SUMMARY

The teaching and learning of mathematics in U.S. schools is in urgent need of improvement. The nation needs a mathematically literate citizenry, but most Americans graduate from high school without adequate mathematical competence. In the 2000 National Assessment of Educational Progress, only 17 percent of grade-12 students nationally performed above a basic level of competence.[1] Furthermore, achievement gaps have persisted between white students and students of color, and between middle-class students and students living in poverty. As both a matter of national interest and a moral imperative, the overall level of mathematical proficiency must be raised, and the differences in proficiency among societal groups must be eliminated.

Improving proficiency in mathematics and eliminating the gaps in proficiency among social groups is and has been the goal of many public and private efforts over the past decade and a half. States and national professional organizations have developed standards for mathematics proficiency and assessments intended to measure the degree to which students attain such proficiency. Various programs have been developed to attract and retain more effective teachers of mathematics. New curricular materials have been developed along with training and coaching programs intended to provide teachers with the knowledge and skills needed to use those materials. However, these efforts have been supported by only a limited and uneven base of research and research-based development, which is part of the reason for the limited success of those efforts.

This report proposes a long-term, strategic program of research and development in mathematics education. The program would develop knowledge, materials, and programs to help educators achieve two goals: to raise the level of mathematical proficiency and to eliminate differences in levels of mathematical proficiency among students in different social, cultural, and ethnic groups. In the short term, the program is designed to produce knowledge that would sup-

[1]National Center for Education Statistics, 2001.

port efforts to improve the quality of mathematics teaching and learning with the teachers and materials that are now in place or that will become available over the next several years. More important, over 10 to 15 years, the program would build a solid base of knowledge for the design and development of effective instructional practice. That instructional practice, in turn, would enable the dual goals of increased levels of proficiency and equity in attaining proficiency to be achieved.

To yield maximum returns from the resources that are available for investment in mathematics education research and development, the program must focus on high-leverage areas of need; employ appropriate and valid methods for developing knowledge and practice; be grounded in and usable for instructional practice; develop and build on prior knowledge; and be coordinated, sustained, and cumulative. These program attributes will require sustained leadership from funders of mathematics education research and development—largely agencies of the federal government, including the U.S. Department of Education, the National Science Foundation, and the National Institutes of Health. Achieving these goals will also require that changes be made in the institutions of the research and development community and in those institutions' activities. In that regard, this report suggests both priorities for research and development activities and institutional arrangements intended to make the program outcomes rigorous, cumulative, and usable.

This report was commissioned by the Office of Education Research and Improvement (OERI, now the Institute of Education Sciences) as part of a larger RAND effort to suggest ways that education research and development could be made more rigorous, cumulative, and usable.[2] The RAND Mathematics Study Panel, which is composed of mathematics educators, mathematicians, psychologists, policymakers, and teachers, addresses the aforementioned concerns about the weak levels of mathematical proficiency of U.S. adults and students, and the inequities in the achievement of students from differing ethnic, cultural, and social groups. The work of the panel was inspired by the conviction that a program of research and development could be designed to help address these problems.

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[2]This report was written before the reauthorization of the research program of the U.S. Department of Education. That reauthorization created an Institute for Educational Sciences (IES) within the department, replacing the OERI. We retained the designation OERI throughout this text. The features of the legislation authorizing the IES do not conflict with the proposals made here.

THE CONTEXT FOR A PRACTICE-CENTERED PROGRAM OF MATHEMATICS EDUCATION RESEARCH AND DEVELOPMENT

The mathematics performance of students and adults in the United States has never been regarded as wholly satisfactory. However, current goals and expectations for mathematics proficiency, as reflected in recent federal legislation such as the No Child Left Behind Act and numerous state policy initiatives, present a new and formidable challenge: Although the educational system has always produced some mathematically proficient individuals, now *every* student must be mathematically competent. The ambitious goal of mathematical proficiency for *all* students is unprecedented, and it places enormous demands on the U.S. educational system.

These new goals and expectations mean that skill in basic arithmetic is no longer a sufficient mathematics background for most adults. Although number sense and computational proficiency are important, other domains of mathematics knowledge and skill play an increasingly essential role in students' educational advancement and career opportunities. For example, the endless flood of quantitative information that people receive requires that they be familiar with statistics and have an understanding of probability. Algebra is vital as a medium for modeling problems, and it provides the tools for solving those problems. To reason capably about quantitative situations, students must understand and be able to use the basic principles of mathematical knowledge and mathematical practice that include, and go beyond, basic arithmetic.

