Chapter Three

TEACHING AND LEARNING MATHEMATICAL PRACTICES

Because expertise in mathematics, like expertise in any field, involves more than just possessing certain kinds of knowledge, we recommend that the second strand of the proposed research and development program focus explicitly on mathematical know-how—what successful mathematicians and mathematics users *do*. We refer to the things that they do as *mathematical practices*. Being able to justify mathematical claims, use symbolic notation efficiently, and make mathematical generalizations are examples of mathematical practices. Such practices are important in both learning and doing mathematics, and the lack of them can hamper the development of mathematics proficiency.

Our rationale for this focus is grounded in our fundamental concerns for mathematical proficiency and its equitable attainment. While some students develop mathematical knowledge and skill, many do not, and those who do acquire mathematical knowledge are often unable to use that knowledge proficiently.[1] Further, debates over the improvement of students' mathematics achievement are often intertwined with questions about what we mean by "proficiency." The work related to mathematical practices that the RAND Mathematics Study Panel proposes should contribute to a better understanding of proficiency and hence to greater clarity and consensus about goals for the improvement of mathematical education.

It is important to note that this focus is the most speculative of the three we propose in this report. After much deliberation, we chose it because we hypothesize that a focus on understanding these practices and how they are learned could greatly enhance our capacity to create significant gains in student achievement, especially among currently low-achieving students who may have had fewer opportunities to develop these practices. Our belief that this focus would contribute to greater precision about what is meant by mathematical proficiency reinforced our decision to make it a priority.

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[1]Boaler, 2002, and Whitehead, 1962.

MATHEMATICAL PRACTICES AS A KEY ELEMENT OF PROFICIENCY

Our choice of the term "practices" for the things that proficient users of mathematics do is rooted in a definition given by Scribner and Cole:

> By a *practice* we mean a recurrent, goal-directed sequence of activities using a particular technology and particular systems of knowledge. We use the term "skills" to refer to the coordinated sets of actions involved in applying this knowledge in particular settings. A practice, then, consists of three components: technology, knowledge, and skills. We can apply this concept to spheres of activity that are predominantly conceptual (for example, the practice of law) as well as to those that are predominantly sensory-motor (for example, the practice of weaving). All practices involve interrelated tasks that share common tools, knowledge base, and skills. But we may construe them more or less broadly to refer to entire domains of activity around a common object (for example, law) or to more specific endeavors within such domains (cross-examination or legal research).[2]

Those of us in the RAND study panel believe that too little attention has been paid to research on the notion of practice as set forth by Scribner and Cole. When considering what it means to *know* mathematics, most people think of one's knowledge of topic areas, concepts, and procedures. Of course, these things are central to knowing mathematics. But mathematics is a domain in which what one does to frame and solve problems also matters a great deal. Simply "knowing" concepts does not equip one to use mathematics effectively because using mathematics involves performing a series of skillful activities, depending on the problem being addressed.

Because the concept of "mathematical practices" will be unfamiliar to many readers, we begin by illustrating what is involved in this concept. We chose an example that involves elementary school students because we want our readers to see that what we are discussing here is important to learning and using mathematics at any grade level. In this example, a third-grade class is dealing with an unexpected claim made by one of the children concerning even and odd numbers.

As you read this example, you may become puzzled, or even impatient. You may be asking, why doesn't the teacher immediately set the student straight by clarifying the definitions of even and odd numbers? Try instead to look for the mathematical practices that students are using and learning to use. Notice, too, that no choice needs be made on the part of the teacher between the mastery of content and the development of practice. The students are developing and using certain mathematical practices at the same time that their understanding of

[2]Scribner & Cole, 1981.

the definitions of even and odd numbers are strengthened and made more explicit. By the end of this classroom episode, the students know what determines whether a number is even or odd. In addition, the practices in which they engage will be important for many other mathematical problems, puzzles, and confusing situations that they will face in the future.

Near the beginning of a class, one of the boys in the class volunteers something that he says has occurred to him. He has been thinking that the number 6 "could be even or it could be odd." Of course, this is wrong—6 is even. His classmates object. The teacher does not immediately correct the child, but instead lets the other students respond. She recognizes that figuring out how to resolve this debate might offer students an important opportunity to learn how to deal with confusion about core mathematical concepts.

