Chapter Two

TEACHERS' MATHEMATICAL KNOWLEDGE: ITS DEVELOPMENT AND USE IN TEACHING

Our proposed research agenda centers on building the resources needed for high-quality mathematics instruction. Given that the quality of instruction depends fundamentally on what teachers *do* with students to develop their mathematical proficiency, and given that what teachers *can do* depends fundamentally on their knowledge of mathematics, we recommend that the first of the three strands of research in the proposed program focus on the mathematical knowledge required for teaching mathematics and on the key resources needed to use that knowledge in teaching. In particular, this strand of research would focus on the materials and institutional contexts that support the deployment of mathematical knowledge in teaching.

Thus, if the program is well managed, its results could have a profound effect on the professional education of mathematics teachers and on various components of the education system, such as certification requirements, teacher assessments, and teachers' guides. Such an effect would require coordination of work across a variety of studies and interventions. Our decision to focus on knowledge of mathematics for teaching furthers the overarching goal of the work of the RAND Mathematics Study Panel: achieving mathematical proficiency for all students. We need better insight into the ways in which teachers' mathematical knowledge, skills, and sensibilities become tools for addressing inequalities in students' opportunities to learn. What role does teachers' mathematical knowledge play in their being able to see the potential mathematical merit in students' spontaneous ideas and strategies for solving mathematics problems? What role does this knowledge play in enabling teachers to connect the mathematics in students' everyday world with school mathematics? How does teachers' knowledge of students' mathematical thinking and students' personal interests combine with teachers' knowledge of mathematical content to shape their presentation and representation of that content, use of materials, and ability to understand their students?

BENEFITS OF A FOCUS ON MATHEMATICAL RESOURCES FOR TEACHING

Our focus on mathematical resources for teaching both extends an existing body of mathematics education research and development and targets important practical problems. Over the past several decades, two refrains have echoed throughout the discourse on teachers' knowledge of mathematics: (1) the mathematical knowledge of U.S. teachers is weak, and (2) the mathematical knowledge needed to enable effective teaching is different from that needed by mathematicians. But efforts to improve our understanding of the mathematical knowledge needed for teaching have lacked an adequate theoretical and empirical basis to guide the connection of mathematical knowledge with the work that teachers need to do.

This lack of a theoretical and empirical basis has created impediments to improvement in the training and development of teachers. Since the late 1980s, new programs, materials, curricular frameworks, standards, and assessments have been developed, all aimed at improving mathematics education. Still, teachers are the crucial element in the learning of mathematics. Teachers require substantial mathematical insight and skill to use new curriculum materials that emphasize understanding as well as skill, open their classrooms to wider mathematical participation by students, make responsible accommodations for students with varying prior knowledge of mathematics, and help more students to succeed on more-challenging assessments.

In light of these requirements, many efforts have been undertaken to help teachers develop a more robust mathematical understanding to support their teaching: Courses and workshops offer teachers opportunities to revisit and relearn the mathematical content of the school curriculum, states have raised the content-knowledge requirements for teacher certification, and programs have been developed to attract mathematically experienced and skilled people into teaching. However, these programs lack theoretical foundations and adequate evidence of their effectiveness. Despite some successful efforts to develop teachers' mathematical knowledge through professional development, teachers participating in those efforts are often no better able to understand their students' ideas, to ask strategic questions, or to analyze the mathematical tasks contained in their textbooks than they were before these efforts.[1]

The need for knowledge of mathematical content seems obvious. Who can imagine teachers being able to explain methods for finding equivalent fractions, answer student's questions on why $n^0 = 1$, or represent place value without un-

[1]See, for example, Borko, Eisenhart & Brown, 1992; Lubinski et al., 1998; Thompson & Thompson, 1994, 1996; and Wilcox et al., 1992.

derstanding the mathematical content? Less obvious, perhaps, is the nature of the knowledge of mathematical content needed for *effective teaching*: What do teachers need to know of mathematics in order to teach it? What are the mathematical questions and problems that teachers face in their daily work? What is involved in *using* mathematical knowledge in the context of teaching? What does it take for teachers to use mathematical knowledge effectively as they make instructional decisions and instructional moves with particular students in specific settings, especially with students who traditionally have not performed well in mathematics?

