly (at the 7% and 3% level respectively) between self-employed and employees. The lack of significant differences for the other variables does not necessarily mean that employees and self-employed do not differ, but may rather be a reflection of the small number of self-employed in the sample.

Table 6 presents results of regressing each of the measures on a number of background variables and the other measures. For the explanation of risk tolerance, careful and precaution the method of analysis is ordered probit. For riskat1 through riskat3 we use regression. The ordinal variables risk tolerance careful and precaution are simply coded from 1 to 7 when used as explanatory variables. Replacing the simple coding by dummies does not alter the outcomes appreciably. Since the variables do not all come from the same interview and because not all

questions are asked to all respondents, the number of observations is fairly small, reflecting the partial overlap in observations. Of course, the regressions should be viewed as purely descriptive. They present a way of showing the nature of the interrelations between the six measures. Although we briefly discuss the effects of background variables, like age, income, wealth, gender, education, and being selfemployed, one should keep in mind that the effects of these variables are all conditional on all the other risk attitude measures in the regressions.

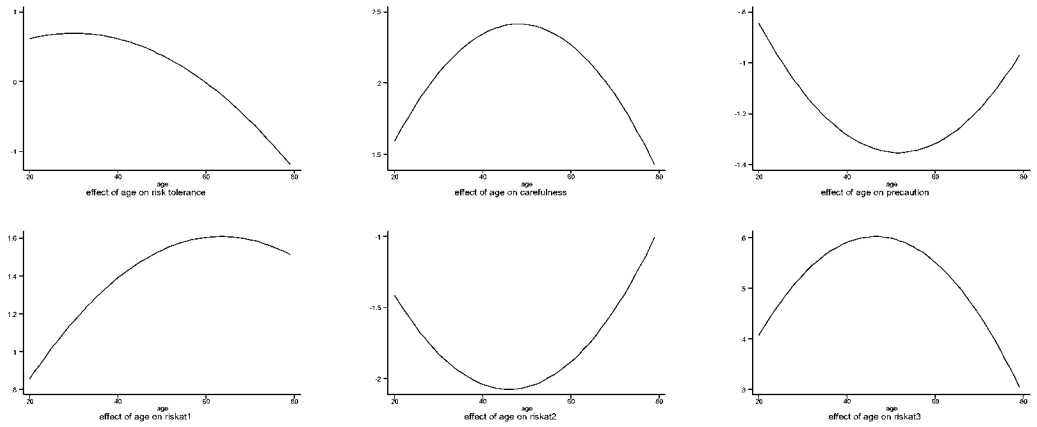
Age is only significant for risk tolerance and riskat1. Figure 2 draws the quadratic age functions implied by the estimates for all six measures. The figure shows that both risk tolerance and riskat1 imply that risk aversion increases with age. The variables income and total wealth are both coded in the form of an inverse hyperbolic sine[3]. Income has a significant effect on risk tolerance and riskat1. The signs of the effects imply in both cases decreasing risk aversion with income. Wealth is (marginally) significant in four out of six regressions. The estimation results for risk tolerance suggest increasing relative risk aversion with increasing wealth, which would be consistent with a constant absolute risk aversion utility function, for instance. The results for riskat3 suggest decreasing risk aversion with increasing wealth, but since we do not know how riskat3 would be related exactly to the parameters of a known utility function, the result is hard to interpret. The two precautionary measures (precaution and riskat2) increase with wealth. Gender is never significant and education and being self-employed are only significant for risk tolerance. One should recall once again, that these effects are to be interpreted as being conditional on the other risk attitude measures.

Turning to the interrelationships between the risk attitude measures, we observe that precaution and riskat2 appear to be significantly related, as one would expect from the interpretation of riskat2. Somewhat less expected, careful, riskat3, and risk tolerance are also significantly related to precaution.

Table 6: Interrelationships between the various measures (ordered probit and ols estimates)

[3]The inverse hyperbolix sine of x is $h(x) = \ln[1 + \sqrt{x^2 + 1}]$

Expl. variables	**Risk tol.**	**Careful**	**Precau.**	**Riskat1**	**Riskat2**	**Riskat3**
Age	.046	.100	-.052	.051	-.090	.026
Age squared	-.0008	-.001	.0005	-.0004	.001	-.0003
p-value age eff.	*.002*	*.133*	*.423*	*.050*	*.186*	*.864*
Income	.533	-.321	.206	-.503	-.110	.004
t-value	*2.79*	*1.67*	*1.09*	*2.91*	*.65*	*.02*
wealth	-.070	.125	-.052	.036	.065	-.075
t-value	*1.90*	*3.37*	*1.39*	*1.05*	*1.94*	*2.27*
Gender	-.047	-.072	-.168	-.072	.379	.204
t-value	*.26*	*.40*	*.93*	*.44*	*2.34*	*1.28*
Self-empl.	.624	-.008	.186	.238	-..497	-.246
t-value	*2.12*	*.03*	*.62*	*.87*	*1.85*	*.93*
Middle educ.	.435	-.131	.164	.067	-.174	.108
High educ.	.331	-.125	.175	.042	-.278	.071
p-value ed eff.	*.023*	*.676*	*.492*	*.901*	*.149*	*.745*
Careful	-.058	-	.531	.189	.055	-.044
t-value	*1.00*	*-*	*9.72*	*3.63*	*1.09*	*.87*
Precaution	-.152	.518	-	-.013	.102	.103
t-value	*2.69*	*9.62*	*-*	*.26*	*2.02*	*2.07*
Risk tolerance	-	-.049	-.127	-.113	.058	-.067
t-value	*-*	*1.09*	*2.85*	*2.77*	*1.44*	*1.71*
Riskat1	-.172	.233	-.009	-	-	-
t-value	*2.90*	*3.90*	*.15*	-	-	-
Riskat2	.083	.086	.134	-	-	-
t-value	*1.33*	*1.40*	*2.20*	-	-	-
Riskat3	-.120	-.021	.130	-	-	-
t-value	*1.94*	*.33*	*2.10*	-	-	-
Number of obs.	342	342	342	342	342	342
(Pseudo) R^2	0.066	.156	.128	.158	.076	.060



Age effects on risk attitude measures

Figure 2: Age functions for the six risk attitude measures

2.5 Assets and liabilities

The CSS collects extensive information on assets and liabilities. Respondents are asked for ownership and quantity of different categories of assets, both real and financial, and of liabilities and mortgages. Table 7 reports data on financial assets at the household level. We group assets in categories that are somewhat homogeneous with respect to their risk profile. We have indicated the category an asset belongs to by NR for non-risky assets, by R for risky assets and by O for "other" assets. Non-risky assets are checking accounts, savings accounts, deposits, and insurances. Risky assets are defined as the sum of growth and mutual funds, options, stocks and business equity. Other assets are the sum of real estate, mortgage, bonds, money lent out and financial debt. The table presents relative frequencies of ownership, mean values and shares of each asset category in the total portfolio. Notice that the data do not only refer to the respondents who have been posed the questions about uncertain life time incomes.