Using the number line above the chalkboard, one girl tries to show the student that his theory creates a problem because if 6 is odd, then 0 would be odd, too. She is relying on her knowledge that the even and odd numbers alternate on the number line. Unconvinced, the boy persists. To show why he thinks that 6 can be thought of as odd as well as even, he draws six circles on the board, divided into three groups of two circles, as such:

"There can be three of something to make six, and three is *odd*," he explains. Many hands go up as the other students disagree with this logic. "That doesn't mean that 6 is odd," replies one classmate. Another classmate uses the definition of even numbers to show that 6 has to be even because you can divide 6 into two equal groups and not break anything in half. Finally, another girl, after pondering further what the first student is saying, asks him why he doesn't also say that 10 is "an even number and an odd number" since it is composed of *five* groups of two, and 5 is odd. She makes a drawing just like his, except with five groups of two circles. When he agrees that 10 could also, like 6, be odd, the classroom erupts with objections, and another girl explains firmly that "it is not according to how many groups it is." She explains that the definition of an even number means that what is important is whether a number can be grouped by twos "with none left over." Using the first student's drawing, she shows that the key point is that no circles are left over.

Over the next few minutes, the children spend time clarifying definitions for even and odd numbers, and they also have a bit of mathematical fun noticing that other even numbers have the property that the first student observed—14 is seven groups of two, 18 is nine groups of two, and so on. Several children contribute examples excitedly, and finally one girl observes that there seems to be a pattern in such numbers—the numbers with this quality of "oddness" appear to be the alternating even numbers.

At first glance, the children in the previous example might seem to be merely helping a classmate who is confused about a simple fact—whether 6 is even or odd—or who is unaware that the definitions of even and odd are mutually exclusive. A closer look at the situation, however, shows that the students are using and learning some important mathematical *practices* that actually enable them to resolve the confusion—i.e., no one ends up believing that 6 can be both even and odd—and they also explore some significant mathematical ideas along the way. For example, several of the children use representations to communicate mathematical ideas—one student uses the number line and another creates a diagram, which is then used by the other children. One student makes a mathematical claim, which is seen as a matter of common concern by the others who deploy their shared knowledge to illustrate the contradictions that are implicit in the claim. If 6 can be odd, another student reasons, then 0 might also be odd. And another student generalizes the first student's reasoning about the number 6, arguing that the first student must have to accept that 10 could also be odd.

Attentive to the importance of language in resolving this problem, the students refer to and use various definitions of even and odd numbers to make and support their arguments. They also seek to compare alternative definitions of even and odd numbers. Further generalizing the first student's argument about the "oddness" of 6, they identify a class of even numbers having the same characteristic—the numbers can be grouped into odd multiples of two—and notice patterns in this class of numbers.

These activities—mathematical representation, attentive use of mathematical language and definitions, articulated and reasoned claims, rationally negotiated disagreement, generalizing ideas, and recognizing patterns—are examples of what we mean by *mathematical practices.* As the mathematician Andrew Wiles endeavored to prove Fermat's last theorem, he engaged in similar practices—representation, reconciliation, generalization, and pattern-seeking—that enabled him to make one of the world's greatest mathematical breakthroughs, as detailed in a number of biographies depicting the famous proof.[3] These practices and others are essential for anyone learning and doing mathematics proficiently.

Competent learning and use of mathematics—whether in the context of algebraic, geometric, arithmetic, or probabilistic questions or problems—depend on the way in which people approach, think about, and work with mathematical tools and ideas. Further, we hypothesize that these practices are not, for the most part, explicitly addressed in schools. Hence, whether people somehow ac-

[3]Singh, 1997.

quire these practices is part of what differentiates those who are successful with mathematics from those who are not. Our proposed research and development program would help to answer key questions in this area, such as: How are these practices learned? What role do they play in the development of proficiency? How does the lack of facility with such practices hamper the learning of mathematics? And what affects their equitable acquisition?

