A RAND NOTE

SIMPLE ANALYTIC SOLUTIONS TO COMPLEX MILITARY PROBLEMS

Michael V. Finn, Glenn A. Kent

August 1985

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Prepared for The United States Air Force

Rand
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PREFACE

This Note documents mathematical solutions to seven military problems of historical significance. The analytic techniques were developed by Lt. Gen. Glenn A. Kent, USAF, now retired, during his years in the Office of the Secretary of Defense, Office of the Under Secretary for Research and Engineering, from 1962 to 1965, and as the Director of Studies and Analysis at Headquarters USAF until 1972. General Kent is now a member of the Rand staff. Michael Finn was a Rand consultant under the summer intern program in 1984, after receiving the AB and AM degrees in mathematics from Harvard University. He is now pursuing the PhD degree at Princeton University.

The first technique described, the optimal mix of offensive and defensive deployments, was used around 1963 to quantify the concept that attacking with a mixed force of ballistic missiles and bombers had decided advantages over attacking with ballistic missiles alone or with bombers alone, because mixed attacks complicated Soviet defensive planning. The analytic technique demonstrated, with game theoretic methods, the utility of mixed attacks and mixed defenses. Thus, it refuted the erroneous conclusion, drawn from analyses comparing pure missile attacks with pure bomber attacks, that a force postured only for missile attacks represented the proper allocation of U.S. resources.

The techniques concerned with multiple aim points were used to gain insight into (1) the merit of deploying a new strategic ballistic missile in multiple shelters and (2) the proper mix of shelters and missiles in such a deployment. The techniques were designed to highlight the utility of redundant shelters as a means of increasing survivability without increasing the number of offensive weapons.

The game theoretic technique to solve the problem of optimal bomber payload, taking into account Soviet defense trade-offs between area defenses and terminal defenses, was developed and used in the late 1960s. The solution demonstrated the efficacy of large bombers carrying around 25 short-range attack missiles over small bombers carrying fewer than ten.

The cost-effectiveness of bomber penetration aids, optimum pattern radius for a tactical munitions dispenser, and allocation of interceptors in a multilayered defense are also analyzed mathematically.

The research for the Note was funded as a concept development effort under the Project AIR FORCE National Security Strategies Program.
The Note should interest defense analysts and planners working on similar problems.

SUMMARY

Analytic techniques, some involving rather elegant game theoretical solutions, have been used to solve or illuminate important military problems. This Note documents seven of these analytic solutions and provides a rigorous derivation of each technique.

Section I demonstrates a technique for deriving the optimal mixes of offensive and defensive deployments. Assume that (1) the offense has a choice of two systems, say, System A (perhaps ballistic missiles) and System B (perhaps bombers), with which to attack and (2) the defense has two countermeasures, each of which works effectively against one of the offense's systems. The problem for each side is how to optimally mix expenditures on the two alternatives. The defense, within a given budget, will try to maximize the cost to the offense of executing the attack; the offense will try to carry out the attack at minimum expense.

A key assumption--the assumption that leads to the game theoretic nature of the solution--is that each side may adjust its deployment after seeing the other side's deployment. That is, neither side can guarantee the secrecy of its plans. Thus, the offense would not deploy System A alone, because the defense, knowing the offense's plan, would spend money only against System A, thereby increasing the cost to the offense. According to the solution, the defense enforces the maximum cost to the offense by choosing its defensive mix so that the cost to attack with System A equals the cost to attack with System B.

Section II demonstrates an algorithm to determine the cost-effectiveness of nonexpendable penetration aids for bombers. This algorithm indicates that for the penetration aids to be advantageous, their use must decrease the attrition rate by at least twice the ratio of their cost per sortie to the cost of the aircraft.

The techniques described in Secs. III and IV demonstrate the utility and optimum solutions of using redundant shelters as a means of increasing survivability without increasing the number of offensive weapons. To minimize the number of shelters required in a multiple aim

point deployment, given a cooperative agreement limiting the number of reentry vehicles (RVs) on each side, the defender should deploy twice as many RVs as his retaliatory policy requires in enough shelters to assure a 50 percent survival probability. Even in the absence of a limit on enemy RVs, a multiple aim point deployment may still be used to increase survivability and thereby preserve retaliatory resources. Formulas for the optimum mix of deploying RVs and shelters are described.

Techniques to determine the optimal number of short-range attack missiles (SRAMs) for the offense to deploy per aircraft when limited by a fixed budget are described in Sec. V. In this case, optimalization means selecting the number of SRAMs per aircraft that allows the attacker to attain his objective at least cost in the presence of the defense having the option to allocate this resource between area and terminal defenses.

The bomblets from a cluster bomb dropped from an aircraft can be made to fall in an approximate circle. By adjusting the dispensing mechanism on the bomb, one can control the radius of the circle containing the bomblets as they hit the ground. **Section VI presents a model for determining the radius that maximizes the probability of a bomblet hitting a specific target, given the relevant characteristics of the target and the cluster bomb.**

The analysis in Sec. VII offers a unique optimal way to apportion the intercept capability among the layers of an imperfect defense so as to minimize the number of missiles penetrating the defense. The model assumes that the defense has a given capability to intercept incoming missiles and can deploy its interceptors in a layered defense. It assumes further that each interceptor in a layer is fired randomly at one of the attacking missiles that have penetrated the previous layers of defense.

ACKNOWLEDGMENTS

Thomas A. Brown, formerly of Rand, reviewed the Note, and David C. McGarvey of Rand provided helpful comments on earlier drafts. John Johnston of Syllogistics, Inc., suggested the present research.

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FIGURES

INTRODUCTION

Analytic techniques, some involving rather elegant game theoretical solutions, have been used to solve or illuminate important military problems. This Note documents seven of these analytic solutions and provides a rigorous derivation of each technique.

The solutions include the optimal mix of offensive and defensive deployments, a simple algorithm to determine the cost-effectiveness of bomber penetration aids, the merit of deploying strategic ballistic missiles in multiple shelters, the optimal bomber payload against mixed defenses, the optimal pattern radius for a tactical munition dispenser, and the optimal capability of a layered defense.

I. OPTIMAL MIX OF OFFENSIVE AND DEFENSIVE DEPLOYMENTS

Suppose that the offense has a choice between two different systems, say, System A and System B, with which to attack. The classic case is the choice between the B-52 and Polaris systems for the United States. The defense possesses two countermeasures, each of which is effective against one of the offense's systems. The problem for each side is how optimally to mix expenditures on the two alternatives. The defense, within a given budget, will try to maximize the cost to the offense of executing the attack; meanwhile, the offense will try to accomplish the task at minimum expense.

A key assumption throughout the analysis--the assumption that leads to the game theoretic nature of the solution--is that each side may adjust his deployment after witnessing the other side's deployment. Another way of stating this condition is that neither side can guarantee the secrecy of his plans. Thus, the offense would not deploy only System A, because the defense, with intelligence about the offense's plan, would spend money only against System A, thereby increasing the cost to the offense.

Let x be the offense's cost to mount a succesful attack with System A against no defense, and x' the offense's cost if the defense spends his complete budget on measures to defeat System A. Let y and y' denote similar costs for System B.

Figure 1 illustrates the offense's cost to attack when the defense divides his budget between the two available countermeasures. The parameter along the x-axis is the fraction of his budget that the defense spends against System A. If the defense spends a fraction f of his budget on defense against System A and a fraction $(1 - f)$ on defense against System B, the cost to the offense to attack with System A alone is $(1 - f)x + fx'$. On Figure 1 this is represented by the line connecting x and x'. Similarly, if the offense attacks only with System B, the cost is $(1 - f)y' + fy$; this is shown by the line connecting y and y'.

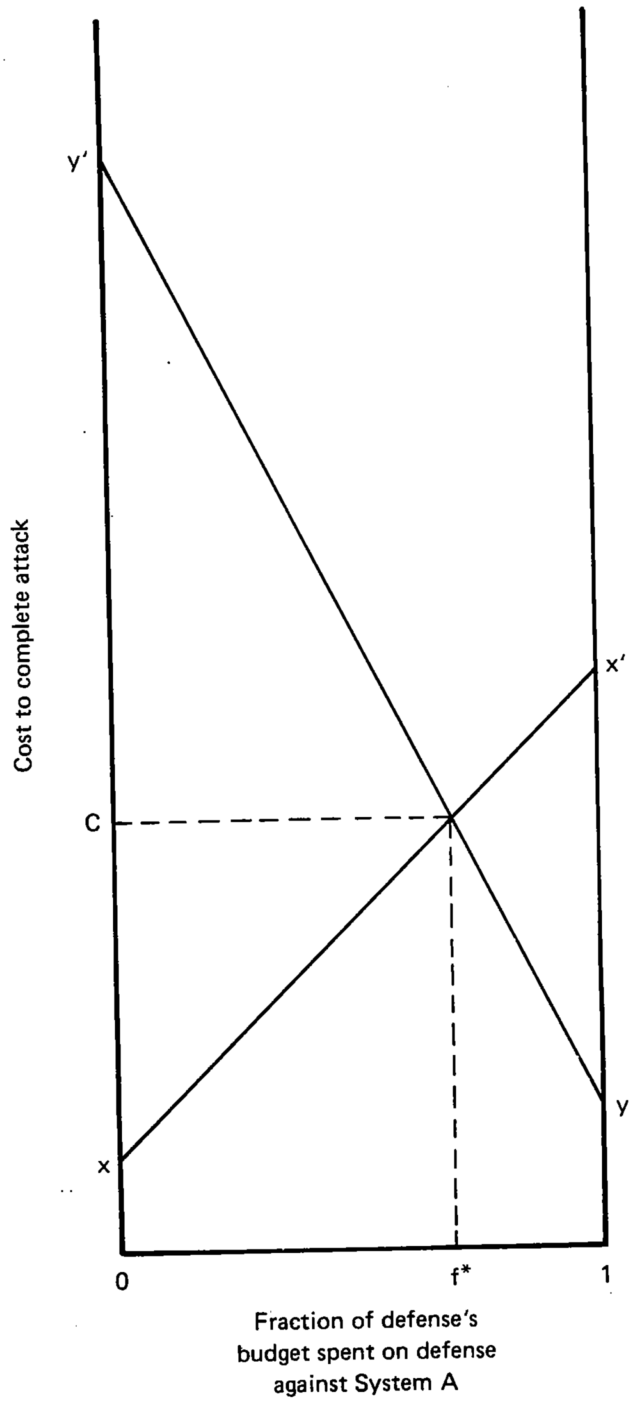

Fig. 1 — Offense's cost to complete attack using either System A or System B against range of defensive alternatives

These functions are linear in the variable f. For example, in the case of attack by System A with a fraction f of the targets defended against System A and a fraction (1 - f) undefended, to destroy the defended targets costs fx' and to destroy the undefended targets costs $(1 - f)x$; the sum of these costs is the total cost.