Checking accounts, savings accounts, deposits, and insurances are held by 96.3% of the sample. This safe asset makes up close to 50% of average financial wealth in the sample. Business equity is another sizeable component of financial wealth, although it is only held by 7.1% of the sample. Generally, the households in the sample use very little credit: Financial debts amount to less than 10% of total financial assets (and hence financial wealth is more than 90% of total financial assets). Only 4.5% of the sample holds bonds and/or mortgage bonds, whereas 16.5% holds stocks; 22.6% of the sample holds growth or mutual funds. Of course the group of stock holders overlaps with the owners of mutual funds or growth funds. 30.4% of the sample households have stocks and/or growth and mutual funds. Clearly, for most households real estate (usually the primary residence) dominates the portfolio. Financial assets are only 27% of total assets and financial wealth is only 33% of total wealth.

At the aggregate level, both risky and non-risky assets are basically 50% of average financial wealth, but the percentage of people owning risky assets (34.5) is far less than half of that of people owning safe assets (96.3). Other assets are widely spread over the sample (82.8%) and they are 175.9% of financial assets.

Table 7: Households' Assets and liabilities in 1998

Assets	**Type**	**Obs.**	**% own**	**Mean**	**% f.ass.**	**% t. ass.**
Checking, dep., ins., etc.	*NR*	1427	96.3	45630	48.7	13.2
Growth, mutual funds	*R*	1427	22.6	11603	12.3	3.3
Bonds	*O*	1427	4.5	2187	2.3	.6
Stocks	*R*	1427	16.5	15683	16.7	4.5
Options	*R*	1427	1.1	115	.1	.03
Money lent out	*O*	1427	8.3	1943	2.1	.6
Business equity	*O*	1427	7.1	16591	17.7	4.8
Total financial assets		1427	-	93753	100	27.0
Real estate	*O*	1422	68.0	252650	-	72.9
Total assets		1422	-	346693	-	100
Financial debt	*O*	1427	-	8976	9.6	2.6
Mortgage	*O*	1422	55.8	82952	-	23.9
Net financial wealth		1427	-	84776	90.4	24.4
Non-risky assets	*NR*	1427	96.3	45630	48.7	13.2
Risky assets	*R*	1427	34.5	43992	46.9	12.7
Other assets	*O*	1422	82.8	346456	175.9	47.6
Net worth		1422	-	254803	-	73.5

NR: non-risky; *R*: risky; *O*: other assets. All amounts are in Dutch guilders (about $.50 in 1998)

3 Some theory

To motivate our empirical model, it is useful to summarize some concepts and results from the literature. In the exposition below we mainly follow the excellent new book by Gollier (2002).

3.1 Comparative risk aversion

Agent 1 is more risk averse than agent 2 if

$$E[u_2(w_0 + \tilde{x}) \leq u_2(w_0) \Rightarrow E[u_1(w_0 + \tilde{x}) \leq u_1(w_0)] \qquad (1)$$

where u_2 and u_1 are utility functions, w_0 is initial wealth and $\tilde{x}$ is a risky asset with zero expected return. This is equivalent with $A_1(w_0) \geq A_2(w_0)$, where $A_i(z) \equiv -\frac{u^{00}(z)}{u^0(z)}$, the coefficient of absolute risk aversion. Of course, if the coefficient of absolute risk aversion is larger for individual 1 at some positive wealth level z, then this is also true of the coefficient of relative risk aversion: $R_i(z) = z.A_i(z)$..

3.2 HARA (harmonic absolute risk aversion) utility functions

$$u(z) = \zeta(\eta + \frac{z}{\gamma})^{1-\gamma} \tag{2}$$

Absolute risk tolerance (the inverse of absolute risk aversion) for this utility function is equal to

$$T(z) = \frac{1}{A(z)} = -\frac{u'(z)}{u''(z)} = \eta + \frac{z}{\gamma} \tag{3}$$

Thus, absolute risk tolerance (inverse absolute risk aversion) is linear in wealth, which explains the name of this class of utility functions. Notice that the coefficient of relative risk aversion then equals

$$R(z) = \frac{z}{\eta + \frac{z}{\gamma}} \tag{4}$$

and the degree of absolute prudence:

$$P(z) = -\frac{u'''(z)}{u''(z)} = \frac{\gamma + 1}{\gamma}(\eta + \frac{z}{\gamma})^{-1} \tag{5}$$

The degree of relative prudence is $zP(z)$.

Notice that if $\eta = 0$, the utility function reduces to

$$u(z) = \zeta(\frac{z}{\gamma})^{1-\gamma} \tag{6}$$

which is the CRRA utility function with coefficient of relative risk aversion γ (cf. (4)). Similarly, if $\gamma \to \infty$, it can be shown that the utility function reduces to:

$$u(z) = -\frac{\exp(-Az)}{A} \tag{7}$$

where A is the coefficient of absolute risk aversion ($A = \frac{1}{\eta}$). Finally, for $\gamma = -1$, we obtain a quadratic utility function.

3.3 Risk aversion and portfolio choice

For CARA preferences (see (7)) the share of wealth to invest in a risky asset is

$$\frac{\alpha^*}{w_0} = \frac{\mu}{\sigma^2}\frac{1}{w_0 A} = \frac{\mu}{\sigma^2}\frac{1}{R(w_0)} \tag{8}$$

where μ and σ^2 are the mean and variance of the distribution of the excess return of the risky asset and α^* is the amount invested in the risky asset. For non-CARA preferences formula (8) is approximate.

HARA preferences (see (2)): For this case no explicit solution is available, but a numerical solution can be found in a rather simple way. Let a be the solution of the equation

$$E\tilde{x}(1+\frac{a\tilde{x}}{\gamma})^{-\gamma} = 0$$

then the general solution for α^* is equal to

$$\alpha^* = a(\eta + \frac{w_0}{\gamma}) = aT(w_0) \tag{9}$$

using (3). So we see that the **amount** invested in the risky asset is directly proportional to the degree of absolute risk tolerance. The **share** of total wealth invested in the risky asset is then inversely proportional to the degree of relative risk aversion. This is qualititatively similar to the result for CARA-preferences.

Next we consider the case of a vector of risky assets. We will restrict ourselves to CARA-preferences. To motivate the econometric model to be used in the sequel, we provide the derivation of the optimal portfolio for this case. Let μ be the $(k-1)$-vector of mean excess returns and Σ the variance covariance matrix of the excess returns. Let W be begin of period wealth, r is the riskfree interest rate. The $(k-1)$-vector α denotes the quantities invested in the risky assets, with stochastic returns given by the vector $\tilde{x}_0$. Let ι be a $(k-1)$-vector of ones. Then $\iota'\alpha$ is the amount of money invested in the risky assets and $W - \iota'\alpha$ is the amount invested in the riskfree asset (No non-negativity restrictions are imposed). Consumption z is equal to the value of the assets at the end of the period. Thus consumption is:

$$z = (W - \iota'\alpha)(1+r) + \alpha'(\iota + \tilde{x}_0) = W(1+r) + \alpha'(\tilde{x}_0 - r) \equiv w_0 + \alpha'\tilde{x} \tag{10}$$

where $w_0 = W(1+r)$ and $\tilde{x} = \tilde{x}_o - r.\iota$ .We assume $\tilde{x}$ to be normally distributed, so that $\tilde{x} \sim N(\mu, \Sigma)$. The consumer wants to maximize the expectation of end

of period utility subject to (10) by choosing α optimally. Inserting (10) in (7), neglecting the multiplicative constant A, and taking expectations yields

$$\begin{aligned} V(\alpha) &= -(2\pi)^{-n/2}\,|\Sigma|^{-1/2}\int \exp(-A(w_0+\alpha' x))\exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu))dx \\ &= \exp(-Aw_0 - A\alpha'\mu + \frac{1}{2}A^2\alpha'\Sigma\alpha).(2\pi)^{-n/2}\,|\Sigma|^{-1/2} \\ &\quad .\int \exp[-\frac{1}{2}(x-\mu+A\Sigma\alpha)'\Sigma^{-1}(x-\mu+A\Sigma\alpha)]dx \\ &= -\exp(-Aw_0 - A\alpha'\mu + \frac{1}{2}A^2\alpha'\Sigma\alpha) \end{aligned} \tag{11}$$