BENEFITS OF A FOCUS ON MATHEMATICAL PRACTICES

Significant research has been conducted on mathematical practices such as problem solving, reasoning, proof, representation, and communication. The ways in which students approach and solve problems of various kinds, the processes used by expert problem solvers, and the heuristics that function to guide the solving of problems all have attracted the attention of researchers, and we know a lot in these areas.[4] For example, some researchers have investigated students' use of diagrams, graphs, and symbolic notation to lend and gain meaning about objects and their relationships with one another.[5] Others have probed students' knowledge of proof.[6] This research has illuminated the importance of these processes in a student's approach to learning and using mathematics. However, many important questions about mathematical practices remain unanswered, and the lack of adequate knowledge about these practices has led to controversy over mathematics education improvement efforts.

New curricula and standards have paid more attention to processes such as problem solving and justifying. However, weak implementation of instruction intended to build facility with these processes has led to contentious debates among educators, mathematicians, and members of the public over whether these curricula and standards are "watering down" mathematics instruction. To build a consensus on what should be taught, and to improve teaching and learning, we need a greater understanding of what it takes to learn and teach mathematical reasoning, representation, and communication in ways that contribute to mathematical proficiency. We hypothesize that people who do well with mathematics tend to have developed a set of well-coordinated mathematical practices and engage in them flexibly and skillfully, whereas those who

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[4] See, for example, Charles & Silver, 1989; Goldin & McClintock, 1984; National Council of Teachers of Mathematics, 1989, 2000; Ponte et al., 1991; Schoenfeld, 1985, 1992; and Vershaffel, Greer, & De Corte, 2000.

[5] See, for example, DiSessa et al., 1991; Even, 1998; Goldin, 1998; Janvier, 1987; Kaput, 1998a; Leinhardt, Zaslavsky & Stein, 1990; Noss, Healy, & Hoyles, 1997; Owens & Clements, 1998; Vergnaud, 1998; and Wilensky, 1991.

[6] See, for example, Balacheff, 1988; Bell, 1976; Blum & Kirsch, 1991; Chazan, 1993; Coe & Ruthven, 1994; De Villiers, 1990; Dreyfus & Hadas, 1996; Hanna & Jahnke, 1996; Maher & Martino, 1996; and Simon & Blume, 1996.

are less proficient have not. We also suspect that such practices play an important role in a teacher's capacity to effectively teach. If we are correct, investing in understanding these "process" dimensions of mathematics could have a high payoff for improving the ability of the nation's schools to help all students develop mathematical proficiency.

A key reason for focusing on practices for learning, doing, and using mathematics is to confront the pervasive—and damaging—cultural belief that only some people have what it takes to learn mathematics. Along the lines of the groundbreaking work that Carol Lee[7] and her colleagues are doing in English literature, which is focused on literary interpretation and on connecting students' prior skills and interests with their evolving literary practices, this focus could enable a serious challenge to the pervasive inequalities seen in school mathematics outcomes. In their work, Lee and her colleagues approach the problem from two directions. On one hand, they seek to uncover and articulate the practices of literary interpretation used in reading poetry or fiction; on the other hand, they study practices that urban youth use in other contexts—for example, in music or in conversation. Lee and her colleagues then build instructional connections between the practices in which students are already engaged and structurally similar practices that are necessary to literary interpretation.

Investigations of mathematical activity in out-of-school contexts similar to those that Lee and her colleagues studied might enable the construction of similar instructional mappings between out-of-school and in-school practices. Students, especially those who traditionally have not acquired mathematical proficiency, could be helped to connect their out-of-school practices of calculating, reasoning, and representing with the mathematical problem-solving practices expected of them in school. For example, the notational systems that some young people invent to keep track of the scores in a complex game can reflect substantial sensitivity and skill in what it takes to construct and use representations of changing quantities. Such representational practices that are developed outside of school could be built upon as teachers help students acquire skill and fluency with mathematical notation.

A second reason for the focus on practices involved in doing and learning mathematics centers on the current demands of everyday life. As we enter the 21st century, many individuals have expressed a renewed concern for the kind of mathematical proficiency needed in a world flooded with quantitative information that requires decisionmaking using such information and that demands

[7] Lee & Majors, 2000.

extensive use of spatial reasoning.[8] Mathematics is increasingly needed for analysis and interpretation of information in domains as varied as politics, business, economics, social policy, and science policy.