If the defender has the option of fractionating his defense and can defend all targets to some degree at different levels of spending--for example, by using different numbers of interceptors at the targets--then the decreasing marginal effectiveness of additional interceptors tends to make the cost function slightly concave with respect to f. However, economies of scale for interceptor production and control/deployment act in the opposite manner--to make the function more convex. The linear approximation therefore seems appropriate.

The defense enforces the maximum cost to the offense by choosing his defensive mix so that the cost to attack with System A is equal to the cost to attack with System B. To see why this statement is true, suppose that the defense is postured so that it is less costly for the offense to attack with System A than with System B. Then, to minimize cost, the offense will choose to attack with A. But the defense can increase the cost to attack with A by slightly increasing the expenditure on defense against A and equally decreasing the expenditure on defense against B. If the change is slight enough, the offense will still have to pay less to attack with System A than with System B, but he will pay a higher price to complete the attack than he would have paid had not the defense increased the expenditure on defense against A.

The equal cost condition establishes that it is best for the defense to use a fraction f^* satisfying

$$(1 - f^*)x + f^*x' = (1 - f^*)y' + f^*y \quad , \tag{1}$$

which leads to the value

$$f^* = \frac{y' - x}{x' - x + y' - y} \quad . \tag{2}$$

The unique optimal deployment of the defense is to put this fraction of his money into defense against System A. When the defense chooses the optimal mix above, the cost to the offense is

$$C = \frac{x'y' - xy}{x' - x + y' - y} \quad . \tag{3}$$

By choosing the optimal mix, the defense can enforce this cost; if the offense spends less than C, then the attack will not succeed. Conversely, if the defense does not choose the optimal mix, the attacker can see the deployment, put all of his money into the less expensive of the two systems, and complete an attack at a cost less than C. Thus, C is the cost that the defender can enforce.

The offense has to allocate money between his two systems so that the defense, seeing the offense's allocation, cannot thwart the attack within his budget by reallocating his defenses. If the offense allocates a fraction α of his budget to System A and the defense uses all of his money to defend against A, the cost to the offense is $\alpha x' + (1 - \alpha)y$. If the defense spends his money solely on defense against System B, the cost will be $\alpha x + (1 - \alpha)y'$.

Figure 2 shows these costs graphically; α is the parameter along the x-axis. By the same reasoning used to determine the defensive mix, the offense will do the best by requiring equal costs:

$$\alpha^* x' + (1 - \alpha^*)y = \alpha^* x + (1 - \alpha^*)y' \quad , \tag{4}$$

which implies that

$$\alpha^* = \frac{y' - y}{x' - x + y' - y} \quad . \tag{5}$$

If the offense divides his budget in the manner dictated by equation (5), then the cost to him will be exactly the cost C determined previously, regardless of the defense's deployment. Thus, the defense, by proportioning his defense according to equation (2), can guarantee

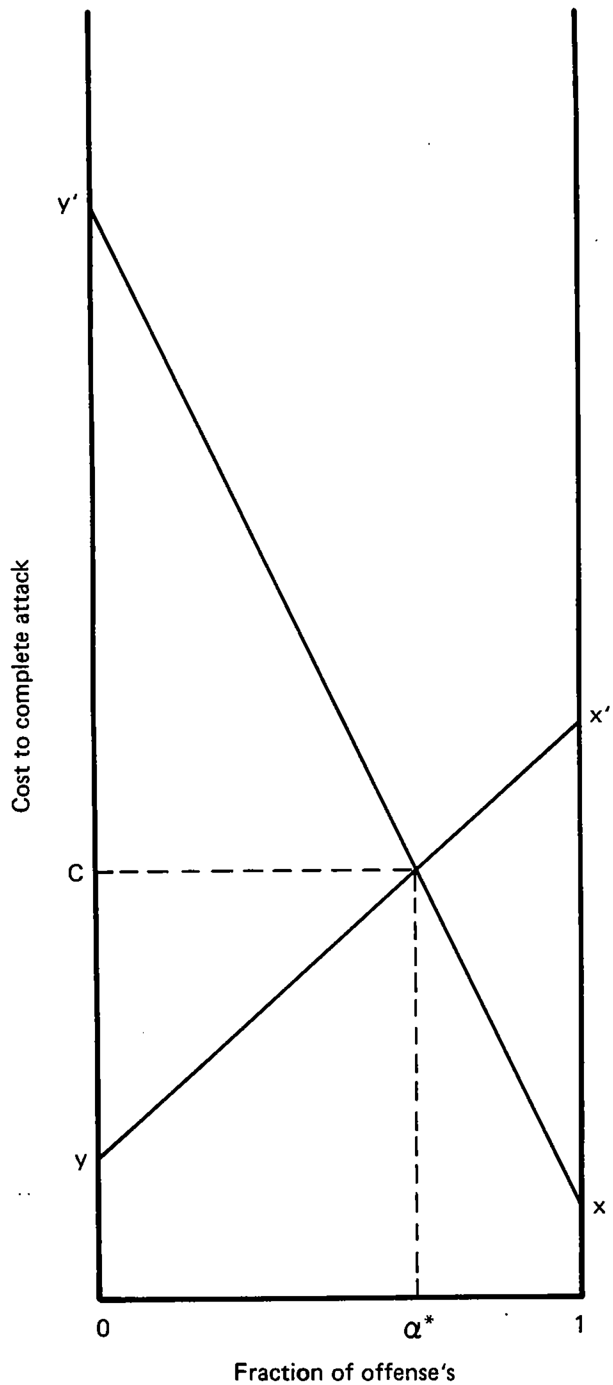

Fig. 2 — Offense's cost to complete attack with range of mixed deployments against pure defensive strategies

that the offense will have to spend C units to complete the attack, and the offense, by following the prescription of equation (5), can guarantee a successful attack with C units in his budget. This statement constitutes the game theoretic aspect of the optimal mix problem.

There is a clearer way of seeing that the costs determined from both the offense's and defense's calculations are equal. The proof uses the theorem of Pappus from projective plane geometry. Let two lines with three points--A, B, C--on the first line and three points --A', B', C'--on the second line be given. Further, let AB denote the intersection point of the line connecting A and B' with the line connecting A' and B, with similar meanings for AC and BC. Then Pappus's theorem asserts that the three points--AB, AC, BC--are colinear. Figure 3 portrays a typical application of Pappus's theorem.

To apply Pappus's theorem, we use the vertical lines in Figs. 1 and 2 as our two lines. We choose three points on each line and label them as shown in Fig. 4. Note that all of the points are obtained by choosing marked points from Figs. 1 and 2. The points of intersection, AC and AB, are the same points of intersection as in Figs. 1 and 2, respectively.

We wish to show that the line through AB and AC is horizontal, which we may do by using Pappus's theorem. We know that AB, AC, and BC are colinear. The point BC however is the point of infinity because the lines connecting B with C' and C with B' are parallel by construction. This implies that the line connecting AB and AC is parallel to the line connecting B and C', which we know to be horizontal. Therefore, we conclude that the two costs, the cost enforced by the defense and the minimum cost for offense, are equal.