Maximizing (11) with respect to α yields:

$$\alpha^* = \frac{1}{A}\Sigma^{-1}\mu \tag{12}$$

We can also write this in terms of portfolio shares. In that case (8) generalizes to:

$$w = \Sigma^{-1}\mu.\frac{1}{R(w_0)} \tag{13}$$

4 An econometric model of portfolio choice

Our interest will be in ownership and portfolio shares of a number of asset categories that vary in riskiness. We want to allow for other factors determining portfolio composition than just the distribution of excess returns. To introduce these other factors in a utility consistent way, we replace (11) by

$$V^*(\alpha) = -\exp(-Aw_0 - A\alpha'\mu - A^2 w_0\alpha'\Sigma z + \frac{1}{2}A^2\alpha'\Sigma\alpha) \tag{14}$$

where z is a vector of taste shifters:

$$z = \Lambda x + \varepsilon \tag{15}$$

where x is a vector of individual (or household) characteristics, Λ is a parameter matrix, and ϵ an i.i.d. error term. We will interpret ϵ as representing unobservable variations in taste across individuals.

Maximizing (14) with respect to the quantity vector α yields the following expression for the vector of risky asset shares:

$$\widetilde{w} = z + \frac{1}{R}\Sigma^{-1}\mu \equiv z + \Gamma\mu^* \tag{16}$$

where $\Gamma = \Sigma^{-1}$ and $\mu^* = \frac{1}{R}\mu$.

Notice that no sign restrictions are imposed on the elements of $\widetilde{w}$. If we impose the condition that assets have to be non-negative -the empirically relevant case- the maximization of (14) has to take place subject to the condition $\alpha \geq 0$. Given that Γ is positive definite, necessary and sufficient conditions for a maximum are then:

$$\begin{aligned} \widetilde{\lambda} &\geq 0 \\ w &\geq 0 \\ \widetilde{\lambda}' w &= 0 \\ w &= z + \frac{1}{R}\Sigma^{-1}(\mu + \widetilde{\lambda}) = z + \Gamma(\mu^* + \lambda) \end{aligned} \tag{17}$$

where $\lambda = \widetilde{\lambda}/R$, and $\widetilde{\lambda}$ is a vector of Lagrange multipliers. The share of the riskless asset in the portfolio is equal to $1 - \iota'_{k-1} w$. Since the share of the riskless asset follows directly from the shares of the risky assets through adding up, we restrict our attention to the shares of the risky assets.

To characterize the Kuhn-Tucker coditions (17) it is convenient to define "virtual prices" $\widehat{\mu} \equiv \mu^* + \lambda$. It follows from the Kuhn-Tucker conditions that the virtual prices are equal to the corresponding elements of μ^* if the corresponding budget share is not equal to zero. To calculate virtual prices for the assets whose share equals zero, we introduce some notation. Let S' (k_1 x $(k-1)$) and D' (k_2 x $(k-1)$) be selection matrices with $k_1 + k_2 = k - 1$, i.e. $\begin{bmatrix} S' \\ D' \end{bmatrix}$ is a permutation of I_{k-1}, the $(k-1)$ x $(k-1)$ identity matrix. The matrix S' selects the elements of w which are zero and D' selects the elements of w which are non-zero. Some useful properties of S and D are:

$$S'S = I_{k_1} \quad D'D = I_{k_2} \quad SS' + DD' = I_{k-1} \quad D'S = 0 \tag{18}$$

Given that S' selects the elements of w that are zero, there holds $S'w = 0$, and similarly $D'\lambda = 0$.

The share equations in (17) can then be written as

$$\begin{bmatrix} 0 \\ D'w \end{bmatrix} = \begin{bmatrix} S' \\ D' \end{bmatrix} z + \begin{bmatrix} S' \\ D' \end{bmatrix} \Gamma [SS' + DD'] \widehat{\mu} \tag{19}$$

The top-half of (19) gives as a solution for the virtual prices:

$$S'\widehat{\mu} = -(S'\Gamma S)^{-1}\{S'z + (S'\Gamma D)D'\mu^*\} \tag{20}$$

using the fact that $D'\widehat{\mu} = D'\mu^*$. Substituting this in the bottom half of (19) yields for the portfolio shares of the non-zero assets:

$$\begin{aligned} D'w &= D'z + \Pi S'z + (D'\Gamma D)D'\mu^* + \Pi(S'\Gamma D)D'\mu^* \\ &\equiv D'z + \Pi S'z + \Psi D'\mu^* \end{aligned} \tag{21}$$

where $\Pi \equiv -D'\Gamma S(S'\Gamma S)^{-1}$, which is a $(k_2 \mathrm{x} k_1)$-matrix. The $(k_2 \mathrm{x} k_2)$ matrix Ψ is defined as $\Psi \equiv (D'\Gamma D) + \Pi(S'\Gamma D)$.

For later purposes, it is useful to rewrite this equation somewhat. Recall the definition of $\widetilde{w}$ (cf. (16)). We will sometimes refer to $\widetilde{w}$ as "latent" portfolio shares, to indicate that they are generally not all observed. Instead w is observed. Using the fact that

$$\begin{aligned} \Psi D' - D'\Gamma &= D'\Gamma DD' - D'\Gamma S(S'\Gamma S)^{-1}S'\Gamma DD' - D'\Gamma \\ &= D'\Gamma(I_{k-1} - SS' - S(S'\Gamma S)^{-1}S'\Gamma DD' - I_{k-1}) \\ &= -D'\Gamma\left\{S(S'\Gamma S)^{-1}S'\Gamma SS' + S(S'\Gamma S)^{-1}S'\Gamma(I_{k-1} - SS')\right\} \\ &= -D'\Gamma S(S'\Gamma S)^{-1}S'\Gamma = \Pi S'\Gamma \end{aligned} \tag{22}$$

we can then write (21) as

$$\begin{aligned} D'w - D'\widetilde{w} &= D'z + \Pi S'z + \Psi D'\mu^* - D'z - D'\Gamma\mu^* \\ &= \Pi S'z + (\Psi D' - D'\Gamma)\mu^* \\ &= \Pi S'z + \Pi S'\Gamma\mu^* = \Pi S'\widetilde{w} \end{aligned} \tag{23}$$

So we observe that the non-zero portfolio shares are equal to their latent counterparts plus a linear combination of the latent budget shares corresponding to the zero assets. Defining $\Delta' \equiv D' + \Pi S'$, this can also be written as $D'w = \Delta'\widetilde{w}$.

