A RAND NOTE

THE DISTRIBUTION OF POWER
IN THE U.S. SUPREME COURT

Arthur Q. Frank, Lloyd Shapley

July 1981

N-1735-NSF

Prepared For

The National Science Foundation

Rand
SANTA MONICA, CA. 90406

The materials incorporated in this work were developed with the financial support of the National Science Foundation under Grant SES 77-23676.

Published by The Rand Corporation

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PREFACE

This Note was written under a grant from the National Science Foundation for investigations into political science using methods of game theory (SES 77-23676). It describes the first application of a new, "attitude-dependent" measure of voting power to a real political institution, namely the 1978-79 term of the U.S. Supreme Court. It was presented at the 1981 Annual Meeting of the Midwest Political Science Association in Cincinnati, Ohio, April 16-18, 1981.

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SUMMARY

The distribution of power among the nine justices of the U.S. Supreme Court is calculated using techniques of factor analysis in conjunction with a generalized Shapley-Shubik power index that takes into account the ideological or philosophical profiles of the voters. A three-dimensional profile space is constructed, based on data from the 1977-78 term. It is found that, because of his central position in this space, Justice Powell has the highest probability--about 35 percent--of being pivotal in a typical decision.

ACKNOWLEDGMENTS

The authors would like to acknowledge the assistance given by Michael Miller and Daniel Shapiro at different points in the computation.

CONTENTS

1. Introduction

Since Shapley and Shubik's seminal paper in 1954, mathematicians, political scientists, and others have used power indices to study the distribution of power in political institutions such as the U.S. Electoral College (Mann and Shapley, 1962; Banzhaf, 1968; Owen, 1975), the United Nations Security Council (Riker and Ordeshook, 1973), the county legislatures of New York State (Banzhaf, 1965; Imrie, 1973), and the proposed new Canadian constitution (Miller, 1973; Straffin, 1977). Both the Shapley-Shubik (S-S) and the Banzhaf power indices, which have been used to study these institutions, depend only on an institution's voting system, not on any political factors, such as the ideologies or predispositions of the members of the voting body.

Owen (1971) and Shapley (1977) have developed slightly different "nonsymmetric" generalizations of the S-S index which measure the power of individual members of a voting body, given prior knowledge of each member's ideological attitudes or predilections. To date, no one has published any application of these indices to the distribution of power in a real political institution. This paper is a first attempt to do so. We use the voting records of the nine Supreme Court justices to estimate their ideological positions, and then apply the new, attitude-dependent index to estimate the distribution of power in the current Supreme Court.

2. The Generalized S-S Index

Consider a simple voting game with a finite set of voters $N = \{1, 2, ..., n\}$ whose "voting rule" is defined by W, the set of all subsets of N which are winning coalitions. Suppose that each issue ranks the voters in order of the degree of their support; the most dedicated advocates first, the less fervid supporters next, then the most persuadable opponents, and so on, down to the most stubborn opponent at the end of the list. For such an ordering of the voters, the person who, together with his predecessors, forms a minimal winning coalition is called the *pivot*. The pivot is the one whose preferences decide how strong a bill gets passed, or how much money actually gets appropriated for a given program. The S-S index of power is defined to be the fraction of orderings for which each voter is pivotal; thus, a voter's power is calculated simply as the number of orderings for which he is the pivot, divided by n!, the total number of orderings. In other words, a voter's power may be regarded as the probability of his being the pivot, given that all orderings are equally likely to occur.

For an abstract measure of constitutional power in a voting system, the S-S index may be very appropriate. If one only wants to measure how the voting system itself distributes power *a priori*, without considering any information one might have about the individual voters in the system, the assumption of equiprobable permutations is unassailable. But if one wants to measure the actual distribution of power in a particular body at a particular time, the information that one has about the political beliefs of the voters should enable one to make better estimates of their probabilities of being pivotal.

For example, if one considers the U.S. Electoral College as a weighted majority game with 51 players (the 50 states and the District of Columbia), then the S-S index yields the same power for Delaware as it does for the District of Columbia, since each has three electoral votes. But Delaware, a politically marginal state which has been carried by the winning presidential candidate in the last eight elections, is much more likely to be pivotal in the 1984 presidential election than is the heavily Democratic District of Columbia, with its predominantly Black electorate. The nonsymmetric, generalized S-S index differentiates among voters according to such identifiable political attributes of the electorate (Shapley, 1977).

