Chapter One

INTRODUCTION

The United States needs to substantially improve the teaching and learning of mathematics in American schools. A growing number of Americans believe not only that the future well-being of our nation depends on a mathematically literate population but also that most adults are weak in mathematics, with some groups disproportionately worse off. The basic level of mathematical proficiency needs to be raised substantially, and the gaps in proficiency across societal groups need to be eliminated.

Despite years spent in mathematics classes learning about fractions, decimals, and percents, many well-educated adults, for example, would respond incorrectly to the following question:

> If the average salaries of a particular group within a population are 16 percent less than the average salary of the entire population, and one wants to give the individuals in that group a raise to bring them up to parity, what should the raise be—16 percent, something more, or something less?[1]

Although they may have been taught the relevant calculation skills, what most American adults remember from school mathematics are rules that are not grounded in understanding. Many adults would be unable to answer this problem correctly or even to attempt to reason through it. Proficiency in formulating and solving even relatively simple percent problems is not widespread.

In the past, mathematical proficiency was regarded as being important primarily for those headed for scientific or mathematical professions. But times have

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[1]Although 16 percent may seem to be the obvious answer, it is not correct. For example, if the average salary is $40,000, then the salaries of the underpaid group are 16 percent of $40,000 ($6,400) less than the average salary—i.e., $33,600. If one had assumed that 16 percent of the lower salary was the required raise, the raise would have amounted to only $5,376 ($33,600 *x* 0.16)—clearly, not enough to make up the $6,400 difference between the higher and lower salaries. Instead, one needs to determine what percentage of $33,600 equals $6,400. A simple calculation ($40,000 divided by $33,600) reveals that $40,000 is approximately 19 percent more than $33,600, so the raise required to bring the lower salaries up to par with the higher ones would be a bit more than 19 percent.

changed. Today, broad agreement exists that mathematical proficiency on a wide scale matters. That few people might be able to solve problems like the one on the previous page is troubling because American adults will require substantial mathematical proficiency to participate fully and productively in society and the economy of the 21st century.

While the mathematics performance of the U.S. population has never been seen as satisfactory, today dissatisfaction with that performance has become intense, and it is growing. Over the past decade or so, we have witnessed a movement to raise educational standards, and we have seen persistent efforts to increase educators' accountability for achievement. The recent legislation entitled "No Child Left Behind" has committed the nation to ensuring that all children meet high standards of mathematical proficiency.[2] The consequences will be enormous if states, school districts, and schools fail to make rapid and continuous progress toward meeting those standards over the next decade.

The American educational system has always been able to develop mathematical proficiency in a small fraction of the population, but current policies create goals and expectations that present a dramatic new challenge: Every student now needs competency in mathematics. This goal of achieving mathematical proficiency for all students is unprecedented, and it places vastly more ambitious performance demands on all aspects of the educational system.

In this report, commissioned by the Office of Education Research and Improvement (now the Institute for Educational Sciences [IES]), we argue that a focused, strategic program of research and development in mathematics education can make a meaningful and essential contribution to achieving America's goals for school mathematics.[3]

GOALS AND EXPECTATIONS

The aims of teaching mathematics in school are rooted in the basic justifications for public education. One element is social: A responsible and informed citizenship in a modern economic democracy depends on quantitative understanding and the ability to reason mathematically. Such knowledge is important in making judgments on public issues and policies of a technical nature. A second element is personal: Mathematics extends the options available in one's

[2] *No Child Left Behind Act of 2001,* 2002.

[3] This report was written before the reauthorization of the research program of the U.S. Department of Education. That reauthorization created an IES within the department, replacing the Office of Educational Research and Improvement (OERI). We retained the designation OERI throughout this text. The features of the legislation authorizing the IES do not conflict with the proposals made here.

career as well as in one's daily life. People's opportunities and choices are shaped by whether they know and are able to use mathematics. A third element is cultural: Mathematics constitutes one of humanity's most ancient and noble intellectual traditions. It is an enabling discipline for all of science and technology, providing powerful tools for analytical thought and the concepts and language for creating precise quantitative descriptions of the world. Even the most elementary mathematics involves knowledge and reasoning of extraordinary subtlety and beauty.