While agreement on the broad goals for mathematics proficiency is widespread, the details of those goals and the means for achieving them are often the subject of disputes among educators, mathematicians, education researchers, and members of the public. These disputes center on the content that should be taught and how it should be taught. Arguments rage over curriculum materials, instructional approaches, and which aspects of the content to emphasize. Should students be taught "conventional" computational algorithms, or is there merit in exploring alternative methods and representations? When and how should calculators be used in instruction? What degree of fluency with mathematical procedures is necessary, and what sorts of conceptual understanding are important? What is the most appropriate approach to algebra in the school curriculum? Too often, questions such as these tend to reduce complex instructional issues to stark alternatives, rather than a range of solutions. More important, the intense debates over the past decade seem to be based more often on ideology than on evidence. In the view of the members of the RAND Mathematics Study Panel, the manner in which these debates have been conducted has hindered the improvement of mathematics education.

Amid this debate, U.S. schools are expected to provide more and better opportunities for students to learn mathematics. Yet, many schools lack the key resources needed to do so. For example, there is an acute shortage of qualified mathematics teachers,[3] and many widely used curriculum programs and assessment instruments are poorly matched with increasingly demanding instructional goals. While there is considerable policy-level pressure to seek "research-based" alternatives to existing programs and practices, the education and research communities lack rigorous evidence about the degree to which alternative existing or proposed curriculum and instructional practices effectively support all students' learning of mathematics.

Improving the effectiveness of school mathematics obviously depends on much more than research and development, but research and development are necessary if resources and energies are to be invested wisely. Future investments in the creation of mathematics education programs and materials, as well as investments in the training of teachers, require knowledge of the problems of instructional practice and the effectiveness of various approaches to addressing those problems.

However, despite more than a century of efforts to improve school mathematics in the United States, investments in research and development have been virtually nonexistent. Recent federal efforts to foster improvement in mathematics education are infrequently based on solid research, and federal funding for mathematics education research and development have been sporadic and uncoordinated. There has never been a long-range programmatic effort to fund research and development in mathematics education, nor has funding been organized to focus on knowledge that would be usable in practice.

This report is based on the premise that the production of knowledge about mathematics teaching and learning, and the improvement of practice based on such knowledge, depend on a coordinated cycle of research, development, implementation in practice, and evaluation, leading in turn to new research and new development. In the absence of such an effort, gaps in the knowledge base will continue to exist, and problems, particularly those associated with the equitable attainment of mathematical proficiency, will not be adequately addressed. Moreover, the success of such an effort requires that explicit attention be paid to the ways in which such knowledge can reach school classrooms in a form that teachers can use effectively to improve students' learning.

To guide such an effort, this report maps out a long-term agenda of programmatic research, design, and development in mathematics education. Rooted in

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[3]National Commission on Mathematics and Science Teaching for the 21st Century, 2000.

practice in both its inspiration and its application, this program would coordinate efforts to create basic knowledge about the learning of mathematics through multiple forms of empirical inquiry. The program would tap the wisdom of practitioners, develop and test theories, and create and test interventions. If successful, such a program would produce resources supporting short-run improvements, and, over the course of 10 to 15 years, yield a strengthened base of knowledge useful for the sustained improvement of instructional practice. The proposed agenda must take into account the reality that public investments in research are a fraction of what is needed to deal with the scale and complexity of the problems. Therefore, difficult choices and careful designs will be required to gain maximum leverage and cumulative impact from available resources.

FOCUS AREAS FOR A LONG-TERM RESEARCH AND DEVELOPMENT PROGRAM

The limited resources that likely will be available for mathematics education research and development in the near future make it necessary to focus those resources on a limited number of topics. Because students' opportunities to develop mathematical proficiency are shaped within classrooms through their interaction with teachers and with specific content and materials, the proposed program addresses issues directly related to teaching and learning. We have selected three domains in which both proficiency and equity in proficiency present substantial challenges, and where past work would afford resources for some immediate progress:

1. Developing teachers' mathematical knowledge in ways that are directly useful for teaching
2. Teaching and learning skills used in mathematical thinking and problem solving
3. Teaching and learning of algebra from kindergarten through the 12th grade (K–12).