Suppose that the members of a voting body all make their voting decisions on the basis of a few underlying political attitudes. If there are m different attitude scales, then we can represent each voter by a point in $\mathbb{R}^m$ [= m-dimensional Euclidean space], sometimes called a "political profile," where each coordinate represents where the player falls on that attitude scale. Now we want to consider how an issue, arising at random, might align the voters in order of their intensity of support or opposition. We represent issues generally as real-valued functions on the profile space $\mathbb{R}^m$, with the interpretation that voter i precedes voter j in the order of support of issue f if $f(x^i) > f(x^j)$, where x^i and x^j denote the profiles of voters i and j, respectively. For convenience and interpretability, we restrict ourselves to homogeneous linear functions on $\mathbb{R}^m$, i.e., functions of the form

$$f(x) = (s, x)$$

[$= \Sigma\, s_i x_i$, the inner product of the m-dimensional vectors s and x]. Thus, issues will be represented by m-dimensional vectors s, and the <u>issue space</u> is the set of all such vectors, in other words, $\mathbb{R}^m$ again. (Note that although the issue space and the profile space are isomorphic, they should not be regarded as the same space. For example, the origin in the profile space has no particular significance, but in the issue space it is the center that determines the direction in which each issue "points.")

We have, then, the definition: $i \underset{s}{<} j$ (read "voter i is more supportive of issue s than voter j") if and only if $(s, x^i) > (s, x^j)$. We think of issues as representing linear combinations of the underlying attitudes in accordance with which voters make their decisions. For example, if we have a voting body in which voters decide how to vote according to their views on only two scales, a liberal/conservative scale and an urban/rural scale, then s = (1, 0) would represent a purely liberal/conservative issue, s = (0, 1) a purely urban/rural issue, and s = (2, 1) an issue in which the liberal and urban goals are in opposition to the conservative and rural goals, with the liberal/conservative factor twice as important as the urban/rural factor. Under this interpretation, only the direction of an issue matters, not its magnitude and so (if we ignore the "null" issue s = 0) we can normalize by $\| s \| = 1$, where $\| \cdot \|$ represents the Euclidean norm. With this assumption, the inner product (x^i, s) has a simple geometrical interpretation: it is the signed distance along the ray $\overrightarrow{0s}$ between the origin 0 and the perpendicular projection of x^i onto that ray. The order of these projections along the ray $\overrightarrow{0s}$ gives the ordering of the voters in support of s. (See Figure 1.) If we assume the profiles x^i to be all distinct points, then with negligible exceptions each s will in fact generate a complete ordering, i.e., one without ties.

As before, for each ordering of the voters, the voting rule will yield a unique voter who is pivotal. For a given s and for each i, define $P_i(s)$ to be the set of voters who are more supportive than i of issue s:

$$P_i(s) = \{j \in N: \ j \underset{s}{<} i\} .$$

Each voter i will then have a region S_i on the surface of the unit sphere $S = \{s \in \mathbb{R}^m: \|s\| = 1\}$, consisting of those normalized issues for which he is pivotal:

$$S_i = \{s \in S: P_i(s) \notin W \text{ and } P_i(s) \cup \{i\} \in W\} .$$

If we assume that the issue direction is a random variable uniformly distributed over S, then i's probability of being pivotal is

$$\varphi_i = \frac{\text{surface area of } S_i}{\text{surface area of } S} .$$

This will be taken as the definition of the "attitude-dependent" (AD) power index that we shall use.