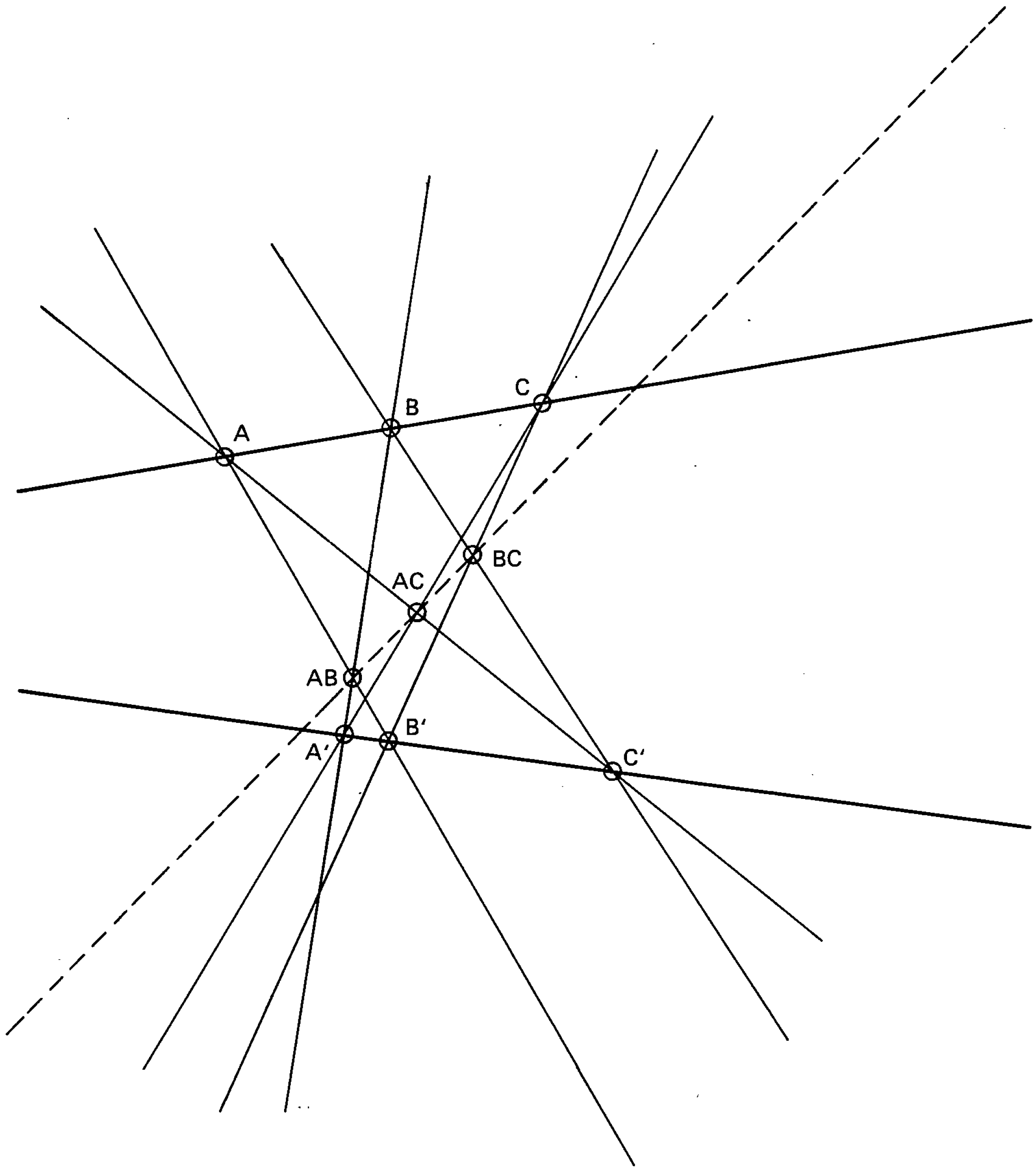

Fig. 3 — Typical application of Pappus's theorem

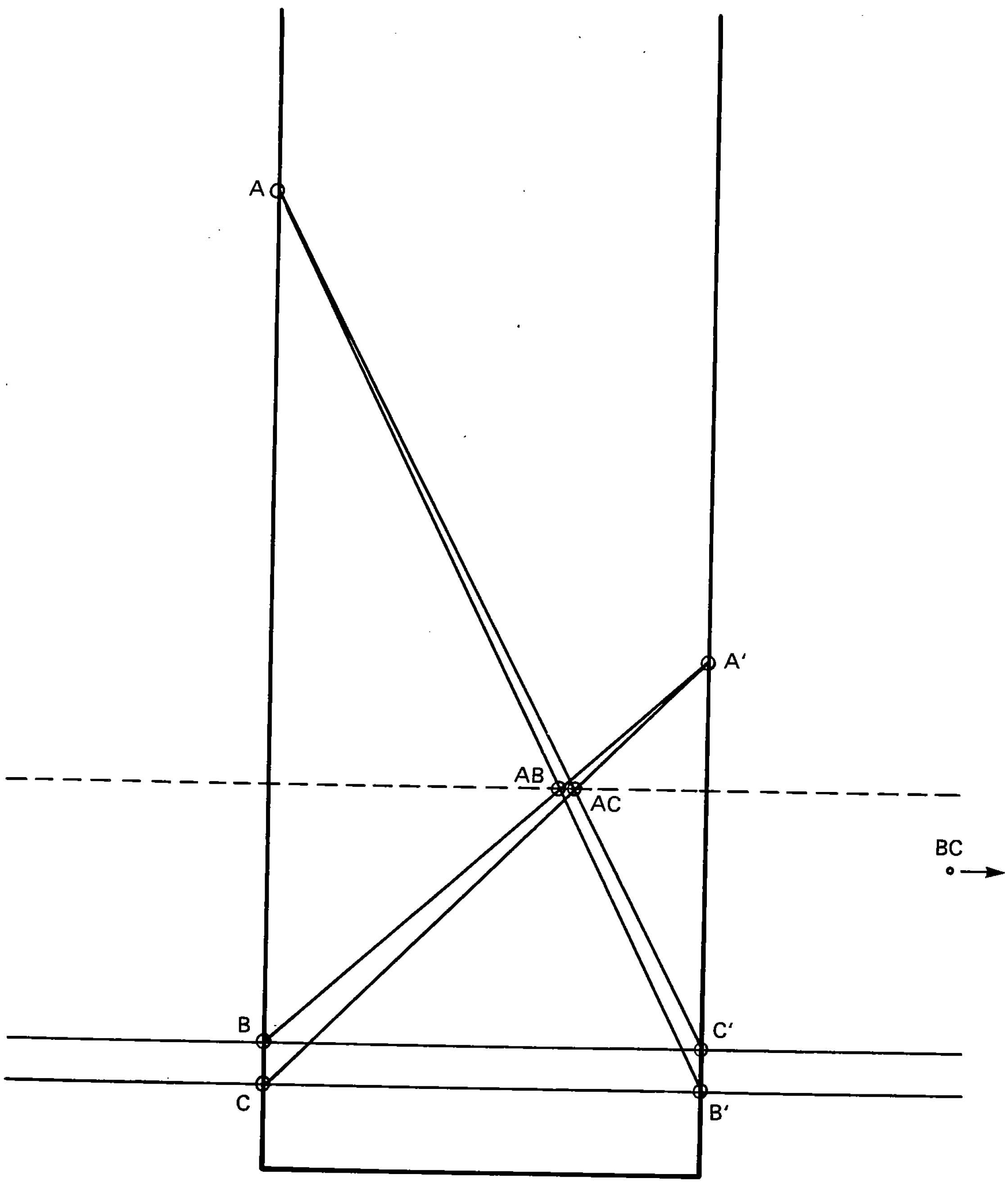

Fig. 4 — Geometric proof, using Pappus's theorem, that defense and offense enforceable costs are equal

II. AN ATTRITION RATE MODEL TO DETERMINE COST-EFFECTIVENESS OF PENETRATION AIDS FOR BOMBERS

A force of N bombers is conducting sorties over enemy territory where the attrition rate is λ. We assume that a fraction $\lambda/2$ of the bombers is lost on the way to the target and another $\lambda/2$ is lost on the return trip. The first sortie will produce $N(1 - \lambda/2)$ attacks on targets. The $N(1 - \lambda/2)^2$ aircraft available for the second sortie will produce $N(1 - \lambda/2)^3$ attacks on targets. Continuing inductively, we see that after c cycles the total number of attacks on targets S will be given by

$$S = N[(1 - \lambda/2) + (1 - \lambda/2)^3 + \ldots + (1 - \lambda/2)^{2c-1}]. \quad (1)$$

This expression may be simplified by using the following well-known formula for the sum of a truncated geometric progression

$$\frac{x^{n+1} - 1}{x - 1} = 1 + x + x^2 + \ldots + x^n \quad . \quad (2)$$

To apply equation (2) to equation (1), we first factor out $N(1 - \lambda/2)$ and then let x equal $(1 - \lambda/2)^2$ and n equal $(c - 1)$. Doing so and simplifying, we see that

$$S = \frac{N}{\lambda} \frac{(1 - \lambda/2)}{(1 - \lambda/4)} [1 - (1 - \lambda/2)^{2c}] \quad . \quad (3)$$

Typically, the values for λ will be quite small (at most 10 percent), in which case the cumbersome formula (3) may be replaced by

$$S = \frac{N}{\lambda} (1 - e^{-c\lambda}) \quad (4)$$

since the second factor on the right side of (3) is close to unity and the term in parentheses in the third factor is nearly $e^{-\lambda/2}$. Extensive

numerical comparisons show that formula (4) approximates the ungainly equation (1) to within 2 percent for λ less than .10 and c less than 20, so it will henceforth be used.

This section seeks to devise a simple algorithm for deciding if the offense should purchase penetration aids for its bombers. Penetration aids will lower the attrition rate, but they must be bought at the expense of some aircraft. The question, then, is to determine how much the attrition rate must be lowered in order for the penetration aids to be cost-effective.

Penetration aids may be classified according to whether or not they are expendable. Expendables include (1) standoff weapons, thanks to which the bombers need not fly as far over enemy territory, and (2) small but highly reflective decoys that the bomber jettisons when sighted by radar. Nonexpendables include, for example, enhanced on-board defense suppression equipment, such as jammers, and stealth technology to inhibit radar acquisition. The simpler case of nonexpendables will be treated first.

Let β be the number of aircraft that must be given up to outfit one bomber with the penetration aid. If the original number of bombers is N, then the number of equipped bombers will be $N' = N/(1 + \beta)$.

Let λ' be the attrition rate necessary to allow N' aircraft to complete S attacks on targets in c cycles. We want to determine the value of λ', or more precisely, the value $\Delta\lambda = \lambda - \lambda'$. By taking the logarithm of equation (4) and then the derivative with respect to λ (S is held constant), we find that

$$\frac{dN/d\lambda}{N} = \frac{1}{\lambda} - \frac{c}{e^{c\lambda} - 1} \quad . \tag{5}$$

The second term on the right has a simple pole at $\lambda = 0$ with residue 1, which exactly cancels the pole of the first term. Also, the pole at $c = 0$ in the denominator of the second term is canceled by the

zero in the numerator. There are no other potential singularities, so the function on the right is holomorphic. Thus, it has a convergent two-variable Taylor series about the origin, and its first few terms are

$$\frac{dN/d\lambda}{N} = \frac{1}{2}c - \frac{1}{12}c^2\lambda + \frac{1}{720}c^4\lambda^3 + \dots \quad . \tag{6}$$

We now integrate this equation with respect to λ between λ' and λ to find that

$$\ln(N/N') = \frac{1}{2}c(\lambda - \lambda') + \frac{1}{24}c^2(\lambda^2 - \lambda'^2) + \dots \quad . \tag{7}$$

As a first approximation, we can take only the first term of the power series on the right, and since we know that $N/N' = 1 + \beta$ and that β is assumed to be small, we can approximate the left-hand side by β. Thus, approximately

$$\beta = (\Delta\lambda)c/2 \quad \text{or} \quad \Delta\lambda = 2\beta/c \quad . \tag{8}$$

If we let α be the cost per aircraft per sortie (i.e., the cost β amortized over c cycles), then equation (8) can be rewritten as

$$\Delta\lambda = 2\alpha \quad . \tag{9}$$

According to this analysis, then, for the penetration aids to be advantageous, their use must decrease the attrition rate by at least twice the ratio of the cost of the penetration aids per sortie to the cost of the aircraft. For example, a group of 99 bombers is reduced to 90 when money is spent on penetration aids, so $\beta = (99 - 90)/90 = 0.10$. If ten cycles are planned, $\alpha = \beta/10 = 0.01$. Then, according to equation (9) the attrition rate must drop by 2 percent. If the initial attrition rate were 5 percent, then equation (4) indicates that 779 attacks would be expected, and if the group of aircraft equipped with the penetration

aids had an attrition rate of 3 percent, 778 attacks would be expected. So, the agreement is fairly good.