In 1985, Lee Shulman and colleagues introduced the term "pedagogical content knowledge" to the teaching and teacher education research lexicon.[2] This term called attention to a special kind of teacher knowledge that links content, students, and pedagogy. In addition to general pedagogical knowledge and content knowledge, Shulman and his students argued,[3] teachers need to know things like what topics children find interesting or difficult and the representations of mathematical content most useful for teaching a specific content idea.[4] This notion of pedagogical content knowledge not only underscored the importance of understanding subject matter for teaching, but it also made plain that ordinary adult knowledge of a subject could often be inadequate for teaching that subject.

Existing investigations of teacher knowledge have painted a large and distressing portrait of teachers' mathematical knowledge. In the late 1980s, researchers at the National Center for Research on Teacher Education developed new and better methods of assessing teacher content knowledge. One new technique was to pose questions in the context of teaching. In this way, the interviews probed how well respondents were able to use their mathematics knowledge for the work teachers have to do—for example, deciding if a student's solution is mathematically valid, spotting an error in a textbook, or posing problems well.

When researchers began to look closely at these issues, their analyses revealed how thin most teachers' understanding of mathematics and mathematics pedagogy are. Both elementary and secondary teachers, whether they entered teaching through traditional or alternative routes, appeared to have some sound mechanical knowledge as indicated by the fact that they were often, although not always, able to solve straightforward, simple problems. When asked to explain their reasoning, however, or why the algorithms that they used worked, neither elementary nor secondary teachers displayed much under-

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[2]Shulman, 1986.

[3]Wilson, Shulman & Richert 1987.

[4]Shulman, 1986, 1987.

standing of the concepts behind their answers. Secondary teachers who had majored in mathematics were, for example, unable to explain why division by zero was undefined or to connect the concept of *slope* to other important mathematical ideas. Other researchers, using the same instruments or similar ones, found similar results.[5] Although teachers participating in these studies often— but surprisingly inconsistently— got the "right answers," they lacked an understanding of the meanings of the computational procedures or of the solutions. Their knowledge was often fragmented, and they did not integrate ideas that could have been connected (e.g., whole-number division, fractions, decimals, or division in algebraic expressions).

These findings are not surprising, given that most teachers have learned mathematics within the same system that so many are seeking to improve. The fact that their understanding is more rule-bound than conceptual, and more fragmented than connected, reflects the nature of the teaching and curriculum that they, like other American adults, experienced in elementary and secondary schools. However, if teachers are to lead the improvement of mathematics teaching and learning, it is crucial that they have opportunities to revise and develop their own mathematical knowledge. To accomplish this, program developers and educators need better insight into the nature of the mathematics used for the work of teaching.

We also need better insight into the ways that materials and institutional contexts can either assist or impede teachers' efforts to use mathematical knowledge as they teach. For example, how can teachers' guides be crafted to provide opportunities for teachers to learn mathematics? How can they be designed such that teachers understand the mathematical purposes pertinent to an instructional goal? How can those guides be designed to help teachers use their mathematical knowledge as they prepare lessons, make sense of students' mistakes, and assess students' contributions in a class discussion? Mentoring, team teaching, lesson study, and other organizational structures provide further opportunities for developing and helping teachers to convey mathematical knowledge, but we have little systematic knowledge concerning the effects of such resources.[6]

Recently, Liping Ma's work[7] has added to our understanding of knowledge of mathematics for teaching and the resources that support its use by proposing an important idea that she calls "profound understanding of fundamental

[5]See, for example, Eisenhart, Borko, & Underhill, 1993; Even, 1990; Simon, 1993; Ma, 1999; Wheeler & Feghali, 1993; and Graeber & Tirosh, 1991.

[6]Gutiérrez, 1996.

[7]Ma, 1999.

mathematics." Ma describes the "knowledge packages" that were evident in the knowledge of the 72 Chinese elementary teachers she interviewed. These knowledge packages represented a refined sense of the organization and development of a set of related ideas in a particular arithmetic domain. The Chinese teachers articulated ideas about "the longitudinal process of opening up and cultivating [a set of ideas] in students' minds."[8] Their knowledge packages consisted of key ideas that "weigh more" than other ideas in the package, sequences for developing the ideas, and "concept knots" that link crucially related ideas. Moreover, the development and use of the knowledge packages is supported by institutional practices of mentoring and socialization, as well as professional collaboration.