Knowing and using mathematics is critical to a functional citizenry and the empowerment of all members of society. We believe that such knowledge and use requires what we have termed "mathematical practices." Because the broad and effective development of mathematical proficiency is the fundamental goal of school mathematics education, we argue that a focus on understanding the mathematical practices that are required beyond one's school years should be a critical component of the proposed research and development agenda.

This requirement for proficiency gives rise to a third reason for the proposed focus on mathematical practices. These practices provide learning resources needed by teachers and students who are engaged in more ambitious curricula and who are working toward more-complex educational goals. Without these resources, ambitious agendas for improvement in mathematics education are unlikely to succeed. When higher standards for student performance are set, educators know little about what students and teachers would have to do, and learn to do, to meet those standards. What it would take for all students and teachers to achieve such ambitious goals has not been adequately examined. Consequently, despite greater expectations and important new goals, student performance may not improve. For example, when a teacher who never before has asked her students to explain their thinking suddenly asks those students to justify their solutions, she is likely to be greeted with silence. When she asks a student to explain a method he has used, he will probably think that he made an error. And when teachers assign more-challenging work, students who are unsure of what to do may ask for so much help that the tasks' cognitive demands on the student are reduced.[9] Teachers who do not know how to produce mathematical explanations or choose useful representations for solving a problem may lack the necessary resources for helping students. Discouraged, teachers may conclude that their students cannot do more-complex work and may return to simpler tasks.

In sum, our emphasis on investigating mathematical practices offers a means for uncovering what has to be understood in order for students to learn and do mathematics proficiently. Uncovering these practices can make it possible to design systematic opportunities for students (and teachers) to develop the

[8]See, for example, Banchoff, 1988; Devlin, 1999; National Research Council, 1989; Paulos, 1988, 1991, 1996; Rothstein, 1995; and Steen, 2001.

[9]Stein, Grover, & Henningsen, 1996.

learning resources needed to build a system in which all students can become mathematically proficient.

WHAT DO WE NEED TO KNOW ABOUT LEARNING AND TEACHING MATHEMATICAL PRACTICES?

The RAND Mathematics Study Panel's proposal for a focus on mathematical practices reflects the conviction that this focus would yield crucial insights that are needed to close the broad gap between those few who become mathematically proficient and the many who do not. Building knowledge about mathematical practices would help to make visible and connect the crucial elements of mathematical proficiency, the acquisition of which has been unsystematic and uneven.

The proposed focus on mathematical practices can build on significant prior research related to specific mathematical practices. While the general notion of "mathematical practices" may be an unfamiliar one to some people, we believe progress can be made by grouping together aspects of mathematical practices that have usually been treated separately and investigating aspects of their use, learning, and instruction that have remained unevenly explored. The proposed research and development on mathematical practices would focus on *activity*—the work of *learning* and *doing* mathematics. It would also take a more social view of these practices by examining them as activities of doing mathematics in interaction with other activities in specific settings, as well as examining them as cognitive processes in which individuals engage when they do mathematics. This perspective is important because practices are often acquired and enacted through interaction with others in a mathematical activity.

Where should investments be made if a focus on mathematical practices is to have the payoff that we envision? First, rather than plunging into an unmapped territory of unnamed and undefined mathematical practices, we argue that the most progress will be made in the short term if the work concentrates on three core practices that have already been the subject of substantial research: representation, justification, and generalization. These three practices are arguably central to the learning and use of mathematics in a wide range of classroom and everyday settings, and new work done in these areas can build on established research.

The domain of *representation* includes the choices one makes in expressing and depicting mathematical ideas and the ways in which one puts those choices to use. The decimal representation of numbers (using place values), for example, is one of the most important historical examples of representation. For example, consider the difference between the Roman numeral representation of the year 1776 (MDCCLXXVI) and its base-ten representation. Representation also includes modeling, in which a physical situation is described in mathematical terms, such as Newton's formulation of gravitational attraction. Representing ideas in a variety of ways is fundamental to mathematical work. No one ideal representation exists because the quality of a representation depends on the purpose of the representation.