Economic considerations are also relevant to the goals of school mathematics: In today's economy, with its emphasis on high technology, most jobs that support a decent standard of living demand strong and flexible quantitative skills. As workplaces evolve, the mathematical ideas that students need on the job will change, and people must be prepared to learn, analyze, and use mathematical ideas they have never encountered in school or used before.

CHALLENGES AND CONFLICTS

Current goals for mathematics proficiency and the accompanying higher expectations that go with them have complicated the task of improving school mathematics. We no longer assume that facility in paper-and-pencil arithmetic is the only mathematics that most adults will ever need. Other domains of mathematics—algebra, in particular—have become increasingly essential to educational advancement and career opportunities. We also do not assume that students can become proficient in mathematics only if they enter school equipped with some special innate abilities and predisposition for math proficiency. Students can be taught strategies and techniques to compensate for their limited experiences outside of school and their inadequate preparation in mathematics. Although comparisons with the mathematics performance of students in other countries demonstrate that U.S. students' performance is inadequate, those comparisons also suggest that this performance could be much greater if we made specific improvements in our curriculum, teaching, and assessment practices.

Further complicating the process of improving school mathematics are disputes about what content should be taught and how it should be taught.[4] Some observers argue that mathematics should be taught primarily by teachers providing clear, organized expositions of concepts and procedures and then giving students opportunities to practice those procedures and apply those concepts. Others contend that teachers should design ways to engage students firsthand in exploring the meaning of mathematical procedures, rather than simply

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[4]See, for example, Loveless, 2001.

showing them how to carry them out. Yet others want students to memorize procedures and develop skills so that understanding can follow from those activities. And others want to put understanding first and foremost, contending that in the computer age, a heavy emphasis on procedural skill is no longer relevant. Arguments also rage over the nature of school mathematics: Should it be mostly abstract and formal or mostly concrete and practical? With these basic issues in play, battles have been waged over curriculum materials. The intense debates that filled the past decade have often impeded much-needed collective work on improvement. Moreover, they have been based more often on ideology than on evidence.

Amid this conflict, U.S. schools are expected to provide more and better opportunities for students to learn mathematics and to do so despite chronic shortages of resources. Most school districts lack a cadre of qualified, mathematically proficient teachers,[5] and it is not clear whether widely used curriculum programs and assessment instruments are adequate for the task of helping schools meet new and more demanding instructional goals. Schools are seeking information on the effectiveness of curriculum materials and instructional practices to help them make better decisions about how to support all children's learning of mathematics. More and more, the public is insisting that the choices the schools make be "research based."

RESEARCH KNOWLEDGE NEEDED TO MEET CURRENT NEEDS

Tackling the problems of school mathematics obviously depends on much more than research, but research is necessary if human energies and other resources are to be invested wisely. Future investments require knowledge about problems of instructional practice and about ways to address those problems. Where such knowledge exists and has been appropriately used, it has paid off.

Examples of research that has made a difference in school mathematics practices include studies of how teachers can use knowledge of students' arithmetic strategies to develop their problem-solving and computational skills,[6] studies of characteristics of professional development that enhance teachers' instruction and their students' learning,[7] and studies of how to improve mathematics instruction in urban schools.[8] But research-based knowledge about mathematics education has often been of little use to teachers. It often does not address

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[5]National Commission on Mathematics and Science Teaching for the 21st Century, 2000.

[6]Carpenter et al., 1989; Carpenter, Fennema & Franke, 1996; Cobb, Wood & Yackel, 1991; Hiebert et al., 1997; Kilpatrick, Swafford & Findell, 2001.

[7]Borko & Putnam, 1996; Cohen & Hill, 2000; Saxe, Gearhart & Seltzer, 1999.

[8]Garet et al., 2001; Silver & Lane, 1995; Silver & Stein, 1996.

problems that concern teachers, for instance, or it is communicated in ways that make it seem esoteric and render its implications unclear or impractical.