Ma's notion of "knowledge packages" is a particularly generative form of pedagogical content knowledge. Central to her ideas of how to make mathematical knowledge usable in teaching is the ability to structure relationships among a set of ideas and to map out the longitudinal trajectories along which ideas can be effectively developed.

In sum, research over the past several decades has clearly indicated that the knowledge of mathematics needed to be an effective teacher is different from the knowledge needed to be a competent professional mathematician or the knowledge that is needed to use mathematics in some other field such as engineering or science. At the same time, research has provided evidence that many teachers of mathematics lack sufficient mathematical knowledge to teach mathematics effectively. Research also suggests that even well-developed materials when used by teachers who neither understand the content or the difficulties that students typically experience in learning that content do not by themselves lead to the development of student proficiency.

Although we know that teaching requires special knowledge of mathematics, we lack robust empirical descriptions of the mathematical knowledge associated with successful teaching. We also lack persuasive theory upon which to base the design of effective programs for teachers' learning. We turn now to the issue of what we need to know to design such programs or experiences.

WHAT DO WE NEED TO KNOW ABOUT MATHEMATICAL KNOWLEDGE FOR TEACHING?

Although common sense suggests that the best preparation for teaching K–12 mathematics might be an undergraduate degree in mathematics, the real answer is not that simple. First, most elementary school teachers are responsible

[8]Ma, 1999, p. 114.

for teaching all subjects, not simply mathematics, and so they cannot major in any single field as undergraduates. Instead, they typically take a few college mathematics courses in a mathematics department. Second, the mathematics of the K–12 curriculum does not map well onto the curriculum of an undergraduate mathematics degree. Even if prospective teachers majored in mathematics as undergraduates, the last time they may have studied the mathematics of the K–12 curriculum was when they were K–12 students themselves. Thus, although it is often overlooked, the problem of developing mathematical knowledge for teaching is important for the preparation and professional learning of secondary as well as elementary and middle school teachers.[9]

Therefore, one area that we have targeted for programmatic work concerns the content-specific knowledge used for teaching mathematics and how and where the use of such knowledge makes a difference for high-quality instruction. In the past decade, numerous studies have probed teachers' knowledge of mathematics in a few key areas, and the findings so far have been sobering. Division has garnered enormous attention, followed by fractions, rational numbers, and multiplication.[10] Moreover, many other key mathematical areas and ideas warrant attention—discrete mathematics, number systems, integers, geometry, place value, probability, algebra—to name a few. We know little about what teachers need to understand specifically within these areas. We do not know much about how teachers need to be able to get inside mathematical ideas to make them accessible to students. And we do not know what they need to know of the mathematics that lies ahead of them in the curriculum. We need studies that would help us learn about the mathematical resources needed to teach mathematics effectively.

Research on teachers' mathematical knowledge has frequently focused on substantive knowledge—or *topics*. As Kennedy points out,[11]

> Because the main goal of [current] reformers is to instill a deeper understanding in students of the central ideas and issues in various subjects and to enable students to see how these ideas connect to, and can be applied in, real-world situations, it therefore makes sense to require that teachers themselves also understand the central ideas of their subjects, see these relationships, and so forth.

To a lesser extent, past research has also probed teachers' knowledge and use of *mathematical thinking and problem solving* (what we termed *mathematical practices* in Chapter One) as a component of mathematical knowledge. Why

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[9]See Conference Board of the Mathematical Sciences (2001) for a thorough examination of and recommendations for the mathematical preparation of teachers at all levels.

[10]See, for example, Post et al., 1991, and Simon & Blume, 1994a, 1994b, 1996.