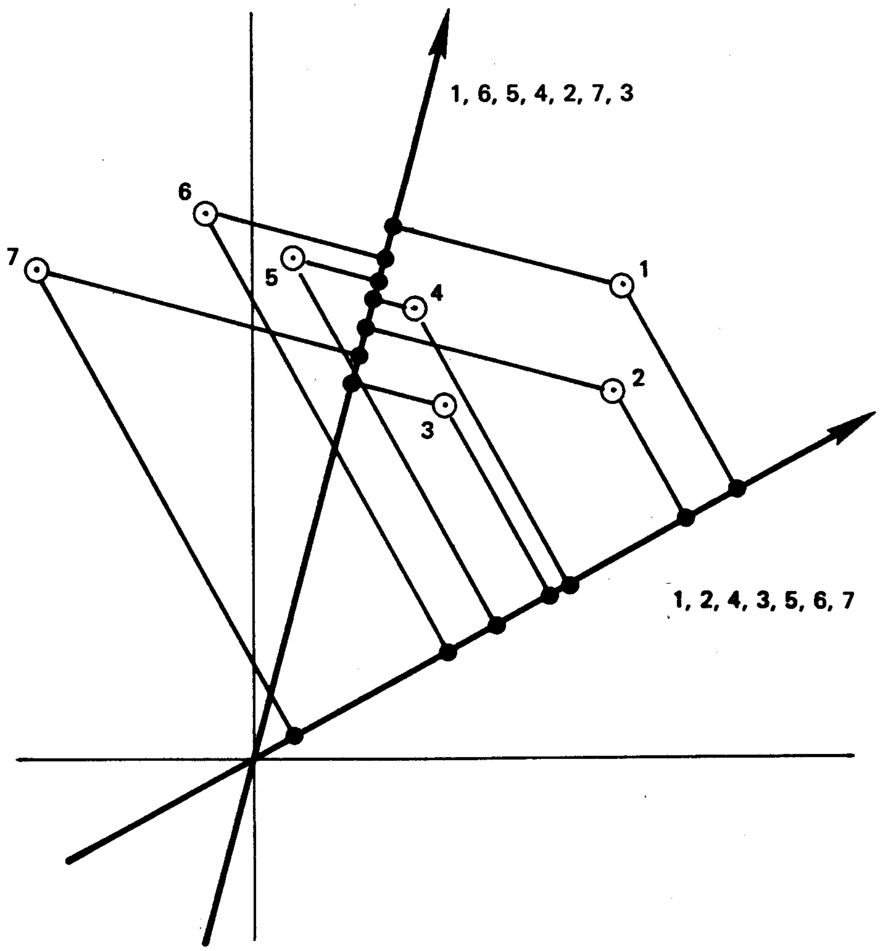

Fig. 1 — Orders generated by two issues in $\mathbb{R}^2$

That this index generalizes the original, symmetric Shapley-Shubik index may be seen as follows. Take the dimension of the profile space equal to the number of voters: $m = n$, and take the profiles themselves to be the unit vectors of $\mathbb{R}^m$. (In this representation, we may think of each coordinate of $\mathbb{R}^m$ representing the corresponding voter's private welfare.) Then, by symmetry, each ordering of the voters is equally likely, and so the AD power index just defined will coincide in this case with the original, symmetric Shapley-Shubik index.

In some applications of the AD power index, the assumption that issue directions are distributed uniformly over the unit sphere may be unreasonable. In this case, one might seek a linear transformation T of the given issue space that yields a new space in which spherical uniformity would be a more reasonable assumption. (Note that linear transformations will generally distort the unit sphere into some sort of ellipsoid, so that "spherical uniformity" is not an invariant quality.) A simple example of such a transformation would be to rescale one of the "ideological" coordinates and thereby increase or decrease its relative importance. Other linear transformations might be employed to compensate for intrinsic correlations between the different coordinates. Applying T to the issue space is equivalent to applying T^{-1} to the profile space. Thus, we can get the effect of any "elliptically uniform" probability measure on the space of issue directions by applying a suitable linear transformation to the profiles

x^i instead, and then using ordinary spherical measure on the resulting transformed data. (See the second appendix in Shapley (1977).)

3. Power in the United States Supreme Court

The Supreme Court is a nine-member voting body which operates under simple majority rule, so that for any ordering of the justices, the one ranked fifth is the pivot. For an extreme example, if the dimension of the attitude space were just 1, so that the justices always considered cases in purely liberal/conservative terms, then the median justice in the center of the liberal/conservative ordering would have all the power. Thus, for $m = 1$ there is no computational problem to speak of, once the x^i have been determined. For $m = 2$, again, once the justices' profiles have been properly located in the plane, the computation of the power index is a fairly routine exercise in plane trigonometry (Shapley, 1977). For $m = 3$, however, the computation is not at all routine; the methods used to get answers will be described presently. For $m \geq 4$, we do not at this time have a systematic method for computing the AD power index.

In order to estimate the attitudes of the justices and the ideological distribution of the cases that they decide, we performed a principal-components factor analysis of the 94 nonunanimous votes that occurred during the Court's 1977-78 term. There were a few more nonunanimous votes than there were cases decided by nonunanimous vote, for in some instances two separate issues in the same case were voted on.* The votes were tabulated from U.S. Law Week, and missing data were handled by pairwise deletion.

From the principal components analysis three factors were retained; together they accounted for 64.7 percent of the variance in the ninety-four votes. The factor pattern matrix gave us a (preliminary) placement of the issues in $\mathbb{R}^3$, and the factor score matrix gave us a placement of the voters. The first principal component was clearly interpretable as a liberal/conservative measure, which by itself accounted for 35.3 percent of the variance. This is the horizontal axis in Figure 2, along which the justices may be seen to line up from left to right much as one would expect from journalistic accounts of their judicial philosophies. Subsequent principal components are less clearly interpretable. The second principal component, represented by the vertical axis in Figure 2, accounts for 15.6 percent of the variance; it seems to load most highly on issues pertaining to judicial restraint in overturning

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*For example, in the most publicized case of the term, Bakke vs. University of California at Davis Medical School, there were two separate votes: one on whether the racial quota system for admission to the Davis Medical School violated Bakke's rights under the Fourteenth Amendment or the 1964 Civil Rights Act, and a different vote on whether any race conscious admissions policy was permissible.