Also note that the non-zero Lagrange multipliers are found as

$$\begin{aligned} S'\lambda &= S'\widehat{\mu} - S'\mu^* = -(S'\Gamma S)^{-1}\{S'z + (S'\Gamma D)D'\widehat{\mu}\} - S'\mu^* \\ &= -(S'\Gamma S)^{-1}\{S'z + S'\Gamma DD'\mu^* + S'\Gamma SS'\mu^*\} \\ &= -(S'\Gamma S)^{-1}\{S'z + S'\Gamma[DD' + SS']\mu^*\} \\ &= -(S'\Gamma S)^{-1}S'\widetilde{w} \end{aligned} \tag{24}$$

The econometric model of portfolio shares can now be written as follows:

$$\begin{aligned}\widetilde{w}_i &= z_i + \Gamma\mu_i^* = \overline{z}_i + \Gamma\mu_i^* + \epsilon_i \\ &\left\{ \begin{array}{c} D_i' w_i = \Delta_i' \widetilde{w}_i \\ S_i'\lambda_i = -(S_i'\Gamma S_i)^{-1} S_i'\widetilde{w}_i \end{array} \right\} \quad \textbf{iff} \quad \left\{ \begin{array}{c} \Delta_i'\widetilde{w}_i \geq 0 \quad \textbf{and} \\ (S_i'\Gamma S_i)^{-1} S_i'\widetilde{w}_i \leq 0 \end{array} \right\} \end{aligned} \tag{25}$$

where a subscript i has been added to index observations and $\overline{z}_i$ is the systematic part of z_i, i.e. $z_i = \overline{z}_i + \epsilon_i$. The selection matrices D_i' and S_i' vary by observation. The Kuhn-Tucker conditions guarantee that for each realization of the latent shares $\widetilde{w}_i$ there is only one unique combination of D_i' and S_i' such that the inequality conditions (25) are satisfied.

4.1 Identification

Using (25) we observe that the vectors z are identified up to a scaling constant from the simple probit equations explaining ownership patterns. Furthermore, we note that the elements of μ^* vary proportionately, so that *given* z we obtain $k-1$ pieces of information on Γ from the probits based on (25). To fully identify all parameters we need to consider the equations for the non-zero shares (21). The number of free elements in Π is equal to $(k_2 \mathrm{x} k_1)$. The number of elements in Ψ is equal to $(k_2 \mathrm{x} k_2)$, but since all elements of μ^* are proportional to each other, we can only identify k_2 elements. Thus for a given pattern of non-zero asset shares, and given z, we have $k_2 + k_2.k_1 = k_2(k_1+1) = k_2(k-k_2)$ pieces of information that can be identified from the rationed equations. To determine the total number of pieces of information on Γ that can be identified from the rationing equations, we have to account for all possible patterns of misssing asssets. We find that the number of restrictions imposed on Γ is equal to:

$$R(k) \equiv \sum_{k_2=1}^{k-1} \binom{k-1}{k_2} (k-k_2)k_2 \tag{26}$$

Since Γ is symmetric, the number of free elements in Γ is equal to $k(k-1)/2$. In addition we need $(k-1)$ scaling constants to identify z, but on the other hand the probits provide $k-1$ pieces of information on Γ, so these cancel out. In total we thus need $k(k-1)/2$ pieces of information. Table 8 presents the number of free elements in Γ and the numbef of restrictions $R(k)$ for different values of k. For $k \geq 2$, the parameters in the model are identified, at least by the simple counting rule we have applied here.

Table 8: Number of assets and restrictions on Γ

k	**Free elements of** Γ	R(k)
2	1	1
3	3	6
4	6	24
5	10	80
6	15	240
7	21	672
8	28	1792
9	36	4608
10	45	11520

4.2 The Likelihood

The likelihood is based on (25). We consider two cases. The first case is where all asset shares are non-zero. In this case the observed shares are equal to the latent shares and the likelihood contribution is the joint density of the asset shares as implied by the first equation in (25). The second case is where one or more of the asset shares are zero. For observed values D_i', $D_i'w_i$, and S_i' the likelihood contribution of this observation is:

$$\begin{aligned} g(D_i'w \quad | \quad & \Delta_i'\widetilde{w}_i \geq 0,\ (S_i'\Gamma S_i)^{-1}S_i'\widetilde{w}_i \leq 0).\Pr(\Delta_i'\widetilde{w}_i \geq 0,\ (S_i'\Gamma S_i)^{-1}S_i'\widetilde{w}_i \leq 0) \\ = \ & \Pr(\Delta_i'\widetilde{w}_i \geq 0,\ (S_i'\Gamma S_i)^{-1}S_i'\widetilde{w}_i \leq 0 \mid D_i'w_i).h(D_i'w_i) \\ = \ & \Pr((S_i'\Gamma S_i)^{-1}S_i'\widetilde{w}_i \leq 0 \mid D_i'w).h(D_i'w_i) \end{aligned} \tag{27}$$

using the fact that $D_i'w_i = \Delta_i'\widetilde{w}_i$ and with obvious definitions for the conditional density g and the marginal density h of $D_i'w_i$. To evaluate the likelihood contribution, we need to find the marginal distribution of $D_i'w$ and the conditional distribution of $(S_i'\Gamma S_i)^{-1}S_i'\widetilde{w}_i$ given $D_i'w_i$. We assume normality of the error vector ϵ_i throughout, with $\epsilon_i \sim N(0,\Omega)$. Given this normality assumption these are straightforward exercises.

We first consider the joint distribution of $(S_i'\Gamma S_i)^{-1}S_i'\widetilde{w}_i$ and $D_i'w_i = \Delta_i'\widetilde{w}_i$. We immediately have that $(S_i'\Gamma S_i)^{-1}S_i'w_i$ and $D_i'w_i$ are jointly normal with variance-

covariance matrix equal to

$$\begin{bmatrix} (S_i'\Gamma S_i)^{-1}S_i'\Omega S_i(S_i'\Gamma S_i)^{-1} & (S_i'\Gamma S_i)^{-1}S_i'\Omega\Delta_i \\ \Delta_i'\Omega S_i(S_i'\Gamma S_i)^{-1} & \Delta_i'\Omega\Delta_i \end{bmatrix} \tag{28}$$

The means of the marginal distributions of $(S_i'\Gamma S_i)^{-1}S_i'\widetilde{w}_i$ and $D_i'w_i$ are equal to:

$$\mathbf{E}[(S_i'\Gamma S_i)^{-1}S_i'\widetilde{w}_i] = (S_i'\Gamma S_i)^{-1}S_i'[\overline{z}_i + \Gamma\mu_i^*] \tag{29}$$

and

$$\mathbf{E}[D_i'w_i] = \Delta_i'[\overline{z}_i + \Gamma\mu_i^*] \tag{30}$$

The conditional variance-covariance matrix of $(S_i'\Gamma S_i)^{-1}S_i'w_i$ given $D_i'w_i$ is given by

$$(S_i'\Gamma S_i)^{-1}[S_i'\Omega S_i - S_i'\Omega\Delta_i(\Delta_i'\Omega\Delta_i)^{-1}\Delta_i'\Omega S_i](S_i'\Gamma S_i)^{-1} \tag{31}$$

and the conditional mean of $(S_i'\Gamma S_i)^{-1}S_i'w_i$ given $D_i'w_i$ is given by

$$(S_i'\Gamma S_i)^{-1}S_i'[\overline{z}_i + \Gamma\mu_i^*] + (S_i'\Gamma S_i)^{-1}S_i'\Omega\Delta_i(\Delta_i'\Omega\Delta_i)^{-1}[D_i'w_i - \Delta_i'[\overline{z}_i + \Gamma\mu_i^*]] \tag{32}$$

Appendix A provides the details of the likelihood for the case $k = 3$, which will be considered in our empirical work.