It is possible to increase the accuracy of our result at the expense of the simple form of the algorithm by using sharper approximations. On the left-hand side of equation (7), we can use the exact value of $\ln(1 + \beta)$, and on the right-hand side we can use more terms of the power series. Figure 5 shows the exact value of the function $(dN/d\lambda)/N$ in equation (5) for the range of λ's and c's of interest. The near linearity of the graphs suggests that in equation (6) two terms of the power series provide a good approximation. We can then integrate these terms, as above, to find a closer approximation than the rule of thumb.

For expendables, the above analysis must be modified slightly, because after the first sortie, only the remaining aircraft will need to be outfitted with penetration aids. If α is the cost to equip a single bomber for a single sortie, then we no longer have the equality that the total cost to outfit the bombers = $N'\beta = N'c\alpha$ bombers. Instead, the total cost, being α multiplied by the number of aircraft equipped, is

$$\alpha N'[1 + (1 - \lambda'/2)^2 + \ldots + (1 - \lambda'/2)^{2c-2}] \quad ,$$

which can be simplified by using equation (2) to

$$\alpha N' \frac{1 - (1 - \lambda'/2)^{2c}}{1 - (1 - \lambda'/2)^2} \quad .$$

We define β' by

$$\beta' = \alpha \frac{1 - (1 - \lambda'/2)^{2c}}{1 - (1 - \lambda'/2)^2} \quad . \qquad (10)$$

We can then use the analysis above in the case of nonexpendables. We replace β in those formulas by the β' defined in equation (10). The result is not as neat, because β' depends on λ', but the analysis remains valid and may be used.

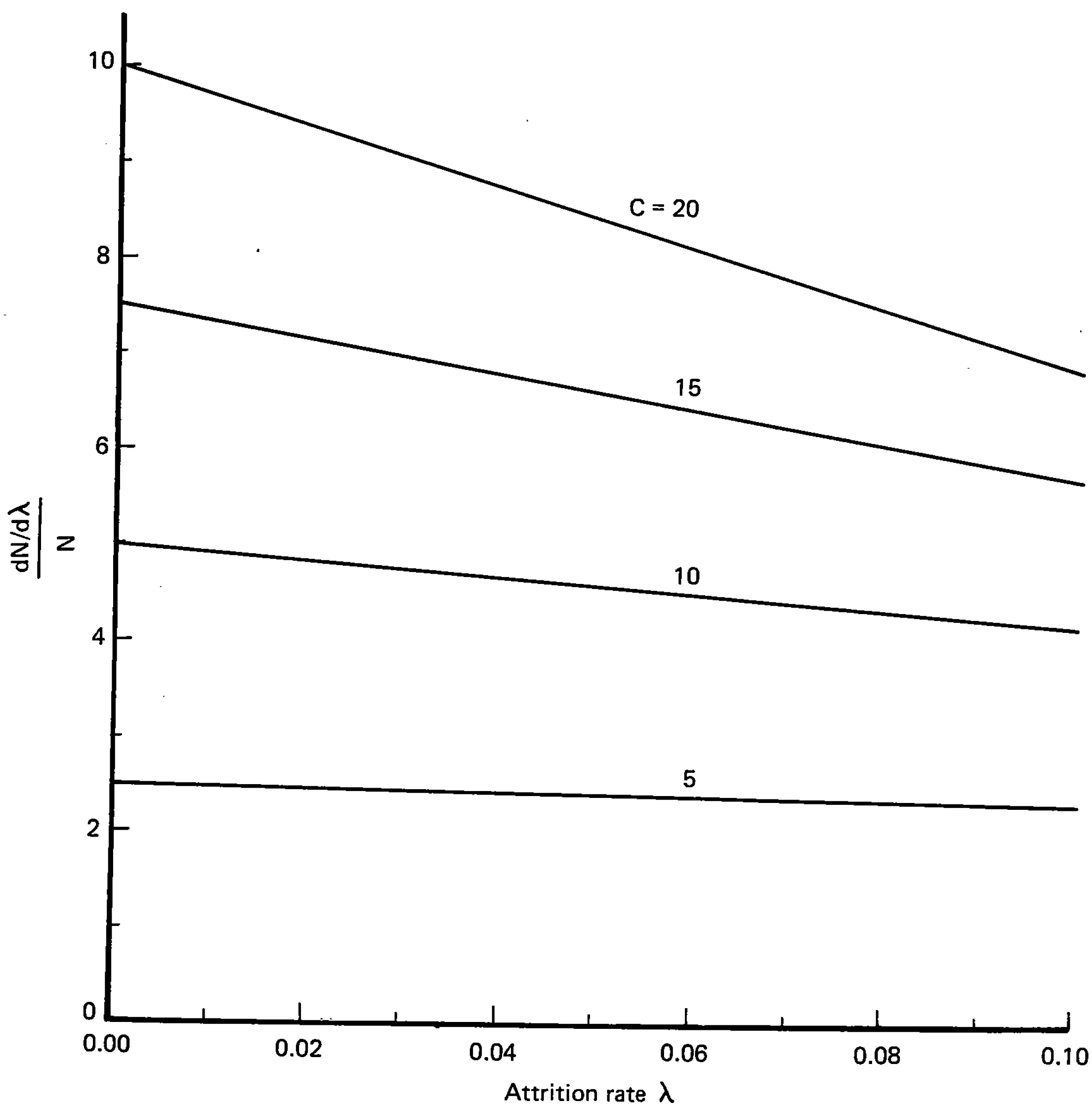

Fig. 5 — Value of $(dN/d\lambda)/N$ for different number of cycles

III. MINIMIZING THE NUMBER OF SHELTERS REQUIRED IN A MULTIPLE AIM POINT DEPLOYMENT UNDER A COOPERATIVE AGREEMENT

A world with mutual assured destuction is stable because neither side by definition has the capability to successfully carry out a disarming first strike. One way to ensure sufficient retaliatory capability is to deploy a large number of missiles so that even in the face of a high casualty rate the remaining missiles and their accompanying warheads suffice for assured destruction. However, the enemy, seeing such a large deployment, would most likely also deploy a large number of missiles to assure his survivability, negating the initial build up and adding another cycle to the arms race.

Missiles with multiple independently targetable reentry vehicles (MIRVs) further impede the quest for survivability by concentrating a number of RVs in a single silo that may be destroyed by a single warhead. Multiple aim points (MAP) operate in the other direction by rarefying the distribution of RVs and providing a means of unilaterally enhancing survivability without increasing the number of RVs.

The following analysis demonstrates the leverage provided by a MAP scheme under a cooperative agreement limiting the number of RVs on each side at a fixed level. Henceforth, all agreements will be assumed to have symmetrical limitations.

With a large number of shelters, the defender can ensure an arbitrarily high survival probability. But, because of cost and space limitations, he would prefer to build as few shelters as possible consistent with the assured destruction task. We consider, then, the problem of minimizing the number of shelters needed to ensure a retaliatory capability at some level, given a cooperative agreement limiting RVs.

The defender's assured destruction plan calls for Q RVs to survive a first strike. If he deploys a number R of RVs, each RV must have a probability of survival P_s satisfying

$$Q = P_s R \quad . \tag{1}$$

If there are A aim points and γ RVs per aim point, equation (1) may be rewritten as

$$Q = P_s \gamma A \quad . \tag{2}$$

Since Q is fixed, to minimize the number of shelters A, the defender will seek to maximize the quantity $P_s \gamma$.

With a cooperative agreement on RVs, the number of incoming RVs will equal the number of RVs that the defender posseses, which is R. Let λ be the single shot kill probability (SSKP) of the attacking RVs. In the case that there are fewer RVs than shelters,

$$P_s = 1 - \gamma\lambda \quad , \tag{3}$$

because the probability that an RV is killed is the product of the probability that it is targeted and the conditional probability that it is killed given that it is targeted. The function that the defender is maximizing,

$$f(\gamma) = P_s \gamma = (1 - \gamma\lambda)\gamma \quad , \tag{4}$$

obtains its maximum when $f'(\gamma) = 0$, which implies that

$$\gamma = 1/(2\lambda) \quad . \tag{5}$$

With this number of RVs per shelter, the optimal defense deployment is:

$$R^* = 2Q \quad , \quad A^* = 4Q\lambda \quad , \quad P_s^* = 1/2 \quad . \tag{6}$$

Thus, to minimize the number of shelters, the defender should deploy twice as many RVs as his retaliatory policy requires in enough shelters to assure a 50 percent survival probability.

Figure 6 illustrates the effect of deploying more or fewer RVs than the optimum for the case $Q = 100$ and $\lambda = 0.8$. With fewer RVs deployed than the optimal 200, the defense needs more shelters than the optimum to meet the higher requisite survival probability. With more than 200 RVs for the attacker as well as the defender under the cooperative agreement, the increased size of the attacker's arsenal is the factor that forces the defender to build more shelters.

If the defender deploys MIRVed missiles in a MAP scheme to make the numbers of RVs greater than the number of shelters, equation (3) above for the probability of survival must be modified to take into account the fact that more than one incoming RV is targeted to a shelter. The attacker will maximize the expected damage by distributing his RVs as evenly as possible to the defender's shelters.

For example, if the attacker has 175 RVs to use against 100 shelters, the maximally distributed attack will target 75 shelters with two RVs and 25 shelters with one RV. In general, when the attacker targets optimally, the probability of survival satisfies

$$P_s = (1 - \lambda)^m \, (1 + \lambda \, \{m - \gamma\}) \quad , \tag{7}$$

where $m = [\gamma]$, the greatest integer less than or equal to γ. Note that when $0 < \gamma < 1$, m is zero, and equation (7) collapses to equation (3), as it should since $\gamma < 1$ means there are fewer RVs than shelters.