These are only the starting points for addressing mathematics proficiency problems. Fundamental problems to be addressed would remain and would be the subject of work in the longer-term collective effort we envision.

Developing Teachers' Mathematical Knowledge for Teaching

The first of the three focus areas in the proposed research and development program is *teachers' mathematical knowledge*. The quality of mathematics teaching and learning depends on what teachers *do* with their students, and what teachers *can do* depends on their knowledge of mathematics. Yet, numerous studies show that many teachers in the United States lack adequate knowledge of mathematics for teaching mathematics. Moreover, research indicates that higher proportions of classrooms in high-poverty areas, compared with classrooms in the nation as a whole, are staffed with poorly prepared teachers, which poses a particular problem from the perspective of the RAND panel's twin goals of mathematical proficiency and the equitable attainment of such proficiency.[4]

The knowledge base upon which to build policy and practice is poorly developed. While it is widely agreed among the mathematics education community that effective mathematics teaching depends on teachers' knowledge of content, the nature of the knowledge required for such teaching is poorly specified, and the evidence concerning the nature of the mathematical knowledge that is needed to improve instructional quality is surprisingly sparse. The same is true for the ways in which such teacher knowledge requirements for effective teaching may differ for diverse student populations. Building an improved understanding of these needs for mathematical knowledge, and developing effective means for enabling teachers to acquire and apply such understanding, would provide crucial help to the mathematics education community and to education policymakers. For these reasons, we propose a programmatic focus on three areas in which to frame fruitful lines of work on the knowledge needed for teaching:

1. Developing a better understanding of the mathematical knowledge needed for the actual work of teaching

2. Developing improved means for making useful and usable mathematical knowledge available to teachers

3. Developing valid and reliable measures of the mathematical knowledge of teachers.

To understand the mathematical knowledge needed for the work of teaching, the research community should investigate a number of key questions. The most central question addresses the role that teachers' knowledge of mathematics, their knowledge of students' mathematics, and their knowledge of students' out-of-school practices play in their instructional capabilities. Answers to this question must be developed in the context of specific mathematical domains. In addition, we feel it is important to develop a clearer delineation of the

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[4]Council of Great City Schools, 2000; Darling-Hammond, 1994; National Commission on Teaching and America's Future, 1996.

knowledge and skills required of teachers to build students' capacity to engage in the kinds of mathematical thinking and mathematical problem solving that we term "mathematical practices."

In short, the purpose of this proposed area of work is to determine the specific knowledge of mathematical *topics and practices* that teachers need to teach particular domains of mathematics to specific students. This learning should ultimately be embodied in preservice programs, curricula, and materials supporting instruction, and professional development programs.

The professional development programs are the target of the second area of the proposed focus on teacher knowledge—developing improved means for making mathematical knowledge that is useful and usable for teaching available to teachers. The most fundamental effort in this area is identifying and shaping professional learning opportunities for teachers (or prospective teachers) to enable them to develop the requisite mathematical knowledge, skills, and dispositions to teach each of their students effectively. However, the challenge is not just to learn what is needed but to create arrangements for professional work that supports continued improvement of teachers' knowledge and their pedagogical skills. Meeting this challenge will involve experimenting with ways of organizing schools and school days to support these professional learning opportunities (e.g., scheduling of the week's classes, scheduling for collaborative planning and critiquing, freeing up time for mentoring, or providing on-demand professional development).

The advancement of professional practice in mathematics instruction can be supported through the development of "tools" that support teachers in their day-to-day work. Such tools include curriculum materials, technology, distance learning, and effective assessments. For example, teachers' manuals may provide teachers with opportunities to learn about mathematical ideas, about student learning of those ideas, and about ways to represent and teach those ideas. A recurring theme in our proposed program is the potential to make knowledge created through research and reflections on teaching practice usable by teachers by embodying that knowledge in tools, materials, and program designs.