This was a rare case in which it was obvious who the pivot was. Four Justices voted to uphold the Davis quota system, four Justices voted to strike down race conscious admission programs at public universities generally, and Justice Powell, by siding with the "liberals" on one vote and the "conservatives" on the other, was able to determine the final content of the decision.

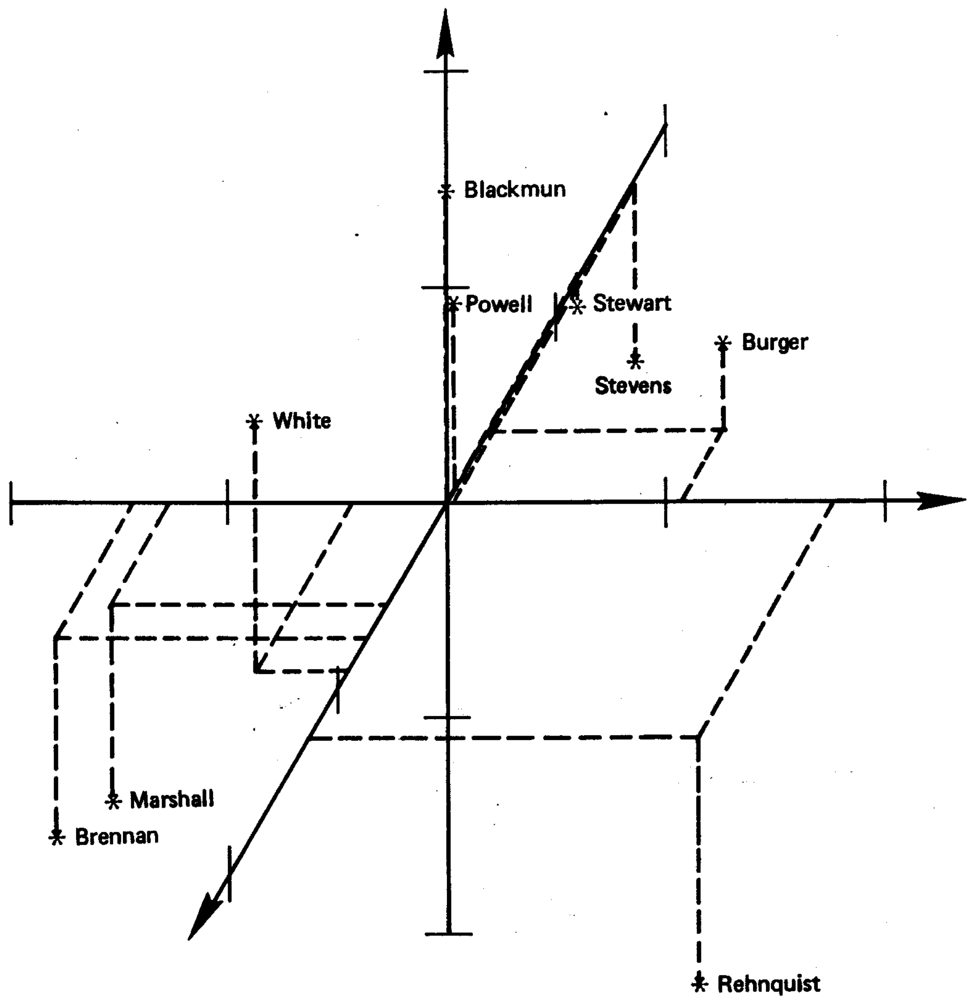

Fig. 2— Unadjusted position of justices in three-dimensional space

state laws and regulations. Justices with a negative score on this dimension are inclined to permit the states to legislate as they please unless the case for federal pre-emption is quite strong, while those with a positive score are more inclined to overturn state actions. The third dimension, represented by the axis perpendicular to the surface of the paper, accounts for 13.8 percent of the variance, and is more difficult to identify heuristically. It loads most highly on technical cases, particularly those having to do with taxes. Justices with a negative score on this dimension tend to support the taxpayer, while those with a positive score tend to support the Internal Revenue Service. Succeeding factors accounted for considerably smaller portions of the total variance and were not included in the analysis.