[11]Kennedy, 1997.

does this component matter? As students learn mathematics, they are engaged in using and doing mathematics, as are their teachers. They represent ideas, develop and use definitions, interpret and introduce notation, figure out whether a solution is valid or not, and recognize patterns. Students and teachers together are constantly engaged in situations in which mathematical practices are essential. Inevitable as this is, teachers and curricula vary enormously in the explicit attention they give to this component of mathematical knowledge. Conceptions of teacher knowledge have seldom considered the kinds of mathematical practices that are central to teaching. For example, rarely do teachers have opportunities to learn about notions of definitions, generalization, or mathematical reasoning.

The use of knowledge, whether of content or of mathematical practices, is an important subject for research. What are strategic ways to conceptualize the work of teaching that are theoretically and empirically based and will effectively guide efforts to improve teaching and learning? What aspects of the work of teaching depend on knowledge of mathematics? For instance, one important set of activities in teaching is identifying, interpreting, and responding to students' mathematical ideas, difficulties, and ways of thinking. Several researchers have profitably mined this domain of teaching.[12] Research in this area needs to be extended to examine what it takes for teachers to hear, understand, and work effectively with the widest possible range of students in mathematics education and to identify other important aspects of teachers' instructional work where mathematical knowledge for teaching is crucial.

Other "resources" can contribute to the quality of mathematics instruction. Recent studies of how people use mathematics outside of school reveal that candy sellers, basketball players, and shoppers, for example, all use mathematics in their everyday lives.[13] However, little work exists on how knowledge of such uses might be effectively used in mathematics classrooms. Such understanding seems likely to have considerable importance to achieving greater equity in the acquisition of proficiency in mathematics.

We know that students bring knowledge from outside of school to the mathematics classroom, and that such knowledge can differ significantly by race and by class.[14] Can information about students' out-of-school practices be recognized and used by teachers so that they can connect mathematical content to students in more meaningful ways? How might such information be used to en-

[12] Barnett, 1991, Barnett et al., 1994; Carpenter, Fennema, & Franke, 1996; Schifter, 1998.

[13] See, for example, Saxe, 1991; Saxe, Gearhart, & Seltzer, 1999; Lave, 1988; Nunes, Schliemann, & Carraher, 1993; Nasir, 2000, 2002; and Civil, 2002.

[14] McNair, 2000.

gage students who traditionally do not perform well in mathematics? Although some professional development efforts emphasize ways in which teachers can build on students' out-of-school practices (e.g., riding the subway) and prepare teachers to help students translate everyday activities into abstract mathematical equations,[15] much remains to be done to extend this line of work in ways that would make it usable for classroom instruction.

Still other research has investigated the role teachers' beliefs and expectations about different kinds of students play in their teaching.[16] Much of this work explores how teachers' knowledge and expectations about students affect students' opportunities to learn as well as their learning. A teachers' ability to see and make good use of their students' mathematical efforts depends in large measure on whether he or she can see and make sense of the mathematics in those efforts. Research suggests that teachers' expectations for their students' performance often shape their assumptions about the correctness or merit of a particular student's work. A teacher whose mathematical knowledge is thin is less likely to recognize the mathematical sense in a student's representation or solution, leading to an inappropriate assessment of the student's capabilities. But, while much has been learned, more remains to be uncovered about how such expectations and beliefs play out for particular topics or mathematical practices.

Although significant progress has been made toward better understanding the mathematical knowledge needed for teaching, we need to know more if we are to improve teachers' mathematical preparation. We identify three areas around which to frame and focus a fruitful line of work on knowledge for teaching: (1) developing a better understanding of the knowledge of mathematics needed in practice for the actual work of teaching; (2) developing improved ways to make useful and usable knowledge of mathematics available to teachers; and (3) developing valid and reliable measures of the mathematical knowledge for teaching.

Developing a Better Understanding of the Mathematical Knowledge Needed for the Work of Teaching

One line of work would extend current research on mathematical knowledge needed for teaching to other mathematical topics and to the realm of mathematical practices and their role in teaching. Another line of work would explore

[15] Moses & Cobb, 2001.