5 Results

We estimate the model for a number of different specifications of the risk aversion measure and for two definitions of wealth. The first definition of total wealth is total assets (cf. Table 7). The second definition is total financial assets.

5.1 Results for shares of gross wealth

We distinguish three asset shares: (1) a riskless asset as defined in Table 7; (2) a risky asset comprising growth and mutual funds, stocks, and options; (3) an "other asset" consisting of bonds, money lent out, business equity, and real estate. Despite the perhaps somewhat confusing terminology, both the "risky asset" and the "other asset" are risky. Thus the three asset shares are shares of total assets (or gross wealth, as we will sometimes call it). Although the theoretical framework presented in Section 3 would suggest to consider shares of net worth, rather than gross wealth, the obvious advantage of using gross wealth

is that we avoid having to deal with shares in negative wealth (about 12% of the sample reports negative net worth). Table 9 presents estimation results for three versions of the model. The first and second version use a combination of risk attitude variables to parameterize risk aversion. For the first version, we specify risk tolerance as

$$\frac{1}{R} = \frac{1}{1 + \exp[\lambda.\text{riskat1} + (1-\lambda).\text{riskat3}]} \tag{33}$$

where the parameter λ can be estimated jointly with the other parameters in the model. For the second version, we use (33) but with riskat1 and riskat3 replaced by careful and precaution. In the third version the term $[\lambda.\text{riskat1}+(1-\lambda).\text{riskat2}]$ is replaced by the variable risk tolerance. The number of observations varies per version, reflecting sample selections and skipping patterns in the questionnaires, as discussed before. In all versions the estimate of γ_{12} (the off-diagonal element of Γ) had to be bounded from below to maintain positive definiteness of Γ. Somewhat arbitrarily we have restricted the quantity $\gamma_{12}/(\gamma_{11}\gamma_{22})$ to be greater than -0.99.

We observe that γ_{11} and γ_{22} (the diagonal elements of Γ) are only jointly significantly different from zero in the first version where we use riskat1 and riskat3 as indicators of risk aversion. The version with the direct risk tolerance measure shows virtually no effect of risk tolerance on portfolio choice. The estimates of the effect of the other explanatory variables are qualitatively similar across the three versions. Income has a positive effect on the portfolio share of the risky asset and a negative effect on the portfolio share of the other asset. The effects of wealth are the opposite of those for income. Age and age squared are always jointly significant, whereas education is never significant. For the version with riskat1 and riskat3, the parameters of the estimated age functions imply that the share of the risky asset will rise monotonically with age, whereas the share of the other asset will fall after age 30. Gender does not exert a statistically significant influence on portfolio choice. The parameter λ is estimated around .5, so that both variables making up the risk tolerance variable are having an almost equal influence.

The parameters of the variance covariance matrix (ω_1, ω_2, ρ) are measured very precisely. Although one might be suspicious of the estimated standard errors, estimation of the model with simulated data (with varying sample sizes) generally produced parameter estimates that were well within two standard errors around the true parameters. Thus, it does not seem likely that the estimated standard

errors are severely biased towards zero[4]

Table 9: Estimation results for the full model (total assets)

	Riskat1/3		**Careful/Prec.**		**Risk tolerance**	
Parameter/variable	**Est.**	**t-val.**	**Est.**	**t-val.**	**Est.**	**t-val.**
Risky Asset						
log-income	.132	3.53	.177	5.26	.096	1.83
gender	.036	1.43	.006	.26	.059	1.07
middle education	.049	1.57	.067	2.30	.050	1.07
higher education	.062	2.07	.077	2.63	.086	1.90
log-wealth	-.177	11.0	-.224	15.7	-.223	10.9
age	-.016	2.41	-.005	.80	-.013	1.25
age squared	.0002	3.46	.0001	1.99	.0002	2.05
constant	1.02	3.11	1.15	3.86	1.978	3.90
Other Asset						
log-income	-.177	4.64	-.214	6.94	-.153	3.33
gender	-.034	1.29	-.023	1.05	-.047	.98
middle education	-.035	1.09	-.034	1.30	.030	.76
higher education	-.058	1.83	.069	2.54	-.070	1.79
log-wealth	.324	23.4	.339	26.9	.329	18.7
age	.009	1.26	.002	.43	.016	1.74
age squared	-.0001	2.30	-.0001	1.70	-.0002	2.42
constant	1.65	5.00	-1.41	5.29	-2.13	4.87
ω_1	.246	20.7	.311	24.9	.280	16.2
ω_2	.318	28.2	.350	29.3	.302	19.4
ρ	-.859	47.4	-.912	99.5	-.904	57.3
γ_{11}	2.59	1.78	3.62	1.36	.098	.27
$\frac{\gamma_{12}}{\sqrt{\gamma_{11}\gamma_{22}}}$	-.99	-	-.99	-	-.99	-
γ_{22}	.319	1.33	.398	1.16	.007	.26
λ	.527	6.71	.607	1.84	-	-
Elements of Γzero?	$\chi^2(2) = 42.2$		$\chi^2(2) = 2.71$		$\chi^2(2) = .07$	
Number of obs.	762		1324		516	
log-likelihood	-247.5		-574.1		-170.8	

The variables riskat1 and riskat3 are linear combinations of the underlying

[4]Of course, this is all predicated on the assumption that the model specification is correct.

responses to the subjective questions listed in Section 2.2 above. In principle therefore, one can also use these responses directly in the definition of the risk tolerance measure analogous to (33). Table 10 provides the estimates of the weights of each of these variables in the risk tolerance measure (see the columns with heading "All included"). The other parameters of this version of the model have been suppressed for reasons of space. They are similar to the results reported in Table 9 for riskat1 and riskat3 (except for the elements of Γ; see below). The likelihood of the model with separate parameters for each of the risk attitude responses is substantially higher than in the model where these measures are combined via principal components (log-likelihood in Table 9 is equal to -247.5, whereas in Table 10 the log-likelihood is equal to -236.8). We observe from Table 10 that the subjective variables are dominated by "spaar2". Recall that this is the response to the question "Investing in stocks is something I don't do, since it is too risky". The fact that the word "stocks" is mentioned explicitly in this question may explain why this variable dominates all others in explaining portfolio choice. Indeed we observe that a joint test of significance of the parameters of the other risk attititude measures does not reject the null that these parameters are all zero ($\chi^2(7) = 10.57, p = .16$). If we re-estimate the model with spaar2 excluded, (see the columns with heading "Spaar2 excluded") the remaining risk attitude variables turn out to be statistically highly significant ($\chi^2(7) = 77.6, p = .000$). Since these other risk attitude variables are much less directly related to investments in stocks, it appears that they may capture genuine risk preferences and that they have a significant effect on observed portfolio choices. We also observe that the likelihood for this case is still somewhat higher than when we include riskat1 and riskat3 (i.e. the first column in Table 9), which include spaar2 as component.