Figure 7 illustrates the effect of the modified formula for P_s in the case $Q = 100$ and $\lambda = 0.6$: Cusps are introduced into the curve at the points where A/R is integral. An analysis similar to that above (although slightly more complicated), followed by a comparison of relative minima on different sections of the curve, shows that the optimal number of RVs and shelters is

$$R^* = \frac{2Q}{(1 + n\lambda)(1 - \lambda)^n} \quad \text{and} \quad S^* = \frac{2\lambda R^*}{(1 + n\lambda)} \quad , \tag{8}$$

where the variable n is given by the formula

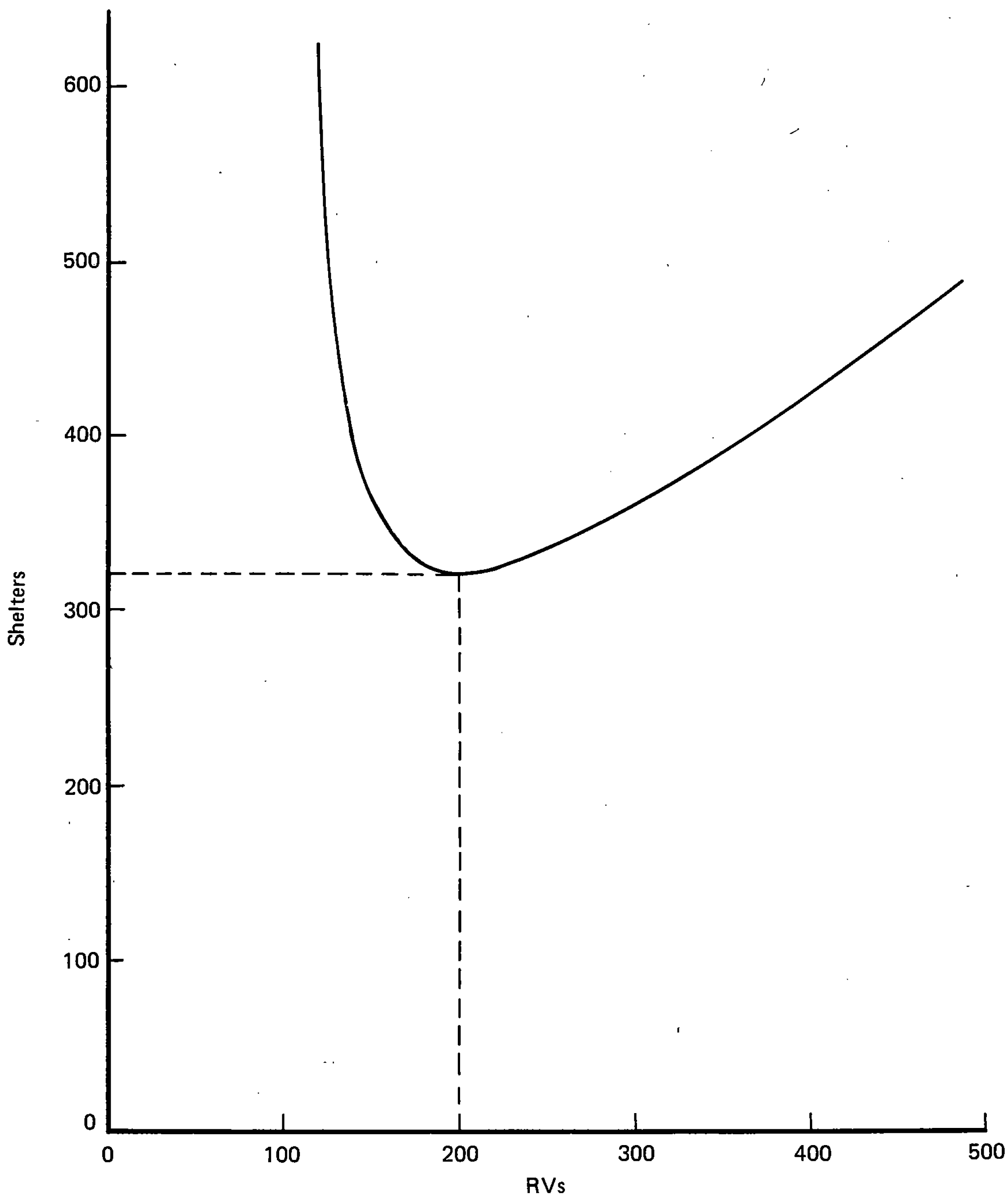

Fig. 6 — Effect of deploying more or fewer RVs than optimum
when 100 RVs are to survive a first strike
and the attacker's SSKP = 0.8

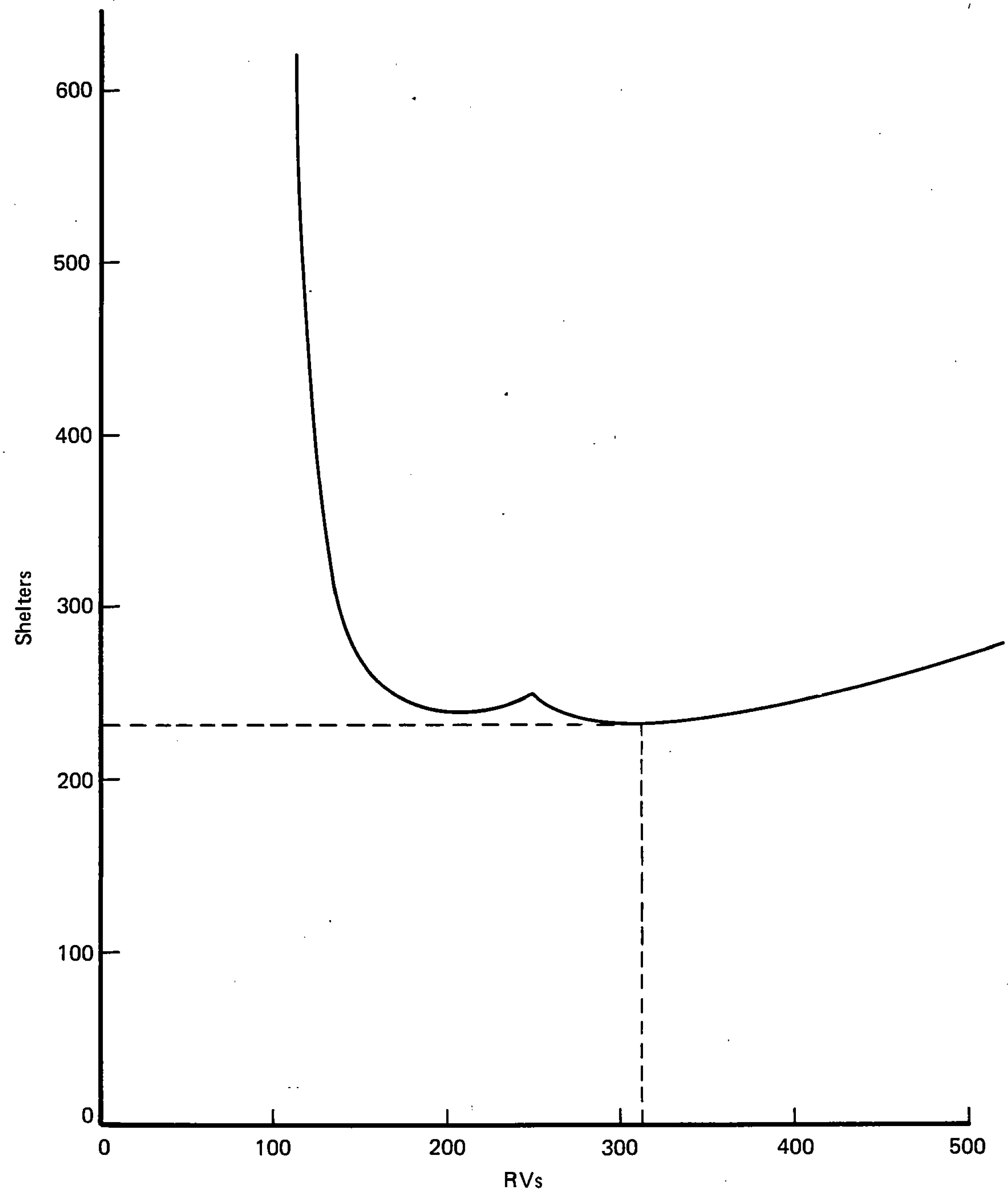

Fig. 7 — Importance of considering cusps for kill probability
of less than 61.8 percent

$$n = [\frac{\sqrt{1 - \lambda}}{\lambda}] \quad . \tag{9}$$

Note that equations (8), as well as equations (6), scale linearly with Q, the retaliatory requirement. So, if the requirement doubles, then the number of RVs and number of shelters also doubles. When n is zero, equations (8) reduce to the previously found optimal values in equations (6). The condition that n = 0 holds provided that

$$\frac{\sqrt{1 - \lambda}}{\lambda} < 1 \quad , \tag{10}$$

which implies that

$$\frac{\sqrt{5} - 1}{2} < \lambda \quad . \tag{11}$$

Thus, the preceding conclusion remains valid provided that the attacker's SSKP is greater than 61.8 percent. If the SSKP is less than this, then the optimal probability of survival is no longer 0.50. Instead, the more general formula

$$P_s^* = \frac{(1 + n\lambda)(1 - \lambda)^n}{2} \tag{12}$$

must be used, along with equations (8) and (9).

To see the leverage provided by a MAP scheme in relation to the traditional basing mode of one silo per missile, assume that the defender uses a single-warhead missile and that the attacker's SSKP is 0.8. Further, assume that the retaliatory requirement is 100 RVs. The optimal MAP deployment is to use 200 RVs in 320 shelters. With the traditional basing mode, 500 RVs would be required in an equal number of silos. The advantage of MAP in reducing the number of weapons is evident. If the defender were to deploy MIRVed missiles with 10 RVs each, the difference would be even greater: He would need nearly one billion RVs in 100 million silos.

IV. MULTIPLE AIM POINTS: LEAST COST VERSUS FIXED THREAT

Even in the absence of a cooperative agreement fixing the number of RVs at equal levels for both sides, as described in Section III, a multiple aim point scheme may still be used to increase survivability and thereby preserve retaliatory resources. This section discusses the problem of determining optimal MAP deployment when the attacking force is postulated at a certain level. As in the preceding section, the defense has an assured destruction plan that requires Q retaliatory RVs for its execution.

The attacker's force is assumed to be capable of destroying K of the defender's shelters. In this situation, the defense will not minimize solely the number of shelters, because that would lead him to deploy an extremely large number of RVs in slightly more than K shelters while still meeting his retaliatory requirement. The defense will instead endeavor to minimize the total cost of the deployment, given that the average cost of an RV is C_R and the average cost of an aim point is C_A.