[16] See, for example, work by scholars such as Aguirre, 2002; Atweh, Bleicher, & Cooper, 1998; Gutiérrez, 1996; Reyes, Capella-Santana, & Khisty, 1998; and Rosebery & Warren, 2001.

the relationship of this knowledge to the instructional contexts in which it is used. In this effort, some important questions need to be answered:

- What specific knowledge of mathematical *topics and practices* is needed for teaching particular areas of mathematics to particular students?
- What mathematical thinking and problem-solving practices are important in the work of teaching? How and where should such practices be developed in the course of teaching? What do teachers need to know about such practices to be able to support students' engagement in and learning of such practices?
- What knowledge and expectations about students' mathematical thinking and capabilities are needed for teaching specific mathematics and mathematics practices to particular students?
- How does or should students' existing content-specific knowledge shape teachers' decisions about the presentation and representation of content, the use of materials, and the ability to hear and understand their students in particular areas of mathematical content?
- What role does teachers' knowledge of mathematics, knowledge of students' mathematics, and knowledge of students' out-of-school practices play in a teacher's ability to address inequalities in students' opportunities to learn?
- What mathematical and student-oriented sensibilities are needed to enable teachers to use their knowledge effectively in practice?

Many opportunities for research are made possible by the adoption of different curricula across the nation. Do the new curriculum series demand more, or different, mathematical knowledge than the textbooks that have more traditionally been used in classrooms? How do different sorts of teachers' guides affect what teachers are prepared to do, and can do, with their students? While studies focused on such curricula would contribute to an understanding of the implementation challenges in the curricula themselves, *collectively* such studies could add much to the body of understanding about teachers' knowledge of mathematics that we have argued is needed.

Developing Improved Means for Making Mathematical Knowledge for Teaching Available to Teachers

A second line of proposed work in this focus area concerns the construction of systems and institutional practices that can make mathematical knowledge for teaching more systematically available. Although we have argued that evidence exists that mathematical knowledge can make a significant impact on instruction, making that knowledge usable and using it in practice remains an important part of the problem to be solved.

We have identified three classes of research and development opportunities. One approach to supporting the effective use of mathematical knowledge in practice focuses on teachers' professional learning opportunities. A second lies in the arrangements for professional work that would support both learning and the use of what is learned. The third class centers on the design of useful tools to support mathematically knowledgeable practice. Issues important to this research and development include:

- **Professional learning opportunities:** What learning opportunities enable teachers to develop the mathematical knowledge, skills, and dispositions needed for teaching? How can teachers be helped to develop the requisite mathematical knowledge, skills, and dispositions in ways that enable them to teach each of their students effectively? What learning opportunities promote teachers' *use* of such mathematical knowledge and skills and their ability to act on such dispositions?
- **Arrangements for professional work:** Over the past decade, many efforts have been made to organize the professional work of teachers to allow them to better develop their mathematics knowledge. Although different approaches have their advocates, we do not know about the relative effectiveness of those approaches in different contexts. For example, arrangements for organizing professional practice that permit some teachers to specialize in mathematics and others to focus on instruction in other content areas could be investigated. Another possibility is to organize teachers' grade-level assignments in ways that facilitate collaboration in learning from teaching. Do organizational arrangements that allow teachers to move through grades together with their students afford the development of mathematical knowledge that is difficult to attain when teachers remain at the same grade level from year to year? Alternatively, does collaboration on lessons with others teaching at the same level and using the same materials develop and facilitate the use of mathematical knowledge (e.g., teachers engaged in practices of "lesson study" similar to those widely used in Japan). This class of work could both systematically examine existing arrangements for professional work and be used to design and test new ones.
- **Tools to support mathematically knowledgeable practice:** What are the characteristics of tools that support the effective use of mathematical knowledge in teaching? Such tools might include curriculum materials, technology, distance learning, and assessments. For example, a wide variety of new curriculum materials have been designed with substantially enhanced teachers' manuals. These manuals are intended to provide teachers

> with opportunities to learn about mathematical ideas, about student learning of these ideas, and about ways to represent and teach these ideas. Some research shows that teachers' use of teachers' guides are shaped by their work conditions—for example, whether they have time for planning—as well as their knowledge of mathematics. How might such materials be designed and used more effectively by teachers, and with what effect on practice?