Table 1

FACTOR SCORES (UNADJUSTED COORDINATES)

	Justice	x_1	x_2	x_3
1.	Burger	1.074	.407	-.384
2.	Brennan	-1.433	-.934	.723
3.	Stewart	-.011	-.145	-.206
4.	White	-.424	1.173	.899
5.	Marshall	-1.262	-.916	.542
6.	Blackmun	-.015	1.415	-.040
7.	Powell	.034	.947	-.001
8.	Rehnquist	1.774	-1.142	1.266
9.	Stevens	.021	-.820	1.694

The factor scores listed in Table 1 were used to estimate the justices' ideological positions, but the factor pattern matrix was not used directly to estimate the distribution of issues. Some of the issues did not load very highly on any of the three factors retained, and for those issues the coordinates given by the factor pattern matrix are not very useful. From the model of voting we are using, if the placement of the justices perfectly reflects their ideologies, then for each issue direction there should be a plane perpendicular to the issue line that clearly divides those who actually voted "nay" on the issue from those who voted "yea." Since the three retained factors do not account for all of the variance in the votes, such a discriminating plane cannot always be found. Nevertheless, we want to choose for each issue a direction, s, which optimizes some measure of how well its best discriminating plane succeeds in explaining the "yeas" and "nays."

Let s be an issue direction, with $\|s\| = 1$, and let x^i, $i = 1, 2, \ldots, 9$ be the profile vectors of the nine justices. Define $t_i = (s, x^i)$, and define $\bar{t}$ to be the mean of the t_i. Then

$$\sigma^2(s) = 1/9 \sum_1^9 t_i^2 - \bar{t}^2$$

measures the dispersion of profiles in the direction s. If t is the (signed) distance from 0 of an arbitrary plane perpendicular to the issue line, then a reasonable measure of its <u>failure</u> to explain the "yeas" and "nays" is

$$V_s(t) = \sum_{i \in \text{YEA}} e^{(t-t_i)/\sigma} + \sum_{j \in \text{NAY}} e^{-(t-t_j)/\sigma} .$$

This measure is to be minimized with respect to t in $(-\infty, \infty)$ for each s, and then minimized with respect to s in S.*

We see that $V_s(t)$ is a convex function of t that goes to $+\infty$ as $t \to \pm\infty$ (since all the votes are nonunanimous), so it has a unique, finite minimum in t for each s. By simple calculus, the minimum is attained at $t = t_o$, where

$$\sum_{i \in \text{YEA}} e^{(t_o-t_i)/\sigma} = \sum_{j \in \text{NAY}} e^{-(t_o-t_j)/\sigma} ,$$

so we have

$$t_o = t_o(s) = \frac{\sigma}{2} \left[\ell n \left(\sum_{\text{NAY}} e^{t_j/\sigma} \right) - \ell n \left(\sum_{\text{YEA}} e^{-t_i/\sigma} \right) \right] .$$

Then the minimum value of V_s is

$$V_s(t_o) = \sqrt{\left(\sum_{\text{YEA}} e^{-t_o/\sigma} \right) \left(\sum_{\text{NAY}} e^{t_j/\sigma} \right)}$$

---

*Note that if the plane separates perfectly, then the exponent of e is negative for every term. The exponential function was chosen because it increases slowly for negative arguments (when t_i is on the "correct" side of t) but rapidly for positive arguments (when t_i is on the "wrong" side of t); it also has very attractive analytical properties.

Call this value $V_{min}(s)$. The issue-direction s which minimizes $V_{min}(s)$ is the one which best explains the "yeas" and "nays," within the restrictions of our model. Since $\|s\| = 1$ by assumption, s may be specified by its <u>longitude</u> θ ($0° \leq \theta \leq 360°$) and its <u>latitude</u> ϕ ($-90° \leq \varphi \leq 90°$). If we measure the longitude of s by the angle it makes in the (x_1, x_2) plane with the x_1 axis, and the latitude by its angle of elevation above the (x_1, x_2) plane--the "north pole" being the point (0, 0, 1)--then the three coordinates of s are given by $x_1 = \cos\phi \cos\theta$, $x_2 = \cos\phi \sin\theta$, $x_3 = \sin\phi$. Thus we have

$$t_i = x_1^i \cos\phi \cos\theta + x_2^i \cos\phi \sin\phi + x_3^i \sin\phi.$$

Plugging this into the above formula for V_{min} gives us a function of the two variables ϕ, θ to minimize. Using the factor pattern matrix to provide our initial guesses, we used Newton's method to minimize the 94 functions V_{min}, one for each issue.