Despite more than a century of efforts to improve school mathematics in the United States, efforts that have yielded numerous research studies and development projects, investments in research and development have been inadequate. Federal agencies (primarily the National Science Foundation and the U.S. Department of Education) have contributed funding for many of these efforts. But the investments have been relatively small, and the support has been fragmented and uncoordinated. There has never been a long-range programmatic effort devoted solely to funding research in mathematics education, nor has research (as opposed to development) funding been organized to focus on knowledge that would be usable in practice. Consequently, major gaps exist in the knowledge base and in knowledge-based development.[9]

The absence of cumulative, well-developed knowledge about the practice of teaching mathematics and the limited links between research and practice have been major impediments to creating a system of school mathematics that works. These impediments matter now more than ever. The challenge faced by school mathematics educators in the United States today—to achieve both mathematical proficiency and equity in the attainment of that proficiency—demands the development of new knowledge and practices that are rooted in systematic, coordinated, and cumulative research.

A PROGRAM OF RESEARCH AND DEVELOPMENT IN MATHEMATICS EDUCATION

To build the resources needed to meet the new challenges outlined above, this report maps out a long-term, strategic program of research and development in mathematics education that connects theory and practice. If successful, the program would produce resources to support mathematics teaching and learning in the near term. After 10 to 15 years, it would have built a solid base of knowledge needed for sustained improvement in effective instructional practice. The proposed agenda for the program must take into account the reality that public investments in research are a fraction of those needed given the scale and complexity of the problems. The proportion of the national education budget spent on research and development is far below the levels of research and development spending in most sectors of the economy. Hence, difficult choices and careful designs will be required to gain maximum leverage and cumulative impact from available resources. In every aspect of the research and development program, attention to the dual themes of mathematical profi-

[9]Wilson, Floden & Ferrini-Mundy, 2001.

ciency and equity is vital, a requirement demanding the development and testing of instruments to assess how well various groups of students are progressing on the road toward proficiency.

We view the production of knowledge and the improvement of practice as being a cycle of research, development, improved knowledge and practice, and evaluation, leading in turn to new research, new development, and so on. This implies that problems can be initially addressed and worked on at different points in that cycle, as shown in Figure 1.1.

A coordinated program of research and development should be designed to strengthen relations among these efforts so that investments in one would contribute to the others. The evolving knowledge base builds on what is being tried in practice, and what is developed for practice draws on new insights from research. Individual projects might work at one or more points in the cycle. No single project by itself would be expected to yield a definitive answer to any significant problem. Instead, program leaders would coordinate a varied progression of projects—interventions, research, and studies of various kinds—in ways that build knowledge and practice. One corollary of this approach is that interventions, whose primary goal is the improvement of practice, could also have, by design, a concomitant goal of testing theoretical ideas and generating new theoretical insights and research questions.