The final component of the focus on teacher knowledge is the development of valid and reliable measures of mathematical knowledge for teaching. The lack of such measures has limited what one can learn empirically about what teachers need to know about mathematics and mathematics pedagogy. Similarly, the research community has lacked the tools to investigate how teachers' mathematical knowledge affects students' learning opportunities and their development of mathematical proficiency over time. As a result, the research and mathematics education communities lack evidence to mediate among the strongly held opinions about the mathematics knowledge that teachers need to have and how that knowledge can be gained and used effectively in teaching. The lack of valid and reliable measure of knowledge for teaching also inhibits the development of evidence-based policies related to teacher credentialing and teacher assignment to schools and classrooms.

Teaching and Learning Mathematical Practices

The second of the three focus areas in the proposed research and development program concerns the *teaching and learning of mathematical practices*. Mathematical practices involve more than what is normally thought of as mathematical knowledge. This area focuses on the mathematical know-how, beyond content knowledge, that constitutes expertise in learning and using mathematics. The term "practices" refers to the specific things that successful mathematics learners and users *do*. Justifying claims, using symbolic notation efficiently, defining terms precisely, and making generalizations are examples of mathematical practices. Another example of mathematical practices is the way in which skilled mathematics users are able to model a situation to make it easier to understand and to solve problems related to it. Those skilled individuals might use algebraic notation cleverly to simplify a complex set of relationships, or they might recognize that a geometric representation makes a problem almost transparent, whereas the algebraic formulation, although correct, obscures it.

Although competent use of mathematics depends on the ways in which people approach, think about, and work with mathematical tools and ideas, we hypothesize that these practices are often not systematically cultivated in school, although they may be picked up by students at home or in other venues outside of school. Moreover, it is likely that students with poorly developed mathematical practices will have difficulties learning mathematics. Thus, it is possible that part of the explanation for differences in students' mathematical proficiency is the degree to which they have had opportunities to develop an understanding of mathematics outside of school.

Thus, we propose a focus on understanding mathematical practices and how those practices are learned because we hypothesize that fostering competency in such practices could greatly enhance the education community's capacity to achieve significant gains in student proficiency in mathematics, especially among currently low-achieving students who may be the least likely to develop these practices in settings outside of school. Moreover, research work on these problems would also contribute to more-precise program goals and a more-precise definition of mathematical proficiency itself. These practices may also supply some of the crucial learning resources needed by teachers and students who are striving to meet increasingly demanding standards.

Significant research and development in mathematics education has already been conducted on processes such as problem solving, reasoning, proof, representation, and communication. Similarly, some researchers have investigated students' use of diagrams, graphs, and symbolic notation to lend meaning to and gain meaning about objects and their relationships to one another, while other researchers have probed students' approaches to proof.

Although past studies have investigated how students engage in particular practices, less is known about how these practices develop over time and how individual practices interact with one another. Little attention has been paid to the implications for the nature of the teaching required and the consequent requirements for teachers' own knowledge and practices in mathematics. To make progress based on past work, this focus area of our proposed research and development program would connect, organize, and expand upon those past studies under the umbrella of "mathematical practices" and address more systematically the question of how mathematical practices can be characterized, taught, and learned. In sum, this work in this focus area would do the following:

1. Develop a fuller understanding of specific mathematical practices, including how they interact and how they matter in different mathematical domains

2. Examine the use of these mathematical practices in different settings (e.g., practices that are used in various aspects of schooling, students' out-of-school practices, or practices employed by adults in their everyday and work lives)

3. Investigate ways in which these specific mathematical practices can be developed in classrooms and the role these practices play as a component of a teacher's mathematical resources.

TEACHING AND LEARNING ALGEBRA IN KINDERGARTEN THROUGH 12TH GRADE

A research and development program supporting the improvement of mathematical proficiency should focus on important content domains within the school mathematics curriculum. Coordinated studies of goals, instructional approaches, curricula, student learning, teachers' opportunities to learn, and policy signals—within a content domain—can be used to systematically investigate how various elements of instruction and instructional improvement affect student learning of that domain. We propose research and development related to the improvement of proficiency in algebra as the initial domain in which to work, and we have made it the third focus of the proposed program. "Algebra" is defined broadly here to include the mathematical ideas and tools that constitute this major branch of the discipline of mathematics, including classical topics and modern extensions of the subject.