Now let us consider these 94 points on the sphere S as a random sample from some underlying issue-generating process. We would like to find a linear transformation on $\mathbb{R}^3$ that makes these 94 points look "most uniform" on S. The untransformed distribution is certainly not uniform, because the first coordinate is so much more prominent than the other two. In issue space, this means that more issues are observed near the x_1 axis (i.e., more issues are almost pure liberal/conservative issues) than are near the other two axes. In the profile space, this means that if we are going to make the assumption of spherical uniformity the configuration of profiles in Figure 2 should be more "one-dimensional," clustered along a line parallel to the x_1 axis, since the second and third coordinates are less significant than the first. This suggests that a simple transformation in issue space whose matrix has the diagonal form

$$(1) \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \qquad \text{with } a, b > 1$$

would improve the representation, since it would magnify the second and third coordinates and cause fewer issues to fall near the "pure" liberal/conservative issue-direction (1, 0, 0). Further examination of the data also suggests that b should be greater than a. Of course, more general transformations than (1) should also be considered; as it turns out, the form

$$(2) \qquad \begin{pmatrix} 1 & c & d \\ 0 & a & e \\ 0 & 0 & b \end{pmatrix} \qquad a, b > 0$$

provides full generality.*

We would like to proceed by minimizing some goodness-of-fit statistic for the transformed issues that measures the "distance" between an empirical distribution of 94 points and the uniform

*Further detail in the matrix would merely introduce rotations, reflections and Euclidean similarities, none of which affect the power index calculations. In other words, we have just five essential degrees of freedom to work with, corresponding to the parameters a, b, c, d, e.

distribution on the sphere S. Unfortunately, there is no multivariate goodness-of-fit statistic with all the optimality properties that the Cramer-von Mises statistic has for the univariate case (Durbin, 1973). In the following we tried two expedients. Our first approach was to use the univariate Cramer-von Mises statistic to measure the goodness-of-fit of the empirical marginal distributions to the theoretical marginal distributions, and then minimize their sum. Now if the joint distribution of (x_1, x_2, x_3) is uniform on S, then the marginal distributions of x_1, x_2, and x_3 are each uniform on the interval [-1, 1]. (This follows directly from the surface area formula for the sphere.) Transform the 94 pairs of angles (ϕ, θ) back to rectangular coordinates by the formula $x_1 = \cos\phi \cos\theta$, $x_2 = \cos\phi \sin\theta$, $x_3 = \sin\phi$, and also double the observations to include the points $(-x_1, -x_2, -x_3)$, since the only difference between an issue direction and its reverse is the way the vote was coded. To find a linear transformation of the diagonal form (1) that makes this distribution look "as uniform as possible," consider now the transformed coordinates

$$\left(\frac{x_1}{\ell_{a,b}} , \quad \frac{ax_2}{\ell_{a,b}} , \quad \frac{bx_3}{\ell_{a,b}} \right) ,$$

where $\ell_{a,b} = \sqrt{x_1^2 + a^2x_2^2 + b^2x_3^2}$. We want to find the a and b with $b > a > 1$ that minimizes the sum of the univariate Cramer-von Mises statistics

$$\int_{-1}^{1} \left[\frac{\text{no. of } (x_1/\ell_{a,b})\text{'s} \le t}{188} - \frac{1}{2}\,(t+1) \right]^2 dt \; +$$

$$\int_{-1}^{1} \left[\frac{\text{no. of } (ax_2/\ell_{a,b})\text{'s} \le t}{188} - \frac{1}{2}\,(t+1) \right]^2 dt \; +$$

$$\int_{-1}^{1} \left[\frac{\text{no. of } (bx_3/\ell_{a,b})\text{'s} \le t}{188} - \frac{1}{2}\,(t+1) \right]^2 dt.$$

By a grid technique, we found that the values $a = 1.85$, $b = 1.96$ minimized the above measure.

We would like to consider the more general class of transformations (2), but the above technique would be substantially more difficult to apply if five variables were involved. So to find the "best" transformation in the larger class, we used a different approach. One property of the uniform distribution on a sphere is that the coordinates are uncorrelated with each other, and that this remains the case if the coordinate system is rigidly rotated or reflected. If we could find a linear transformation that would, when applied to our 188 points on the sphere, make their coordinates uncorrelated even when rotated or reflected, we would have a good candidate for a transformation that makes the issue distribution "look uniform" on S. It is plausible to expect that a transformation that does this will exist and be essentially unique (i.e., up to rigid motions); this question is discussed in Shapley (1977, Appendix B).

We have developed an algorithm for finding such a transformation which will be explained fully in a later edition of this paper. Applied to our present problem, the transformation matrix that resulted is

$$\begin{pmatrix} 1 & -.133 & .204 \\ 0 & 1.789 & -.327 \\ 0 & 0 & 1.930 \end{pmatrix}$$

This is reassuringly close to the matrix obtained by the previous method. The inverse of this matrix, when applied to the political profiles in Table 1 and Figure 2, yields the results shown in Table 2 and Figure 3. As stated earlier, the geometric effect is to bring the profiles of the nine justices in closer alignment to the x-axis, making it more likely that a randomly-chosen issue in S will order the justices in accordance with their positions on the liberal/conservative axis.