Finally, we note that by inverting Γ we obtain an estimate of the variance covariance matrix of excess returns Σ, as perceived by households. That is, $\Sigma = \Gamma^{-1} = \begin{bmatrix} 1.62 & \\ -.563 & .198 \end{bmatrix}^{-1} = \begin{bmatrix} 30.6 & \\ .99 & 251.1 \end{bmatrix}$, where the off-diagonal element is the correlation rather than the covariance The elements of Γ are somewhat smaller than reported in Table 9 for the specification with riskat1 and riskat3. Due to the near-singularity of Γ, the off-diagonal elements of Σ^{-1} are very large and very sensitive to the arbitrary lower bound we imposed on the off-diagonal element of Γ. The scale of Σ is arbitrary at this point, as the scale of the risk tolerance measure (33) is arbitrary. To the extent that one accepts the lower bound on the off-diagonal of Γ as reasonable, one observes that the variance

covariance matrix implies that the second risky asset has a higher estimate of perceived variance in returns than the first one. This is a direct consequence of the fact that the share of the risky asset is much more sensitive to the risk attitude of a respondent than the share of the other asset. For this specification, the shares of the assets do not vary significantly with age.

Table 10: The estimates for the separate risk attitude variables

	All included		Spaar2 excluded	
Variable	**Estimate**	**t-value**	**Estimate**	**t-value**
spaar1	-.079	1.26	.483	2.10
spaar2	.696	6.86		
spaar3	-.0004	.01	-.697	3.85
spaar4	.068	.93	-.052	.19
spaar5	.094	1.52	.101	.47
spaar6	.127	1.44	1.55	5.75
spaarm03	-.0004	.01	-.920	3.05
spaarm12	.04	.56	-.015	.05
spaarm13	.053	.49	.553	2.91
Tests of joint significance				
All risk att. vars.	$\chi^2(8) = 429.8$	$p = .000$		
All except spaar2	$\chi^2(7) = 10.57$	$p = .16$	$\chi^2(7) = 77.6$	$p = .000$
log-likelihood	-236.8		-249.1	

5.2 Results for shares of gross financial wealth

The shares in financial wealth are defined as follows: (1) the non-risky asset defined in Table 7; (2) a risky asset comprising growth and mutual funds, stocks, and options; (3) an "other asset" consisting of bonds and money lent out. Tables 11 and 12 present results analogous to Tables 9 and 10, but now for shares of financial assets. In contrast to the case of shares in total assets, the elements of Γ need not be restricted to guarantee positive definiteness. The estimates of Γ are jointly significant for both the version with riskat1 and riskat3 and the version with careful and precaution, but not for the version with the direct risk tolerance measure. In the latter case we moreover had to restrict the off-diagonal element of Γ to avoid indefiniteness. The pattern of the signs of the estimated parameters

is almost identical in Tables 9 and 11. Hence our discussion of Table 9 carries over to Table 11. The estimates of Γ cannot be compared directly across the first two specifications in Table 3, as the scales of the risk attitude measures differ. The joint test of significance of the elements of Γ yields smaller p-values for the second specification than for the first one, but the number of observations is also considerably larger in the second specification.

Regarding Table 12, the variable spaar2 once again dominates, but now even in the version with spaar2 included the other risk attitude variables remain significant. In the version with spaar2 excluded that is even more the case. The estimates of Γ and Σ are now $\Sigma = \Gamma^{-1} = \begin{bmatrix} 1.26 & \\ -.419 & .298 \end{bmatrix}^{-1} = \begin{bmatrix} 1.48 & \\ .682 & 6.27 \end{bmatrix}$ Once again, the estimated perceived variance in returns of the second risky asset is larger than the first one. The numbers in Σ are much more reasonable than for total assets due to the fact that the matrix Γ is much better conditioned.

Table 11: Estimation results for the full model (financial assets)

	Riskat1/3		**Careful/Prec.**		**Risk tolerance**	
Parameter/variable	**Est.**	**t-val.**	**Est.**	**t-val.**	**Est.**	**t-val.**
Risky Asset						
log-income	.189	1.90	.267	2.06	-.021	.08
gender	.030	.43	-.018	.19	-.051	.18
middle education	.046	.54	.035	.30	.113	.44
higher education	.085	1.02	.048	.41	.190	.80
log-wealth	-.030	.56	-.333	5.93	-.451	4.27
age	-.018	.98	-.023	.94	-.011	.22
age squared	.0002	1.36	.0003	1.19	.0001	.25
constant	-1.11	1.12	3.06	2.41	6.97	2.50
Other Asset						
log-income	-.201	2.28	-.266	2.71	-.090	.45
gender	-.003	.04	.002	.03	.016	.07
middle education	-.025	.33	.047	.52	-.074	.39
higher education	-.090	1.19	-.006	.06	-.138	.78
log-wealth	.255	8.85	.421	10.9	.496	6.27
age	.007	.42	.012	.67	.010	.26
age squared	-.0001	.59	-.0002	.87	-.0001	.21
constant	-1.47	1.90	-3.24	3.55	-5.76	2.80
ω_1	.553	9.95	.934	11.9	1.09	6.84
ω_2	.566	14.5	.860	15.2	.974	7.85
ρ	-.771	11.1	-.955	107.9	-.970	100.3
γ_{11}	.933	6.46	23.8	2.85	1.30	.58
$\frac{\gamma_{12}}{\sqrt{\gamma_{11}\gamma_{22}}}$	-0.125	.36	-.916	3.77	-.99	-
γ_{22}	.011	.21	6.86	3.06	.437	.59
λ	.536	7.92	.380	1.81		
Elements of Γ zero?	$\chi^2(3) = 91.43$		$\chi^2(3) = 180.8$		$\chi^2(3) = .36$	
Number of obs.	755		1310		512	
log-likelihood	-629.4		-1071.2		-403.0	

Table 12: The estimates for the separate risk attitude variables (financial assets)

	All included		Spaar2 excluded	
Variable	**Estimate**	**t-value**	**Estimate**	**t-value**
spaar1	-.077	1.21	.681	2.24
spaar2	.668	6.43		
spaar3	.038	.77	-.445	1.80
spaar4	.077	1.09	-.263	.83
spaar5	.103	1.69	-.012	.05
spaar6	.180	1.98	1.76	7.06
spaarm03	-.067	.79	-1.29	3.92
spaarm12	.055	.67	-.134	.40
spaarm13	.022	.20	.701	3.22
Tests of joint significance				
All risk att. vars.	$\chi^2(8) = 416.2$	$p = .000$		
All except spaar2	$\chi^2(7) = 21.5$	$p = .003$	$\chi^2(7) = 73.3$	$p = .000$
log-likelihood	-621.5		-629.22	

6 Concluding remarks

We have explored the explanatory power of a number of different subjective measures of risk aversion for the explanation of portfolio choice. The variable risk tolerance, which has the firmest grounding in economic theory, appears to have very little explanatory power. There are a few different possible explanations for this. First of all, the question is quite complicated and many respondents may have a hard time understanding the exact meaning of the question. Secondly, the question conditions on a respondent's current situation. So for instance a risk tolerant individual with a risky portfolio may be induced to choose a safe income stream, since she is already exposed to considerable risk. Conversely, a risk averse individual with a very safe portfolio can afford to choose a riskier income path. In both cases the observed relationship between the measured risk tolerance and portfolio choice is attenuated.