We chose algebra as an appropriate initial mathematical domain for intensive focus for several reasons. One is that algebra is foundational in all areas of mathematics because it provides the tools (i.e., the language and structure) for representing and analyzing quantitative relationships, for modeling situations, for solving problems, and for stating and proving generalizations. These tools clearly are important for mathematically intensive professions. But algebraic notation, thinking, and concepts are also important in a number of workplace contexts and in the interpretation of information by Americans on a daily basis.

A second reason for selecting algebra lies in its gatekeeper role in kindergarten through 12th grade (K–12) schooling. Without proficiency in algebra, students cannot access a full range of educational and career options. Failure to learn algebra is widespread, and the consequences of this failure are that far too many students are disenfranchised. This curtailment of opportunity falls most directly on groups that are already disadvantaged and exacerbates existing inequities in our society.

Finally, many states now require students to demonstrate substantial proficiency in algebra in order to graduate from high school. These requirements are driven largely by statutory initiatives at both state and federal levels that are embodied, for example, in high-stakes accountability tests adopted by many states and in the federal No Child Left Behind legislation. This significant escalation of performance expectations in algebra creates challenges for students and teachers alike.

As a result of the enactment of new standards and a variety of mathematics education reform initiatives, the nation is in the midst of a major change in school algebra, including changes in views about who should take it, when they should learn it, what it should be about, and how it should be taught. As recently as ten years ago, the situation was relatively stable: Generally, algebra was studied by college-bound students, primarily those headed for careers in the sciences. Today, algebra is required of all students, and it is taught not only in high school but across all grades. A coordinated program of research and development could contribute evidence to mediate the debates surrounding the new policy moves. Moreover, the program could provide resources for the improvement of teaching and learning and for eliminating inequities in opportunities to become proficient in algebra.

Algebra is an area in which there has already been significant research. Since the 1970s, researchers in the United States and around the world have systematically studied questions about student learning in algebra and have accumulated useful knowledge about the thinking patterns, difficulties, and misunderstandings that students have in parts of this mathematics domain. This previous research work is invaluable as a foundation for what is needed now.

Despite the strong history of work in this area, we lack research about what is happening today in algebra classrooms; how innovations in algebra teaching and learning can be designed, implemented, and assessed; and how policy decisions shape student learning and affect equity. Because most studies have focused on algebra at the high school level, we lack knowledge about younger students' learning of algebraic ideas and skills. Little is known about what happens when algebra is viewed as a K–12 subject, what happens when it is integrated with other subjects, or what happens when it emphasizes a wider range of concepts and processes. Research could inform the perennial debates surrounding the algebra curriculum: what to include, emphasize, reduce, or omit. Three major components frame the recommended research agenda in algebra:

- Analyses and comparison of curriculum, instruction, and assessment
- Studies of relationships among teaching, instructional materials, and learning
- Studies of the impact of policy contexts on equity and student learning.

BUILDING THE INFRASTRUCTURE FOR A COORDINATED PROGRAM OF RESEARCH AND DEVELOPMENT

Our analysis of current issues related to mathematics education leads us to argue that achieving both mathematical proficiency and equity in the acquisition of mathematical proficiency should be fundamental goals for the nation. But mounting a program of research and development to support efforts to attain these goals will not be easy. It requires making judgments about where to focus efforts to build useful knowledge about mathematics education and to develop new designs for instruction and instructional improvement. The program will require workable means of gathering and deploying high-quality evidence to inform the debates on what constitutes effective instructional practice in school mathematics.

Because solutions to the problems we have identified are not the province of any single community of experts, it will be important to build a community of multidisciplinary professionals who have experience and expertise. Producing cumulative and usable knowledge will require the combined efforts of mathematicians, researchers, developers, practitioners, and funding agencies. This community must work together to size up the problems, set priorities, and plan useful programs of research. Thus, we believe the proposed program must also be conducted in such a way as to also increase the capacity of the mathematics education research and development community to carry out high-quality work.

Drawing on the work of the National Research Council and other groups, the RAND Mathematics Study Panel proposes several criteria to judge whether a mathematics research and development program is likely to meet high standards of rigor and usefulness. One set of criteria deals with the strategic framing, design, and conduct of relevant projects. A high-quality program of research and development should respond to pressing practical needs. It should build on existing research and be informed by relevant theory. Research methods should be appropriate to the investigation of a particular question and reflect the theoretical stance taken by the investigator. A coordinated program of research and development would also support groups of researchers to investigate significant questions from different theoretical and conceptual frames using methods consistent with both the questions and the frames.