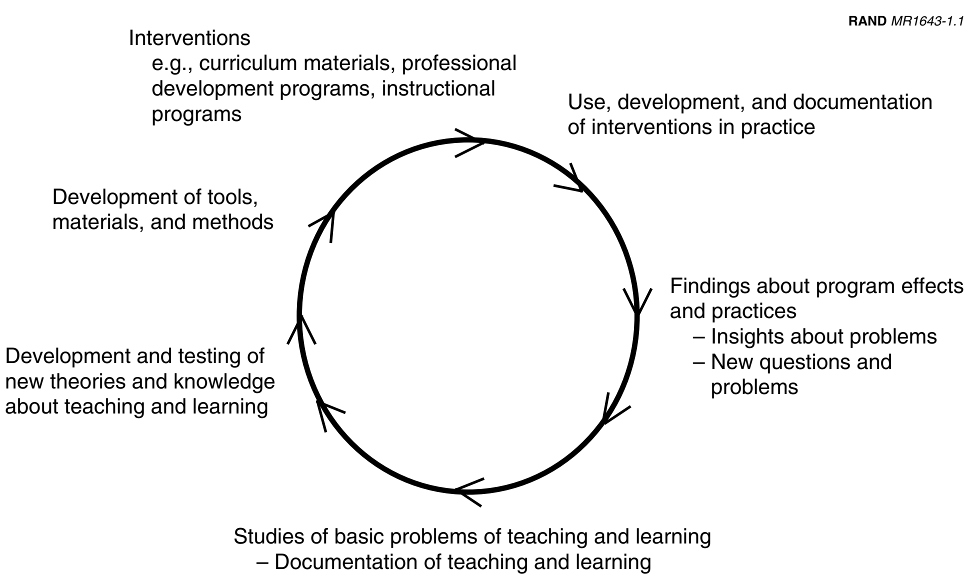

Figure 1.1—Cycle of Knowledge Production and Improvement of Practice

The program should be designed as a joint undertaking involving cooperation among researchers, practitioners, developers, and funders. This requires coordinated funding. But cooperation and coordination does not mean that all work would be guided by some rigid program design. Nevertheless, projects would be linked in a system that coordinates different kinds of work, from survey research and descriptive work, to basic inquiry, to small-scale developments, to efforts to develop and use programs or approaches across contexts.

For a cooperative undertaking such as this to be successful requires direct investments of not just money but also time and imaginative thinking. Significant responsibility for the design of the effort must lie with the sources of funding of the work and the management authority that resides in that function. Among the challenges the funders face are creating ways to commission work that stimulate the field's imagination and initiative, and creating institutional structures that can engage the communities of research and practice in taking collective and disciplined responsibility for this work. A proposal for those structures is outlined in Chapter Five.

FOCUS AREAS OF THE PROPOSED PROGRAM

The overarching goal of the proposed research and development program is to achieve mathematical proficiency for all students. Because students' opportunities to develop mathematical proficiency are shaped within classrooms through interaction with teachers and interaction with specific content and materials, the program must address issues directly related to teaching and learning. In selecting specific areas for a research and development focus, we sought to identify areas in which the goals of both greater mathematics proficiency for all students and greater equity in the levels of proficiency attained by students from differing backgrounds present substantial challenges requiring a long-term collective effort. We also sought to identify areas of research and development in which past research would provide a basis for some immediate progress. In outlining a proposed program, we focused on three areas (which are discussed in greater depth in Chapters Two through Four):

1. Developing teachers' mathematical knowledge in ways directly useful for teaching

2. Teaching and learning skills for mathematical thinking and problem solving

3. Teaching and learning of algebra from kindergarten through 12th grade (K–12).

Our aim is to map out a coordinated agenda of research and development that, by the end of a decade and a half, would provide the nation with the knowledge, materials, and programs needed to make the overarching goal of mathematical proficiency and equity attainable.

PROGRAM GOALS

The first goal of the proposed research and development program is to address the critical problems surrounding the teaching and learning of mathematics in the United States. Our proposals are based on the RAND Mathematics Study Panel's hypotheses about where and how investments in research and development will yield the greatest opportunities for improving American mathematics education. Rooted in practice in both its inspiration and its application, this program seeks to coordinate and combine theory building, multiple forms of empirical inquiry, interventions, and the wisdom of experience. Research-based knowledge should be justified in ways that help to warrant its intended use, and the problems that research addresses should be derived from the problems and goals of practice. We do not intend to neglect basic research in this endeavor, but we argue that what is often understood as "basic research" would be enhanced by more considered attention to its relationship to the activities of students and teachers.

Because solutions to the problems of mathematics education require multiple types of expertise, a second goal is to build a multidisciplinary professional community of people who have experience and expertise in practice, research, development, and policy. This community would work together to size up problems, set priorities, and plan useful programs of research and development. The work of the RAND Mathematics Study Panel represents one such effort to bring together some of the diverse groups of people who have a stake in the improvement of mathematics education: scholars from various disciplines, practitioners, developers, and policymakers. Reaching these aims will require the creation of an infrastructure for research and development to build the capacity for such work.

Although we are optimistic that the proposed program is both appropriate and promising, we recognize that implementing the results of the program in the schools will raise significant policy issues. We do not attempt to delineate these policy issues in any detail, but in several places in this report we do make observations about the possible need for policy research.