The variables riskat1 and riskat3 extracted from a factor analysis of ad hoc measures appear to be doing considerably better in terms of explanatory power, both for total assets and for financial assets. Yet, closer analysis reveals that

most of the explanatory power of these variables comes from one question, which asks directly for the subjective evaluation of the riskiness of investing in stocks. Omitting this variable shows that both in a model with all assets and in a model with just financial assets the ad hoc measures still retain a highly signficant influence on portfolio allocation. Interestingly, in the model for financial assets the measures careful and precaution also contribute significantly to the explanation of portfolio shares. This reinforces the notion that intuitive ad hoc measures may be more powerful in explaining portfolio choice than theory based, but complicated, risk tolerance measures.

Thus, the paper provides evidence that individually measured risk preference variables help explain portfolio allocation in line with economic theory. Perhaps surprising to economists, simple intuitive measures of risk preferences appear to be more powerful predictors of portfolio allocation than more sophisticated measures with a firmer basis in economic theory. Yet, for empirical analysis this may be good news. The ad hoc questions are much easier to ask and demand considerably less imagination of respondents, which may be exactly why the simple measures work better.

The modelling of incomplete portfolios by explicit use of Kuhn-Tucker conditions is fairly straightforward and estimation of the resulting model yields well determined estimates of a number of key parameters. So far we have restricted ourselves to just three broad group of assets. The set-up can be easily generalized to more than three assets, although the computational cost will undoubtedly increase.

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A The likelihood for $k = 3$

For $k = 3$ there are four possible ownership patterns for the two risky assets: (1) $w_1 = 0$, $w_2 = 0$; (2) $w_1 \neq 0$, $w_2 = 0$; (3) $w_1 = 0$, $w_2 \neq 0$; (4) $w_1 \neq 0$, $w_2 \neq 0$. We will discuss these consecutively. For later use we introduce some scalar notation:

$$\begin{aligned} \Gamma &\equiv \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \quad \Omega \equiv \begin{bmatrix} \omega_1^2 & \rho\omega_1\omega_2 \\ \rho\omega_1\omega_2 & \omega_2^2 \end{bmatrix} \quad w_i \equiv \begin{pmatrix} w_{i1} \\ w_{i2} \end{pmatrix} \\ \overline{z}_i &\equiv \begin{pmatrix} \overline{z}_{i1} \\ \overline{z}_{i2} \end{pmatrix}, \quad \mu_i^* \equiv \begin{pmatrix} \mu_{i1}^* \\ \mu_{i2}^* \end{pmatrix} \end{aligned} \tag{34}$$

For the various normal distributions and densities the following notation is adopted. $B\Phi\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; v, \Sigma\right]$ represents the joint probability that two normally distributed random variables, with mean vector v and variance-covariance matrix Σ, are less than or equal to x_1 and x_2, respectively. $B\phi\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; v, \Sigma\right]$ is the value of the corresponding density at the point $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. $\Phi[x; \mu, \sigma]$ is the probability that a normally distributed variable, with mean μ and variance σ^2, is less than or equal to x. $\phi[x; \mu, \sigma]$ is the value of the corresponding density at the point x.

A.1 $w_1 = 0$, $w_2 = 0$

For this case the likelihood contribution is the bivariate probability that both Lagrange multipliers are non-negative, or in other words that $\Gamma^{-1}\widetilde{w}_i \leq 0$. Thus the likelihood contribution for this case is:

$$\mathcal{L}_{1i} = B\Phi\left[(0); \left(\Gamma^{-1}[\overline{z}_i + \Gamma\mu_i^*]\right), \Gamma^{-1}\Omega(\Gamma^{-1})'\right] \tag{35}$$

The matrix Γ is assumed symmetric, so the transposition sign is superfluous, strictly speaking. Define $D \equiv |\Gamma| = \gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}$. Then we can write

$$\Gamma^{-1}[\bar{z}_i + \Gamma\mu_i^*] = \frac{1}{D}\begin{bmatrix} \gamma_{22}\bar{z}_1 - \gamma_{12}\bar{z}_1 + \mu_1^* \\ \gamma_{11}\bar{z}_2 - \gamma_{21}\bar{z}_1 + \mu_2^* \end{bmatrix} \tag{36}$$

$$\Gamma^{-1}\Omega(\Gamma^{-1})' \equiv \frac{1}{D^2}\begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12} & \Theta_{22} \end{pmatrix} \tag{37}$$

with

$$\begin{aligned} \Theta_{11} &= (\gamma_{22})^2\omega_1^2 - 2\gamma_{12}\gamma_{22}\rho\omega_1\omega_2 + (\gamma_{12})^2\omega_2^2 \\ \Theta_{12} &= -\gamma_{22}\gamma_{21}\omega_1^2 + (\gamma_{11}\gamma_{22} + \gamma_{12}\gamma_{21})\rho\omega_1\omega_2 - \gamma_{11}\gamma_{12}\omega_2^2 \\ \Theta_{22} &= (\gamma_{11})^2\omega_2^2 - 2\gamma_{11}\gamma_{21}\rho\omega_1\omega_2 + (\gamma_{21})^2\omega_1^2 \end{aligned} \tag{38}$$

A.2 $w_1 \neq 0$, $w_2 = 0$

For this case we have $S_i' = (\ 0\ \ 1\ )$, $D_i' = (\ 1\ \ 0\ )$, $S_i'\Gamma S_i = \gamma_{22}$. Thus, $(S_i'\Gamma S_i)^{-1}w_i = \frac{w_{i2}}{\gamma_{22}}$. Hence, $(S_i'\Gamma S_i)^{-1}w_i \leq 0$ is equivalent with $w_{i2} \leq 0$. Furthermore we have $\Delta_i' = D_i'[I - \Gamma S_i(S_i'\Gamma S_i)^{-1}S_i'] = (\ 1\ \ -\frac{\gamma_{12}}{\gamma_{22}}\ )$, $S_i'\Omega\Delta_i = -\frac{\gamma_{12}}{\gamma_{22}}..\omega_2^2 + \rho\omega_1\omega_2$, and $\Delta_i'\Omega\Delta_i = \omega_1^2 - 2\rho\frac{\gamma_{12}}{\gamma_{22}}\omega_1\omega_2 + \left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\omega_2^2$. Hence, the marginal density of w_{i1} is normal with variance

$$\sigma_1^2 \equiv \Delta_i'\Omega\Delta_i = \omega_1^2 - 2\rho\frac{\gamma_{12}}{\gamma_{22}}\omega_1\omega_2 + \left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\omega_2^2 \tag{39}$$

and mean

$$\chi_{i1} \equiv \Delta_i'\bar{z}_i + \Delta_i'\Gamma\mu_i^* = -\frac{\gamma_{12}}{\gamma_{22}}\bar{z}_{i2} + \bar{z}_{i1} + (\gamma_{11} - \frac{\gamma_{21}\gamma_{12}}{\gamma_{22}})\mu_{i1}^* \tag{40}$$