For example, a rational number can be represented as many different fractions and also as a repeating or terminating decimal. Three-fourths can be represented as 3/4, but also as 6/8, 9/12, or 273/364. It is easier to compare 4/5 with 13/16 if the numbers are represented in decimal form (0.8 and 0.8125, respectively). What is less apparent from the decimal form is the commensurability of the two numbers—i.e., the difference between 3/4 and 2/3 is more readily apparent if the fractions were represented with 12 as the denominator. Likewise, whole numbers—take 60, for instance—can be represented in base-ten place value notation (i.e., 60). But the prime factorization of 60 (i.e., 2^2 *x* 3 *x* 5) is more informative for some purposes—such as finding the greatest common divisors—because it makes the multiplicative structure of the number visible in a way that the place-value representation does not. Choosing which representations to use depends on the work one wants to do with the mathematical objects in question. Fourth-graders learn that representing 5,002 as 499 tens and 12 ones makes it easier to compute 5,002 minus 169. Rewriting numbers is a critical part of the practice of representation.

Another critical practice—the fluent use of symbolic notation—is included in the domain of representational practice. Mathematics employs a unique and highly developed symbolic language upon which many forms of mathematical work and thinking depend. Symbolic notation allows for precision in expression. It is also efficient—it compresses complex ideas into a form that makes them easier to comprehend and manipulate. Mathematics learning and use is critically dependent on one's being able to fluently and flexibly encode ideas and relationships. Equally important is the ability to accurately decode what others have written.

A second core mathematical practice for which we recommend research and development is the work of justifying claims, solutions, and methods. *Justification* centers on how mathematical knowledge is certified and established as "knowledge." Understanding a mathematical idea means both knowing it and also knowing why it is true. For example, knowing that rolling a 7 with two dice is more likely than rolling a 12 is different from being able to explain why this is so. Although "understanding" is part of contemporary education reform rhetoric, the reasoning of justification, upon which understanding critically depends, is largely missing in much mathematics teaching and learning. Many students, even those at university level, lack not only the capacity to construct proofs—the mathematician's form of justification—but even lack an appreciation of what a mathematical proof is. Mathematical justification involves reasoning that is more general than what we typically call “proof.” In everyday situations, being able to support the validity of a mathematical conclusion also matters.

Justification is a practice supported by both intellectual tools and mental “habits.” These tools and habits are grounded in valuing a cluster of questions about knowing something and what it takes to be certain: Why does this work? Is this true? How do I know? Can I convince other people that it is true? Such questions apply not only to sophisticated mathematical claims but also to the results of the most-elementary observations and procedures.

The third core area of practice the panel proposes for research and development is generalization. Generalization involves searching for patterns, structures, and relationships in data or mathematical symbols. These patterns, structures, and relationships transcend the particulars of the data or symbols and point to more-general conclusions that can be made about all data or symbols in a particular class. Hypothesizing and testing generalizations about observations or data is a critical part of problem solving.

In one of the simpler common exercises designed to develop young students’ capabilities to generalize, students are presented with a series of numbers and are asked to predict what the next number in the series will be. To do this, they must find the pattern in the number series that permits them to calculate the next number in the series.

Representational practices play an important role in generalizing. For example, being able to represent an odd number as $2k+1$ shows that the general structure of an odd number is such that when dividing the number into two equal parts, there is always one left over. This structure is true for all odd numbers. Representing the structure using symbolic notation permits a direct view of the general form. This example suggests that a variety of discrete practices are often combined in mathematical reasoning or practice.

As with representation, the capabilities for generalization are based on both tools and habits that guide individuals or groups in identifying patterns in the world around them. Mathematics education provides a domain in which tools and habits can be developed concerning generalization that can be applied to commonplace tasks in everyday life.

Using these three core practices as a basic starting point, we recommend support for three lines of concentrated work:

- Developing a fuller understanding of specific mathematical practices, including how they interact and how they matter in different mathematical domains
- Examining the use of these mathematical practices in different settings: practices that are used in various aspects of schooling, students' out-of-school practices, and practices employed by adults in their everyday and work lives
- Investigating ways in which these specific mathematical practices can be developed in classrooms and the role these practices play as a component of a teachers' mathematical resources.