Let R denote the number of RVs the defender deploys and A the number of aim points. Since the number of aim points the defender builds will certainly be larger than K, the probability of survival satisfies

$$P_s = 1 - \frac{K}{A} \quad . \tag{1}$$

In order to have the required number Q of RVs survive, the defense's number of RVs (R) must be given by

$$R = \frac{Q}{P_s} = \frac{QA}{A - K} \tag{2}$$

from equation (1). The total cost to the defense is the sum of the costs for RVs and shelters:

$$C_{Total} = RC_R + AC_A = \left(\frac{QA}{A - K}\right)C_R + AC_A \quad . \tag{3}$$

By taking the derivative of this expression with respect to A and setting it equal to zero, we find that the cost is minimized when

$$A^* = K + \sqrt{KQ}\ \sqrt{C_R/C_A} \quad \text{and} \quad R^* = Q + \sqrt{KQ}\ \sqrt{C_A/C_R} \quad . \tag{4}$$

As a numerical example, consider the case where the attacker can destroy 1000 shelters, the defense needs 100 RVs to survive, and a shelter costs twice as much as an RV. Then, for the least cost to the defense

$$A^* = 1224 \quad \text{and} \quad R^* = 548 \quad .$$

This analysis can also be applied to determining the optimal defense deployment to minimize the number of shelters plus the number of RVs--a formula that might have application in an arms control agreement. In equation (3), the total cost will equal R + A if we take $C_R = C_A = 1$. Then equations (4) simplify to

$$A^* = K + \sqrt{KQ} \quad \text{and} \quad R^* = Q + \sqrt{KQ} \quad . \tag{5}$$

Dividing the second equation by the first and canceling common factors, we find that

$$R^*/A^* = \sqrt{Q/K} \quad . \tag{6}$$

From this formula, we can easily derive the optimal number of shelters and RVs in a particular case. For example, if the attacker can destroy 1600 shelters and the retaliatory requirement is 100 RVs, we see from equation (6) that the defense should use four aim points for every RV. The first 1600 shelters--containing 400 RVs--will be destroyed. In order to have 100 RVs for retaliation, the defense must deploy 400 additional shelters. Thus, the total number of shelters is 1600 + 400 = 2000, and the total number of RVs is 400 + 100 = 500. The direct use of equations (5) will yield the same values.

V. OPTIMAL SIZE OF BOMBER PAYLOAD

The assured destruction task requires that A_0 reliable short-range attack missiles (SRAMs) be delivered to the target. For a given class of bombers, the cost of operating a bomber equipped with W SRAMs is approximately

$$C_{a/c} = K_1 + K_2W \quad . \tag{1}$$

K_1 represents the cost of the bomber, which is fixed for bombers of the class under discussion, and K_2 denotes the cost of an individual SRAM. If the attacking force consists of B bombers carrying W SRAMs each, then the cost for the whole force is

$$C_{Total} = B(K_1 + K_2W) \quad . \tag{2}$$

The problem at hand is to determine the optimum number of SRAMs for the offense to deploy per aircraft when limited by a fixed budget. Here optimality means choosing the number of SRAMs that forces the defense to spend the greatest amount of money to deny the attacker his assured destruction objective of A_0 SRAMs on target.

Faced with the offense's threat of B bombers and W SRAMs, the defense will allocate his money between area defense S_A and terminal defense S_T,

$$S = S_A + S_T \quad , \tag{3}$$

in such a manner as to minimize his total expenditure S. For a variety of attack/defense scenarios, the effectiveness of area defense can be approximated by the formula

$$P_s = e^{-[\rho I/B]} \quad , \tag{4}$$

where P_s is the fraction of bombers surviving the area defense, I is the

number of interceptors used against the bombers during the attack, and ρ is a parameter measuring the kill potential, or effectiveness, of each interceptor. The value of ρ depends on a large number of parameters, including the types of interceptors and bombers, the sensitivity of the defensive acquisition equipment, the geography of the attack arena, and the tactics of the offense and defense.

If one interceptor costs the defense C units, then the number of SRAMs penetrating the area defense is

$$BWe^{-[\rho S_A/CB]} \quad , \tag{5}$$

since BW is the total number of SRAMs and $I = S_A/C$. To deny the offense his assured destruction objective, the terminal defense must negate infinitesimally more than

$$BWe^{-[\rho S_A/CB]} - A_0 \tag{6}$$

SRAMs. If destroying a single SRAM at the terminal end costs T units, then the expenditure on terminal defense, accordingly, must be

$$S_T = T(BWe^{-[\rho S_A/CB]} - A_0) \quad . \tag{7}$$

The defense's total cost is the sum of the costs for area and terminal defense, which from the previous equation is

$$S = S_A + TBWe^{-[\rho S_A/CB]} - A_0T \quad . \tag{8}$$

The defense will minimize the cost in (8) by choosing the optimal S_A, subject to the constraints that S_A and S_T are greater than or equal to zero. Setting the derivative of S with respect to S_A equal to zero, we find that the minimum occurs when the area defense is

$$S_A = \frac{BC}{\rho} \ln \left(\frac{\rho TW}{C}\right) \quad , \tag{9}$$

and the terminal defense is

$$S_T = \frac{BC}{\rho} - A_0T \quad , \tag{10}$$

which implies that the total cost is

$$S = \frac{BC}{\rho} \left[1 + \ln\left(\frac{\rho TW}{C}\right)\right] - A_0T \quad . \tag{11}$$

This is the defense's cost to thwart an attack of B bombers equipped with W SRAMs each. The number of interceptors is S_A/C, so using equations (9) and (4), we find that the probability of survival is $C/\rho TW$.

The attacker, knowing the optimal defense deployment above, will try to maximize the defense's total cost by choosing the best mix of bombers and SRAMs within a given budget. With a budget of Λ units, the offense will be able to deploy

$$B = \frac{\Lambda}{K_1 + K_2W} \tag{12}$$

bombers, each equipped with W SRAMs, since we know from equation (1) that the cost of one loaded bomber is $K_1 + K_2W$. Substituting (12) into (11), we can rewrite the cost to the defense as

$$S = \frac{\Lambda C}{\rho} \left[\frac{1 + \ln(\rho TW/C)}{K_1 + K_2W}\right] - A_0T \quad . \tag{13}$$

At the maximum, $dS/dW = 0$, implying that W^*, the optimal number of SRAMs per bomber, satisfies

$$W^* = \frac{C}{\rho T} e^{[(K_1/K_2)/W^*]} \quad . \tag{14}$$

This transcendental equation does not have a closed form solution for W^* in terms of the elementary algebraic, trigonometric, exponential, and logarithmic functions. Instead, it must be solved numerically or with the aid of a nomogram as in Figure 8. To use Figure 8, one locates the input values of K_1/K_2 and $C/\rho T$ along the ordinate and abscissa, respectively, and then finds the value of W^* among the family of curves. For example, if $K_1/K_2 = 32$ and $C/\rho T = 7$, the optimum value of W^* is slightly larger than 25.

Figure 8 also demonstrates some qualitative facts about the bomber-SRAM tradeoff. If the price of SRAMs with respect to bombers decreases (meaning that K_1/K_2 increases), while the defensive parameters do not change, then the optimal number of SRAMs per bomber increases, and vice versa. If the area defense becomes more effective relative to the terminal defense, either by decreasing the cost C of an interceptor or increasing the kill potential ρ, then the optimal number of SRAMs per bomber decreases.

This analysis also exposes the marginal effectiveness of increasing the number of bombers. If the attacker adds one additional loaded bomber, then, from equations (9) and (10), the defense will have to increase his area defense expenditure by $(C/\rho)(\ln[\rho T W^*/C])$ and his terminal defense expenditure by C/ρ to negate the SRAMs aboard the additional bomber. Since the offense is deploying the optimal number of SRAMs per bomber, using equation (14), we simplify the sum of these two numbers to

$$\Delta S = \frac{C}{\rho}\left(\frac{K_1/K_2}{W^*} + 1\right) \tag{15}$$

or

$$\Delta S = \frac{C}{\rho K_2 W^*}(K_1 + K_2 W^*) \quad . \tag{16}$$

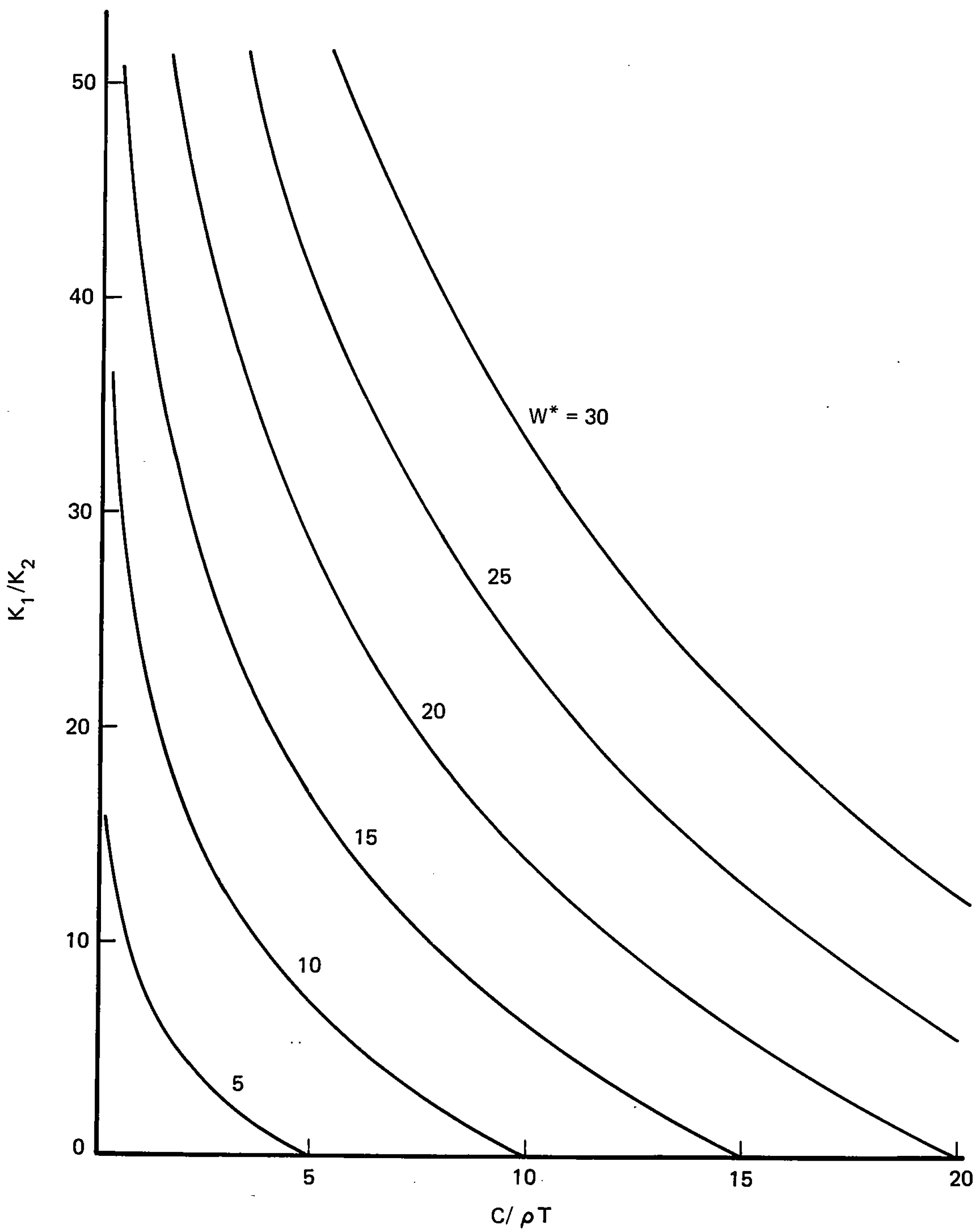

Fig. 8 — Nomogram for solving equation (14) to determine optimal number of SRAMs per bomber