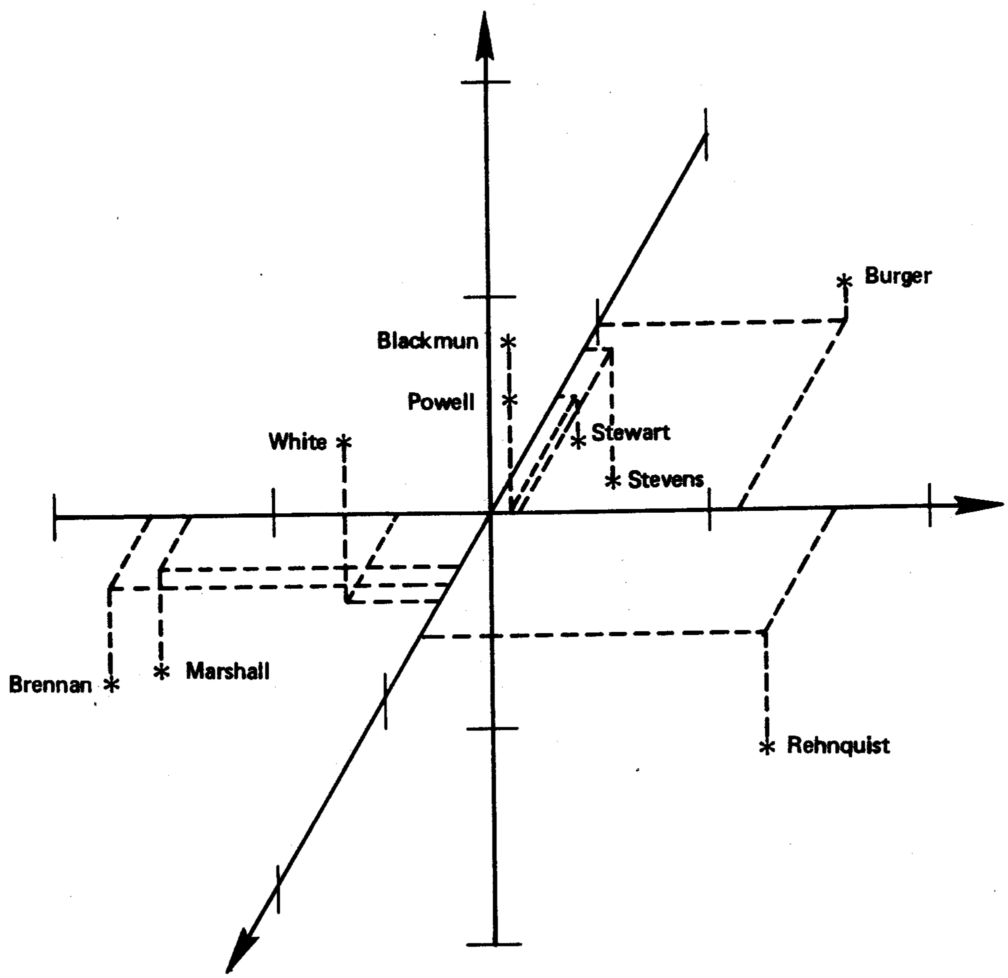

Fig. 3 — Adjusted position of justices in three-dimensional space

Table 2

FACTOR SCORES (ADJUSTED COORDINATES)

	Justice	x_1	x_2	x_3
1.	Burger	1.136	.191	-.999
2.	Brennan	-1.561	-.453	.375
3.	Stewart	.094	-.196	-.625
4.	White	-.436	.741	.466
5.	Marshall	-1.371	-.461	.281
6.	Blackmun	.078	.787	-.021
7.	Powell	.094	.529	-.001
8.	Rehnquist	1.582	-.518	.656
9.	Stevens	.130	-.619	-.877

Now all that remains is to determine the partition of the sphere S into the regions $\{S_i\}$ and to compute their areas. Computing the areas is not difficult, since each S_i consists of a number of spherical polygons, each spherical polygon can be cut up into spherical triangles, and there is a standard formula for finding the area of a spherical triangle given its vertices. So all that needs to be done is to locate and identify the regions S_i.