The conditional variance of the latent budget share $\tilde{w}_{i2}$ given w_{i1} (cf. (31), but without pre- and postmultiplication by $(S_i'\Gamma S_i)^{-1}$) becomes

$$\eta_2^2 \equiv S_i'\Omega S_i - S_i'\Omega\Delta_i(\Delta_i'\Omega\Delta_i)^{-1}\Delta_i'\Omega S_i = \omega_2^2 - \frac{\left(-\frac{\gamma_{12}}{\gamma_{22}}..\omega_2^2 + \rho\omega_1\omega_2\right)^2}{\omega_1^2 - 2\rho\frac{\gamma_{12}}{\gamma_{22}}\omega_1\omega_2 + \left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\omega_2^2} \tag{41}$$

Similarly we have $S_i'\Omega\Delta_i(\Delta_i'\Omega\Delta_i)^{-1} = \frac{-\frac{\gamma_{12}}{\gamma_{22}}.\omega_2^2 + \rho\omega_1\omega_2}{\omega_1^2 - 2\rho\frac{\gamma_{12}}{\gamma_{22}}\omega_1\omega_2 + \left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\omega_2^2}$. Thus it follows from (32) (omitting premultiplication by $(S_i'\Gamma S_i)^{-1}$) that the conditional mean of w_{i2}

given w_{i1} is given by

$$\begin{aligned}\nu_{i2} &\equiv S_i'\bar{z}_i + S_i'\Gamma\mu_i^* + S_i'\Omega\Delta_i(\Delta_i'\Omega\Delta_i)^{-1}[D_i'w_i - \Delta_i'\bar{z}_i - \Psi_i S_i'\mu_i^*] \\ &= \bar{z}_{i2} + \gamma_{21}\mu_{i1}^* + \gamma_{22}\mu_{i2}^* + \\ &\quad \frac{-\frac{\gamma_{12}}{\gamma_{22}}.\omega_2^2 + \rho\omega_1\omega_2}{\omega_1^2 - 2\rho\frac{\gamma_{12}}{\gamma_{22}}\omega_1\omega_2 + \left(\frac{\gamma_{12}}{\gamma_{22}}\right)^2\omega_2^2}\{w_{i1} - \chi_{i1}\} \end{aligned} \tag{42}$$

For the likelihood contribution of this case we obtain:

$$\mathcal{L}_{2i} \equiv \Phi[0; \nu_{i2}, \eta_2].\varphi[w_{i1}; \chi_{i1}, \sigma_1] \tag{43}$$

A.3 $w_1 = 0$, $w_2 \neq 0$

For this case we have $S_i' = (\ 1\ \ 0)$, $D_i' = (\ 0\ \ 1)$. Similar to previous case, the condition that the Lagrange multiplier for the binding constraint is non-negative is equivalent with $w_{i1} \leq 0$. Furthermore, $\Delta_i' = D_i'[I - \Gamma S_i(S_i'\Gamma S_i)^{-1}S_i'] = (\ -\frac{\gamma_{21}}{\gamma_{11}}\ \ 1\ )$, $S_i'\Omega\Delta_i = -\frac{\gamma_{21}}{\gamma_{11}}..\omega_1^2 + \rho\omega_1\omega_2$, $\Delta_i'\Omega\Delta_i = \omega_2^2 - 2\rho\frac{\gamma_{21}}{\gamma_{11}}\omega_1\omega_2 + \left(\frac{\gamma_{21}}{\gamma_{11}}\right)^2\omega_1^2$. Hence the marginal density of w_{i2} is normal with variance

$$\sigma_2^2 \equiv \Delta_i'\Omega\Delta_i = \omega_2^2 - 2\rho\frac{\gamma_{21}}{\gamma_{11}}\omega_1\omega_2 + \left(\frac{\gamma_{21}}{\gamma_{11}}\right)^2\omega_1^2 \tag{44}$$

and mean

$$\chi_{i2} \equiv \Delta_i'\bar{z}_i + \Delta_i'\Gamma\mu_i^* = -\frac{\gamma_{21}}{\gamma_{11}}\bar{z}_{i1} + \bar{z}_{i2} + (\gamma_{22} - \frac{\gamma_{21}\gamma_{12}}{\gamma_{11}})\mu_{i1}^* \tag{45}$$

.The conditional variance of latent budget share w_{i1} given w_{i2} as given in (31) (but without pre- and postmultiplication by $(S_i'\Gamma S_i)^{-1}$) becomes

$$\eta_1^2 \equiv S_i'\Omega S_i - S_i'\Omega\Delta_i(\Delta_i'\Omega\Delta_i)^{-1}\Delta_i'\Omega S_i = \omega_1^2 - \frac{\left(-\frac{\gamma_{21}}{\gamma_{11}}..\omega_1^2 + \rho\omega_1\omega_2\right)^2}{\omega_2^2 - 2\rho\frac{\gamma_{21}}{\gamma_{11}}\omega_1\omega_2 + \left(\frac{\gamma_{21}}{\gamma_{11}}\right)^2\omega_1^2} \tag{46}$$

Similarly we have $S_i'\Omega\Delta_i(\Delta_i'\Omega\Delta_i)^{-1} = \frac{-\frac{\gamma_{21}}{\gamma_{11}}.\omega_1^2 + \rho\omega_1\omega_2}{\omega_2^2 - 2\rho\frac{\gamma_{21}}{\gamma_{11}}\omega_1\omega_2 + \left(\frac{\gamma_{21}}{\gamma_{11}}\right)^2\omega_1^2}$. Thus it follows from (32) (omitting premultiplication by $(S_i'\Gamma S_i)^{-1}$) that the conditional mean of w_{i1} given w_{i2} is given by

$$\begin{aligned}\nu_{i1} &\equiv S_i'\bar{z}_i + S_i'\Gamma\mu_i^* + S_i'\Omega\Delta_i(\Delta_i'\Omega\Delta_i)^{-1}[D_i'w_i - \chi_{i2}] \\ &= \bar{z}_{i1} + \gamma_{11}\mu_{i1}^* + \gamma_{12}\mu_{i2}^* + \\ &\quad \frac{-\frac{\gamma_{21}}{\gamma_{11}}.\omega_1^2 + \rho\omega_1\omega_2}{\omega_2^2 - 2\rho\frac{\gamma_{21}}{\gamma_{11}}\omega_1\omega_2 + \left(\frac{\gamma_{21}}{\gamma_{11}}\right)^2\omega_1^2}\{w_{i2} - \chi_{i2}\} \end{aligned} \tag{47}$$

For the likelihood contribution of this case we obtain:

$$\mathcal{L}_{3i} \equiv \Phi[0; \nu_{i1}, \eta_1].\varphi[w_{i2}; \chi_{i2}, \sigma_2] \tag{48}$$

A.4 $w_1 \neq 0$, $w_2 \neq 0$

This case is straightforward. The likelihood contribution is simply the bivariate density of w_1 and w_2 as generated by (25). Thus the likelihood contribution for this case is:

$$\mathcal{L}_{4i} = B\phi \left[\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}; \begin{pmatrix} \overline{z}_{i1} + \gamma_{11}\mu^*_{i1} + \gamma_{12}\mu^*_{i2} \\ \overline{z}_{i2} + \gamma_{21}\mu^*_{i1} + \gamma_{22}\mu^*_{i2} \end{pmatrix}, \Omega \right] \tag{49}$$