The term $(K_1 + K_2W^*)$ in equation (16) is the price of the offense's optimally loaded bomber. This observation allows us to conclude that the factor $C/\rho K_2W^*$ is the amount that the defense must spend on the margin to negate a unit increase in the offense's budget. Thus, at high levels of spending where average cost is nearly equal to marginal cost, the defense spending will be $c/\rho K_2W^*$ times the offense spending.

One small adjustment remains to this analysis because of the fact that the defense may not spend a negative amount of money on either of its components. The two constraints to the optimization problem posed in equation (8) affect the previously obtained solution in the case of a small offense budget, as shown below. Viewing the solution obtained for the free optimization, equations (9) and (10), one can see that the two constraints translate, respectively, into

$$\frac{C}{\rho T} \leq W \tag{17}$$

and

$$A_0 \frac{T\rho}{C} \leq B \quad . \tag{18}$$

In the region of Ws and Bs where these two inequalities hold, the above analysis is applicable and the optimum for W is given by equation (14).

If the inequality in (17) is violated, then the defense will use no area defense and put all of his resources into terminal defense. In this case, an analysis similar to the above shows that the offense, when constrained within a budget, does best by deploying as many SRAMs as possible per bomber consistent with violating (17), i.e., $W^* = C/\rho T$. A direct computation shows that this optimization, however, forces a lesser cost on the defenses than does the optimization with W^* satisfying equation (14), so the best value for W is given by the solution to equation (14).

In the other case, if inequality (18) is violated, the defense will use no terminal defense, since the marginal cost of negating a SRAM with area defense is less than T up to the budget constraint. The optimal number of SRAMs per bomber, W^*, within a budget of Λ then satisfies the transcendental equation

$$\frac{\Lambda W^{*}}{A_0} = e^{[(K_1/K_2)/W^{*}]}(K_1 + K_2W^{*}) \tag{19}$$

A short calculation shows that the value of W^{*} satisfying this equation will force a higher cost to the defense than that found by using the W^{*} satisfying (14) only in the case that the offense's budget is less than

$$A_0\frac{T\rho}{C}(K_1 + K_2W^{*}) \quad . \tag{20}$$

To summarize, if the offense's budget is limited to be less than this amount, then the optimum number of SRAMs per bomber is given by the solution to (19). Otherwise, the constraints do not affect the analysis and the optimum number of SRAMs is that given previously by the solution to equation (14).

VI. MUNITIONS PATTERN SIZE

The pattern of a cluster bomb carrying a number of bomblets N dropped from an aircraft is approximately a circle with a radius that we can take to be W meters. By adjusting a mechanism on the bomb that controls its angular velocity in free fall, the bombardier can alter the pattern radius of the bomb. With little angular velocity, the bomblets will be grouped tightly together; with a great deal of spin, the bomblets spread across a large area. The bombardier will set the value of W that maximizes the probability of a bomblet hitting the target, given the relevant characteristics of the target and the bomb.

The bomblets are independent of one another once they are free of the bomb casing, so they will disperse according to a uniform probability distribution within the circular pattern. The probability that the bomb destroys a fixed single-point target on the ground, such as a tank or a radar van, is the product of two probabilities: first, that the target is included in the circular pattern and second, that the target is destroyed, given that it is within the pattern.

The aiming error of the bomb can be modeled by a bivariate circular normal distribution whose dispersion is measured by the circular error of probability (CEP). The first probability above, then, is that the center of the pattern is within W meters of the target, namely,

$$1 - e^{-[\ln 2 (W/CEP)^2]} \quad . \tag{1}$$

The second probability is the complement of the probability that all of the bomblets miss the target. If the effective target area is A square meters, then the conditional probability of survival is

$$\left(1 - \frac{A}{\pi W^2}\right)^N \quad . \tag{2}$$

Under the circumstances of concern, the ratio $A/\pi W^2$ is small, so the above expression may be approximated by

$$e^{-[AN/\pi W^2]} \quad . \tag{3}$$

Thus, the probability of destroying the target P_k as a function of W, the pattern radius, is

$$P_k = (1 - e^{-[\ln 2\{W/CEP\}^2]})(1 - e^{-[AN/\pi W^2]}) \quad . \tag{4}$$

The optimum W is one that maximizes the probability of kill; the analysis below demonstrates that the optimum W satisfies

$$W^{*4} = \frac{AN(CEP)^2}{\pi \ln 2} \quad . \tag{5}$$

Directly attacking equation (4) as an exercise in one-variable calculus, by setting the first derivative equal to zero, leads to lengthy equations that are not amenable to solution. To avoid such impediments, we simplify the problem by introducing two new variables and considering a related two-variable optimization problem with one constraint. The new function to maximize is

$$f(x,y) = (1 - e^{-x})(1 - e^{-y}) \tag{6}$$

subject to the one constraint that

$$g(x,y) = xy - K = 0 \quad , \tag{7}$$

where K is a positive constant and x and y must be positive. Such a formulation is equivalent to the one-variable problem in equation (4) through the substitutions

$$x = \ln 2 \left(\frac{W}{CEP}\right)^2 \tag{8}$$

and

$$y = \frac{AN}{\pi W^2} \quad . \tag{9}$$

There is no maximum on the boundary at infinity because, owing to the constraint, as one variable becomes arbitrarily large, the other becomes arbitrarily small, and the probability of kill approaches zero. Therefore, the standard Lagrange multiplier technique is applicable; when applied to (6) and (7), the technique shows that at an interior critical point

$$\frac{e^x - 1}{x} = \frac{e^y - 1}{y} = \lambda e^{x+y} \quad . \tag{10}$$

The function $(e^x - 1)/x$ is strictly increasing for positive arguments, so the first equality in (10) implies that $x^* = y^*$. By equating the expressions for x and y in equations (8) and (9), we arrive at the optimal value for W shown in equation (5).

As a numerical example, consider an attack against a single isolated tank of vulnerability area A = 15 square meters by a cluster bomb carrying 250 bomblets and having a CEP of 50 meters. From equation (5), the optimal pattern radius is

$$W^* = \left(\frac{15 \cdot 250 \cdot (50)^2}{\pi \ln 2} \right)^{1/4} = 45 \text{ meters} \quad . \tag{11}$$

The kill probability for the 45-meter-pattern radius is computed using equation (4) to be

$$P_k = 19\% \quad . \tag{12}$$

VII. OPTIMIZING THE CAPABILITY OF LAYERED DEFENSE

If the defense has a given capability to intercept incoming missiles and can deploy the interceptors in an n-layered defense with "unordered" fire in each layer, then there is a unique optimal way to proportion the intercept capability among the layers so as to minimize the number of missiles penetrating the defense. Unordered fire means that each interceptor in a layer is fired randomly at one of the attacking missiles that have penetrated the previous layers of defense.

The unit of account for the defense potential is "intercepts," which is the sum over all interceptors of the product of reliability and the SSKP of the interceptor. For example, if the defense has 1000 interceptors, each with a reliability of 0.9 and SSKP versus attacking missiles of 0.8, the defense potential would be 1000 x 0.9 x 0.8 = 720 intercepts.

If the interceptors have a reliability of r and an SSKP of λ, the probability of a given missile surviving the fire of a given reliable interceptor is $1 - \lambda/M$. The compound probability of the missile surviving the rI reliable interceptors is thus

$$P_s = (1 - \lambda/M)^{rI} \quad , \tag{1}$$

which can be written as

$$P_s = ([1 - \lambda/M]^{M/\lambda})^{Ir\lambda/M} \quad . \tag{2}$$

For large M, the base of the exponent $Ir\lambda/M$ is close to 1/e. In fact, for M equal to or greater than 20, it is accurate to at least two decimal places. Since we shall be considering large M, equation (2) becomes

$$P_s = e^{-Ir\lambda/M} \quad . \tag{3}$$

Note that the product $Ir\lambda$ in the exponent is what we are calling intercepts.

Let x_i denote the number of intercepts deployed in the i^{th} layer, and define M_j ($j = 1, 2, \ldots, n$) recursively by

$$M_0 = M$$

and

$$M_j(x_1, \ldots, x_j) = M_{j-1} \, e^{-[x_j/(M_{j-1})]} \quad . \tag{4}$$

M_j is the expected number of missiles penetrating the first j layers of the defense, if the attacker's original arsenal is M missiles.

Each of the M_js, with the exception of M_0, is the expected value of a random variable ranging from zero to M. Ideally, we should keep track of the complete distribution of the random variables and take an expectation value of $\underline{M}_n$, but for our problem, the stepwise expectation value model is only slightly inaccurate.