Widespread curriculum development and new textbook adoptions provide opportunities to design systematic investigations to uncover how various forms of professional development—with different structures, content, and pedagogical approaches—interact with new text adoptions to affect the quality of instruction. For example, a school district that adopts a new curriculum series could offer three distinct forms of professional support for the development of teachers' usable knowledge and skill. This would provide the opportunity to compare the effects on teachers' capacity to use curricula skillfully and their students' learning from those curricula within the same environment.

Developing Valid and Reliable Measures of Knowledge for Teaching

A third element of this agenda centers on the need for reliable and valid measures of the content knowledge required in teaching. Such measures are needed for certifying teachers, designing and assessing professional training programs, and the redesign of programs for preservice preparation of teachers. On the whole, existing measures are weak.

Typical approaches to measuring content knowledge include using a major or minor in mathematics or mathematics courses that were taken as proxy measures of teachers' mathematics knowledge. For example, some researchers have investigated the relationship between a major in mathematics and gains in student achievement. Several studies found a (slight) positive correlation between teachers' majors in a subject matter and gains in student achievement,[17] but in another study, researchers found a "ceiling effect"—that is, increases in student achievement were positively correlated with teachers' courses in mathematics up to about five courses, after which the benefits of taking more mathematics courses appeared to be negligible.[18] To complicate matters further, other studies have found some positive correlations between mathematics-specific edu-

[17]Ferguson, 1991.
[18]Monk, 1994.

cation course work and student achievement, but not between mathematics courses and student achievement.[19]

A major problem with proximate measures for teachers' knowledge—such as undergraduate degrees or number of courses taken—is that they are poor indicators of what teachers actually know and how they use that knowledge in teaching. Complicating things further is the variation in what constitutes a "major" across institutions of higher education (e.g., at some colleges and universities, one does not have a "major" but rather fulfills a "concentration"). This variability makes it difficult to know exactly what teachers have had the opportunity to learn in their master's program or undergraduate courses. In recent years, researchers have developed protocols for probing teachers' mathematics knowledge more directly. They have developed interview protocols, written assessments, and observational instruments. These tools have proved useful for learning what teachers know and how they think, and, in some cases, how they reason about a situation in teaching mathematics, analyze a student's response, evaluate a student's work, or make judgments about goals for future instruction. Although some of these instruments have been shared across research projects, few have been tested or validated.

The lack of sophisticated, robust, valid, and reliable measures of teachers' knowledge has limited what we can learn empirically about what teachers need to know about mathematics and mathematics pedagogy. The lack of measures also limits our understanding about how such knowledge affects the learning opportunities of particular students and their development of mathematical proficiency over time.

To identify the mathematical knowledge needed in teaching, and to study the impact of various kinds of learning opportunities, the field needs reliable and valid measures of teachers' knowledge and of their use of such knowledge in teaching. A range of tools is needed, including survey measures, performance tasks, and written and interactive problems. This line of work should build on the past 15 years of work on teacher assessment.[20] Such measures would permit teacher knowledge to be investigated as a variable in virtually all studies of mathematics teaching.

Questions worth pursuing in this area include the following:

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[19] Begle, 1979; Monk, 1994.

[20] For example, Shulman and colleagues at Stanford University with the Teacher Assessment Project (1985–1990); National Board for Professional Teaching Standards; Interstate New Teacher Assessment and Support Consortium; The Praxis Series developed by Educational Testing Service; the National Center for Research on Teacher Education and the Teacher Education and Learning to Teach study at Michigan State University (1986–1991); and the Study of Instructional Improvement currently under way at the University of Michigan.

- Building on previous studies of the knowledge used in effective instruction, what are the domains in which to measure teachers’ knowledge? How can those domains be sampled? How homogeneous or topic-specific is teachers’ knowledge of mathematics?
- How should reliable measures of teachers’ mathematical knowledge be developed, piloted, and validated?
- What assessment tools are needed to carry out research on how teachers’ knowledge of mathematics interacts with their other knowledge, such as their knowledge of particular students, and how it shapes their instruction?
- Would specific measurement instruments enable us to better understand how various sorts of professional development affect teachers’ usable content knowledge and their ability to use that content knowledge in particular settings?
- How can research on models of instruction and their effects on student learning be enhanced by the use of the measures of mathematical knowledge discussed in this chapter?