FOUNDATIONAL ISSUES

Underlying the proposed mathematics research and development program are two foundational issues: proficiency and equity. Not only must the overall level of student proficiency be raised, but also differences in proficiency should no longer be associated with race, social class, gender, language, culture, or ethnicity. We see these as challenging but compatible goals.

Mathematical Proficiency

The notion of mathematical proficiency that we use in this report is based on a conception of what it means to be competent in mathematics.[10] This concept is represented by five separate but intertwined strands:

- Conceptual understanding—comprehension of mathematical concepts, operations, and relations
- Procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- Strategic competence—ability to formulate, represent, and solve mathematical problems
- Adaptive reasoning—capacity for logical thought, reflection, explanation, and justification
- Productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in the value of diligence and in one's own efficacy.

These strands of proficiency are interconnected and coordinated in skilled mathematical reasoning and problem solving. Arguments that pit one strand against another—e.g., conceptual understanding versus procedural fluency—misconstrue the nature of mathematical proficiency. Because the five strands are interdependent, the question is not which ones are most critical but rather when and how they are interactively engaged. The core issue is one of balance and completeness, which suggests that school mathematics requires approaches that address all of the strands. Mathematical proficiency is more complex than the simplistic or extreme positions in current debates over curriculum recognize.

Because mathematical proficiency is a foundation of this research and development program, it is central to each of the areas proposed for intensive, programmatic focus. A major part of the knowledge teachers need for teaching relates to mathematical proficiency and how it can be developed in their students. If teachers hold a restricted view of proficiency and are not themselves proficient in mathematics as well as in teaching, they cannot bring their students very far toward current goals for school mathematics. Thus, addressing

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[10] Kilpatrick, Swafford & Findell, 2001.

the development and use of teacher knowledge is the first critical priority of the proposed research and development program. A second critical priority, if teachers are to help all students attain mathematical proficiency, is the identification, analysis, and development of mathematical practices. In fact, our conception of practices can be seen as another way of framing important aspects of these strands of proficiency. Third, by making algebra a subject matter focus of this program, we are calling for coordinated research and development to probe the nature of mathematical proficiency in a major area of mathematics and to investigate what is necessary to develop it.

Although we regard the above conception of mathematical proficiency as being foundational, we also recognize that it needs further specification. A research program focused on proficiency must work toward a clearer articulation of what the strands of mathematical proficiency mean and how they relate to each other and interact over the course of a student's learning of mathematics. The program also must foster the development of measures or assessments that better capture these conceptions of proficiency and how proficiency grows over time. And the program should work to provide evidence of how performance on these measures relates to the ability of students and adults to function effectively in other aspects of their lives. Such research would help the policy and practice communities develop a better understanding of what proficiency is and what the standards for proficiency should be. This knowledge would provide a stronger basis for making wise decisions about how to improve mathematics education in this country.

Equity

Defining the goal of mathematics education as providing everyone with the opportunity to gain mathematical proficiency brings the issue of equity front and center. The harsh reality is that our educational system produces starkly uneven outcomes. Although some students develop mathematical proficiency in school, most do not. And those who do not have disproportionately been children of poverty, students of color, English-language learners, and, until recently, girls.[11] Recent National Assessment of Educational Progress (NAEP) results show that the gaps in mathematics achievement by social class and ethnicity have not diminished over the years.[12] In 2000, over 34 percent of white students in grade 8 attained either "proficient" or "advanced" performance on the NAEP, up from 19 percent in 1990. Among African-American students, results were dismal, with the percentage holding steady at 5 percent. And

[11] Abt Associates, 1993; Kenney & Silver, 1997; Orland, 1994; Silver & Kenney, 2000.

[12] Braswell et al., 2001; Silver & Kenney, 2000.

although the percentage of Hispanic students who attained "proficient" or "advanced" status more than doubled, the result was still low, at 9 percent.

Lack of success in mathematics has significant consequences: Algebra, for example, plays a significant gatekeeping role in determining who will have access to college and certain career opportunities. The "gates" tend to be closed to the less advantaged, either by default—when the schools they attend simply do not offer advanced mathematics courses—or by discrimination—when low expectations for student performance lead to educational tracking that differentiates among students and therefore further limits students' opportunities to develop math proficiency.