Past work on problem solving, reasoning, and other processes has tended to view the three practices separately and, consequently, knowledge of how these processes interconnect with one another is not well developed. How, for example, do students' approaches to representation shape their efforts to prove claims? Additionally, little research has compared specific processes across mathematical domains. For example, how do students' efforts to use representations in algebra differ from their use of pictorial representations in arithmetic? How do students approach proof in arithmetic versus proof in geometry? How does learning to employ the representational tools of algebra — i.e., symbols— help students to engage in justification of claims in probability? We should seek to uncover how these particular practices differ, and how they are similar, across different mathematical domains.

Because we want to develop insights that can help students make connections between the mathematics they use outside of school and what it means to do mathematics skillfully, research is needed to uncover the mathematical practices that students use in settings outside of school. In particular, activities that involve patterning and repetition, notation and other systems of recording, calculation, construction, and arrangement could be identified. How explanations are sought and developed and how conclusions are justified also would be of interest. Children's activities and performance in various settings could be observed, described with precision, and analyzed to uncover the mathematics-related practices that are important in these settings.

Similar investigations are needed of adults' everyday practices and their practices in the work world. Better understanding of the ways in which adults use (or could use) mathematics in a variety of settings—in their work and in the course of their everyday adult life—would extend the knowledge about practices that are important to mathematical proficiency. Situations that call for mathematical reasoning arise in domains as varied as personal health (e.g., weighing the costs and benefits of new drug treatments), citizenship (e.g., understanding the effect of changes in voting procedures on election outcomes), personal finance, professional practices, and work tasks. The (often-invisible) uses of mathematics and mathematical practices in everyday situations are fascinating. Some common examples involve money (e.g., calculating tips), cooking (e.g., measuring ingredients), home decorating (e.g., figuring out the number of tiles needed for a bathroom floor), playing games of chance (e.g., estimating probabilities), and reading newspapers and magazines (e.g., interpreting data in tables and graphs).

Many adult jobs require the use of mathematics. Some are in mathematically intensive professions such as engineering, nursing, banking, and teaching, but some are in a host of other occupations in which workers must employ a range of mathematical skills and practices (e.g., waiting tables, carpentry, tailoring, or even operating a sandwich cart).[10] This focus in the proposed research and development program should include investigations of the practices used in various work environments to build a broad perspective on mathematical practices that are important to learning and using mathematics. One role of such investigations is that they can make important contributions to setting future standards for mathematics proficiency.

While most existing research has focused on how students engage in these practices, less is known about how their use of particular practices develops over time. Even less attention has been paid to the teaching involved in developing these practices. For example, most research on the subject of proof examines how students approach the task of proving a claim, what they accept as a proof, and what convinces them that a statement is true. Much less has been developed to inform instructional practice related to proof. What does it take to help students to learn to engage in practices of justification? What sorts of tasks contribute to learning, and are there characteristics of instruction that help to build students' effectiveness with particular practices? What are the features of classrooms and classroom activities that make it possible for students to develop and engage in mathematical practices? What features, specifically, shape the learning of different students? How can opportunities for the development and use of mathematical practices be designed to engage students who have traditionally avoided or not performed well in mathematics in school?

Many educators assume that simply offering students instructional tasks that implicitly call for such practices will lead students to engage in those practices. We hypothesize that such practices must be deliberately cultivated and developed, and therefore research and development should be devoted to addressing this challenge.

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[10]See, for example, Hall & Stevens, 1995; Hoyles, Noss, & Pozzi, 2001; and Noss & Hoyles, 1996.

Finally, the way in which mathematical practices affect the knowledge it takes to teach remains largely unexamined. How do teachers' own capacities for representation or justification shape their instructional effectiveness? Under what circumstances does the need to use these practices appear in the course of teaching? For example, when teachers use a board to set up problems, display solutions, or record students' work, how well are they able to represent mathematical ideas, how skillful are they with notation, and how well do they use representations to support students' discussions and classroom work? The place of mathematical practices in the resources that teachers deploy in teaching has been, for the most part, unexplored and should prove to be a fruitful area for investigation.