A second set of criteria concerns the kinds of communication, information sharing, and critiquing that are vital to building high-quality knowledge and evidence-based resources for practice. To support syntheses of results, replication of results, and generalization of results to other settings, researchers and developers must make their findings public and available for critique through broad dissemination to appropriate research, development, and practice communities. The chains of reasoning that lead from evidence to inference should be made explicit so that claims can be inspected. Publicizing claims and evidence will make it possible to compare and synthesize findings, methods, and results from various projects. This comparison and synthesis can help support a dynamic exchange between researchers and developers, leading to better designs coupled with better evidence of the consequences of using those designs.

A research and development program meeting these criteria will require a significant design and management effort. The funders of mathematics education research and development must play the central role in this effort, but they should perform that role in collaboration with both the research and development and the mathematics education communities. We envision an approach that would coordinate research, development, and expertise resources to build the systematic knowledge necessary for making mathematical proficiency an attainable goal for all students. Reaching these goals requires the establishment of a research infrastructure to develop the capacity for such work, and that infrastructure, in turn, requires the following:

- Active overall leadership for the design and organization of the program
- Management of the process of solicitation and selection of projects in a way that promotes work of high scientific quality and usefulness

- Deliberate development of individual, institutional, and collective capacity within the field.

In this report, we present a possible organizational structure to meet these requirements. The organization would consist of an overarching group, the Mathematics Education Research Panel, comprising a wide range of individual expertise and interests, which would advise the OERI on possible directions for the program. From time to time, this panel would assess the progress of the program as a whole, synthesize the program's results, and suggest any new initiatives that are needed. In addition, we propose the formation of three subpanels who would provide planning and guidance for each of the three focus areas of the program—mathematical knowledge for teaching, mathematical practices, and algebra. The membership of these subpanels should represent a wide range of viewpoints and include mathematics education researchers, mathematicians, mathematics educators, cognitive scientists, developers and engineers, experts in measurement, and policymakers. The subpanels would play an active and continuing role in advising OERI on the management of the focus area programs.

A cornerstone of good research and development program management is an effective process for supporting and maintaining the quality of the work that is funded. We recommend the creation of a peer review system that involves individuals with high levels of expertise in relevant subjects and research methods. We believe such a system will be most effective if it is separate from the research planning, synthesis, and advisory functions that we have proposed for the panels. A peer review system that has the confidence of the field (and of the scientific community in general) is likely to attract high-quality researchers and provide reasonable assurance that quality proposals are supported.

Investment in infrastructure will contribute significantly to the quality of the program. Key infrastructure elements include the development of common measures that can be used to gather evidence across projects and deliberate nurturing of new scholars and developers. Modes of communication and opportunities for communication among and between researchers and practitioners should be developed and supported. High-quality work depends on open debate unconstrained by orthodoxies and political agendas. It is crucial that the composition of the panels and the extended research communities be inclusive, engaging individuals with a wide range of views and skills.

CONCLUSIONS

Mathematics education is an area of vital national interest, but it is also a subject of considerable controversy. Claims and counterclaims abound concerning the value of distinctive curricular strategies and specific curricula, requirements for teacher knowledge, and standards that students should meet. For the most part, these debates are poorly informed because research evidence is lacking. The program we propose in this report is most likely to gain the necessary political support if it begins with activities intended to reshape these debates into empirically based investigations of the issues that underlie important competing claims. Thus, we recommend that work be initially supported in three key areas:

1. Studies providing evidence to inform the necessarily political decisions concerning standards of mathematical proficiency to be met by students
2. Studies of current instructional practice and curriculum in U.S. classrooms
3. Studies that collect and adapt existing measures of mathematical performance or develop new ones that can be used across studies in the proposed program.

While such initial investigations would necessarily be broad, they can contribute to understanding in the three proposed focus areas and lay the foundation for an improved relationship between research and practice and more enlightened public discourse.

The program we describe is both ambitious and strategic. Shaped by hypotheses about what will yield payoffs in increased mathematical proficiency for all students, it is a program that will have high scientific rigor and an emphasis on the usability of the knowledge that it produces. The program will involve unprecedented scrutiny, testing, and revision of instructional interventions, building evidence on how those interventions work and what it takes to make them effective.