The three areas on which we focus in this study were chosen largely because they directly relate to the issue of equity:

- First, we focus on the mathematical knowledge needed for teaching because there is no more strategic point at which to address inequity in opportunities to learn mathematics. Schools in the highest-poverty, most ethnically diverse areas of the United States tend to have teaching forces with the poorest preparation in mathematics.[13] Paying teachers more and tapping new pools of potential teaching talent are important, but those measures will not help less-advantaged students as long as their teachers lack the understanding of mathematics needed to engage, inspire, and educate these students. We need to understand exactly what mathematical knowledge is needed for teaching, especially for teaching diverse groups of students, and we need to understand how that knowledge is learned and used together with knowledge of students (their backgrounds, existing skills, interests, and such) and pedagogy. Some studies suggest the importance of teachers being able to understand the use of mathematics in the everyday lives of their students and to use that understanding in their lessons. But just how important these things are, and what way they are important, are empirical questions and, therefore, are vital issues to be addressed in a research and development program addressing inequity in mathematics instruction.
- Our second focus explores mathematical practices: the mathematical activities in which mathematically proficient people engage as they structure and accomplish mathematical tasks. This focus on practices calls attention to aspects of mathematical proficiency that are often left implicit in instruction, going beyond specific knowledge and skills to include the habits, tools, dispositions, and routines that support competent mathematical activity.

[13]Council of Great City Schools, 2000; Darling-Hammond, 1994; National Commission on Teaching and America's Future, 1996.

Owing at least in part to differing opportunities across societal groups to learn these mathematical practices, skill in these practices is unequally distributed in the population and therefore need to be addressed in school. Yet, far too often, mathematics instruction in less-advantaged schools remains a matter of simply drill and practice rather than also trying to initiate students into mathematical practices—learning what it means to create, understand, do, use, and enjoy mathematics.[14] The inclusion of this focus in the proposed program is based on our hypothesis that a fuller understanding of this implicit dimension of proficiency, and the corresponding development of support for teachers in making mathematical practices a specific component of classroom instruction, could lead to major advances in closing the performance gap between various groups.

- Third, we focus on algebra as a strategic content area in the program we envision for many reasons, some of which are cognitive, some of which are disciplinary, and others more social and cultural. But a key reason for this focus is the role that algebra plays in controlling access to further education and good jobs. The reasons for algebra's gatekeeper role are both disciplinary and historical. First, algebra functions as a language system to express ideas about quantity and space, and therefore serves as a foundation, as well as prerequisite, for all branches of the mathematics discipline. Second, algebra has come to play a prominent role in the organization of schooling, school subjects, and curriculum in the United States, within and beyond mathematics. Its role as a gatekeeper has divided students into classes with significantly different opportunities to learn. Currently, disproportionately high numbers of students of color and students living in poverty are not adequately prepared in algebra and do not have access to serious mathematics beyond this level.[15] A close look at the issue of equity in the teaching and learning of algebra will provide valuable specifics for understanding and dealing with inequities in this preparation.

ORGANIZATION OF THIS REPORT

In Chapters Two through Four, we discuss the three proposed focus areas for investments in research and development: developing teachers' mathematical knowledge and the use of that knowledge in teaching (Chapter Two), the teaching and learning of mathematical practices (Chapter Three), and the teaching and learning of algebra throughout grades K–12 (Chapter Four). Each area directly supports the proposed program's goals of building knowledge for

[14]Anyon, 1981; Haberman, 1991.

[15]Payne & Biddle, 1999.

improved practice aimed at developing mathematical proficiency across the country's population of school children. In those chapters, we show how the proposed program is strategically designed to mobilize existing resources and to build on previous research in the area to make substantial progress in the short term and to achieve fundamental changes in the quality of mathematics education in the long term. In Chapter Five, we begin by presenting the elements of the proposed program, and then outline the criteria for a strategic program that is built on existing research and linked to relevant theory, and end with the initial steps in creating the program. Chapter Six summarizes our conclusions and recommendations.