Suppose that at one layer of the defense the number of intercepts is I and the random variable indicating the number of incoming missiles is $\underline{M}$ which has an expected value which we denote by $\widehat{\underline{M}}$. Define the function f(x) by

$$f(x) = x \, e^{-I/x} \quad . \tag{5}$$

Our analysis will compare $f(\widehat{\underline{M}})$ with $\widehat{f}(\underline{M})$. We expand $f(\underline{M})$ as a power series about $f(\widehat{\underline{M}})$ to second order as

$$f(\underline{M}) = f(\widehat{\underline{M}}) + (\underline{M} - \widehat{\underline{M}})f'(\widehat{\underline{M}}) + (\underline{M} - \widehat{\underline{M}})^2 f''(\widehat{\underline{M}})/2 + \ldots \tag{6}$$

and take expected values of both sides to arrive at

$$\widehat{f}(\underline{M}) = f(\widehat{\underline{M}}) + \widehat{(\underline{M} - \widehat{\underline{M}})^2} f''(\widehat{\underline{M}}) + \ldots \quad . \tag{7}$$

Simplification occurs because the second term on the right-hand side of equation (6) is eliminated, owing to the fact that $\widehat{(\underline{M} - \widehat{\underline{M}})} = 0$.

We can model $\underline{M}$ with a standard Poisson distribution, truncate the power series in equation (7) after the second term, and compute the second derivative of f to arrive at

$$\hat{f}(\underline{M}) = f(\hat{\underline{M}})(1 + I^2/2\hat{\underline{M}}^3) \quad , \tag{8}$$

since $f''(x) = I^2/x^3 \; e^{-I/x}$ and $\sigma^2\underline{M} = \hat{\underline{M}}$, for a Poisson distribution. For the situations considered in this problem, the ratio $I^2/2\hat{\underline{M}}^3$ will be small, so the expected value model is a good approximation.

In an n-layered defense, the problem is to minimize the function $M_n(x_1, \ldots, x_n)$ subject to the constraint that

$$g(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n - I = 0 \quad , \tag{9}$$

and that each x_i be nonnegative.

The Lagrange multiplier technique dictates that at the minima,

$$\nabla M_n = \lambda \nabla g \quad , \tag{10}$$

for some constant λ. There is no problem with the boundary, since if one of the x_i's were zero, we could increase the defense effectiveness by taking an interceptor from an adjacent layer and placing it in the previously vacant layer; in any configuration, two layers are always better than one. Since the gradient of g is

$$\nabla g = (1, 1, \ldots, 1) \quad , \tag{11}$$

the system of equations in (10) becomes

$$\frac{\partial M_n}{\partial x_i} = \frac{\partial M_n}{\partial x_j} \tag{12}$$

for all i and j less than or equal to n. Using the multivariable chain rule,

$$\frac{\partial M_n}{\partial x_i} = \left(\prod_{k=i+1}^{n} \frac{\partial M_k}{\partial M_{k-1}} \right) \frac{\partial M_i}{\partial x_i} \quad . \tag{13}$$

By equating the derivatives of M_n with respect to x_i and x_{i-1} and canceling nonzero derivative factors, we find that equation (12) reduces to

$$\frac{\partial M_i}{\partial x_i} = \frac{\partial M_i}{\partial M_{i-1}} \frac{\partial M_{i-1}}{\partial x_{i-1}} \quad . \tag{14}$$

From the definition of the functions M_j, equation (14) becomes, upon substituting the expressions for the derivatives,

$$-e^{-[x_i/M_{i-1}]} = e^{-[x_i/M_{i-1}]} \left(1 + \frac{x_i}{M_{i-1}}\right) \left(-e^{-[x_{i-1}/M_{i-2}]}\right) \quad . \tag{15}$$

Equation (15) simplifies to

$$1 = \left(1 + \frac{x_i}{M_{i-1}}\right) e^{-[x_{i-1}/M_{i-2}]} \quad . \tag{16}$$

We know from equation (4) that

$$\frac{M_{i-1}}{M_{i-2}} = e^{-[x_{i-1}/M_{i-2}]} \quad , \tag{17}$$

which, when substituted into (16), yields

$$x_i = M_{i-2} - M_{i-1} \quad . \tag{18}$$

Thus, to maximize the effectiveness of the defense, the number of intercepts deployed in the i^{th} *layer should equal the expected number of missiles destroyed by the* $(i-1)^{th}$ *layer.* Figures 9, 10, and 11 illustrate the consequenses of this statement for the cases of two-layer, three-layer, and four-layer defenses. To use the figures, one calculates the defense potential in terms of intercepts, divides it by the number of attacking missiles, and locates the resulting ratio along the x-axis. Reading vertically from that point, one can determine the various percentages of intercepts to deploy in each layer.

We were able to calculate these curves by a simple algorithm. For example, in the three-layer case, we assume a value for x_1 and calculate values for x_2 and x_3 from equation (18). This gives us the optimal allocation for $I = x_1 + x_2 + x_3$ intercepts. To get an optimal allocation for a larger value of I, we start the process with a larger value of x_1.

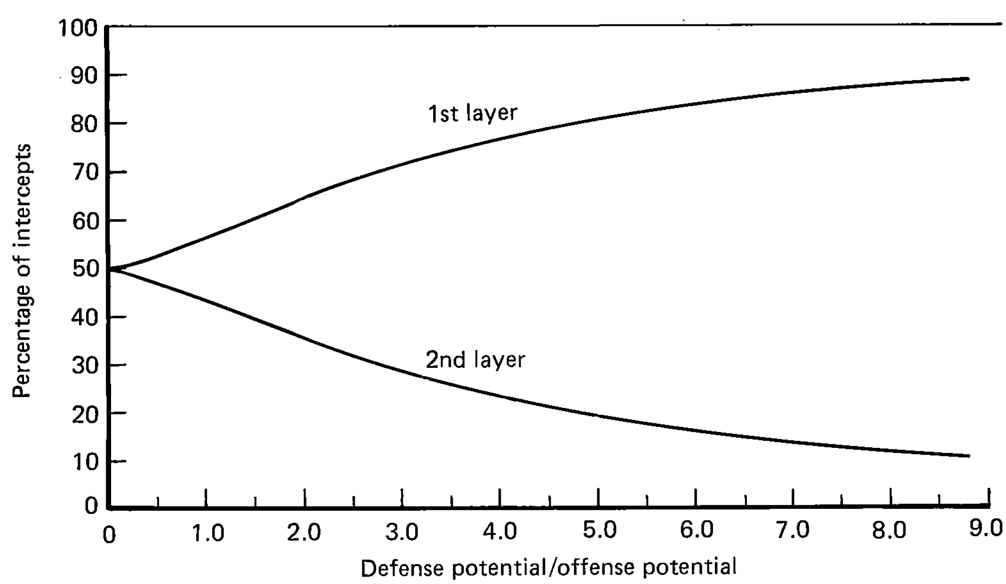

Fig. 9 — Proportioning intercepts in two-layer defense

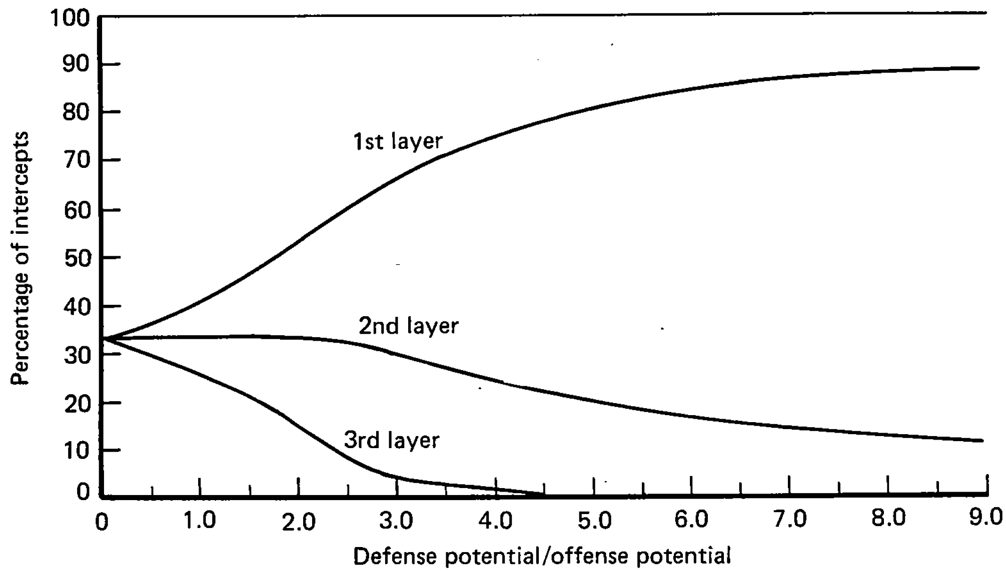

Fig. 10 — Proportioning intercepts in three-layer defense

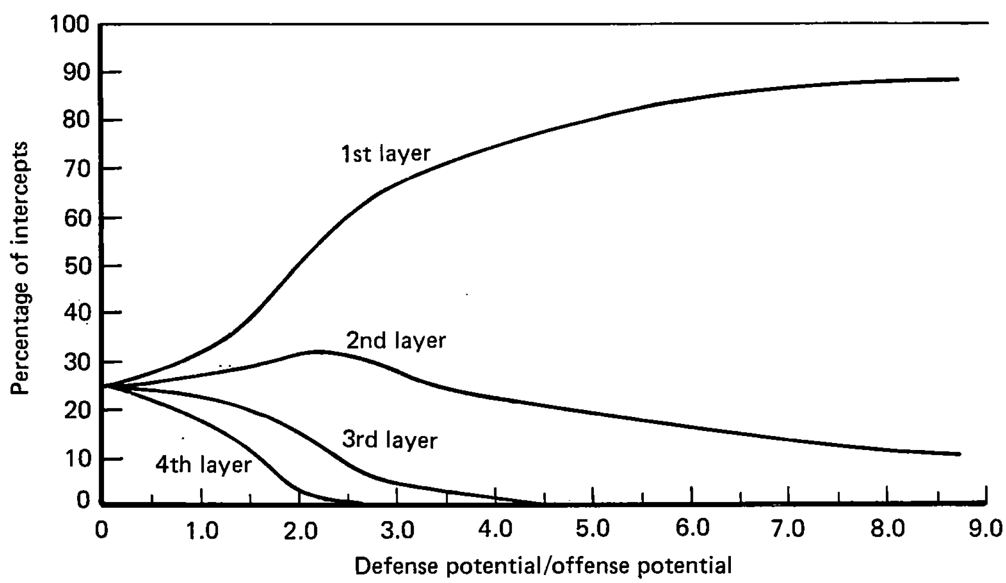

Fig. 11 — Proportioning intercepts in four-layer defense