For the pivotal role to pass from voter i to voter j, the great circle defined by the equation $(s, x^i) = (s, x^j)$ must be crossed. Thus the <u>edges</u> of the partition of S into the nine regions S_i must lie along the $\binom{9}{2} = 36$ great circle paths where there is a two-way tie between voters. The <u>vertices</u> of the partition must be among the intersections of these great circle paths that represent three-way ties among the voters, i.e., the $\binom{9}{3} = 84$ points V_{ijk} defined by

$$(V_{ijk}, x^i) = (V_{ijk}, x^j) = (V_{ijk}, x^i)$$

where $1 \leq i < j < k \leq q$.* To discover which of these points are actually partition vertices we compute, for each triplet (i, j, k), the number n_{ijk} of ℓ's such that $(V^{ijk}, x^\ell) > (V^{ijk}, x^i)$, in other words, the number of voters that precede the tied voters. If n_{ijk} is 0, 1, 5, or 6, then V_{ijk} is not a vertex of the partition, because the triple tie does not include fifth place. If $n_{ijk} = 2$ or 4, then we get a partition vertex of degree 3, exhibiting a pattern illustrated by the vertex V_{ABC} at the left of Figure 4. Three of the spherical polygons of the partition meet at such a vertex. Finally, if $n_{ijk} = 3$, we get a partition vertex of degree 6, like V_{ABD} at the right of Figure 4. Six of the polygons meet here, two belonging to each of the three voters that are tied at that vertex.

Now to start piecing the puzzle together we look for a vertex of degree 6. At least one exists on each of the 36 great circle paths, for i and j cannot get from being tied for, say 3^{rd} and 4^{th} to being tied for 6^{th} and 7^{th} ** without passing through a point where they are changing from being tied for 4^{th} and 5^{th} to being tied for 5^{th} and 6^{th}.

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*We are assuming throughout this discussion that the data are "in general position," so that all 84 points are distinct. Thus we do not worry about four-way ties. Note that the great circles will also intersect at points where four voters are tied in pairs; these points have no significance at present.

**Note that if i and j are tied for k^{th} and $(k+1)^{th}$ at a point s, then at the diametrically opposite point $-s$ they are tied for $(9-k)^{th}$ and $(9-k+1)^{th}$.

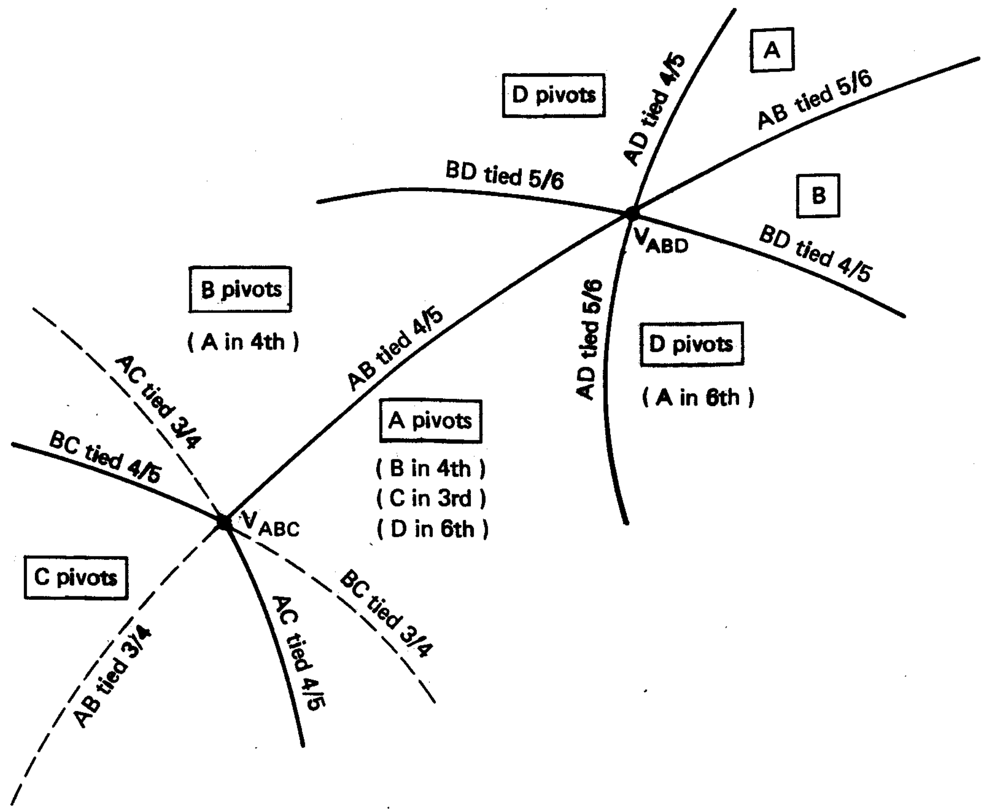

Fig. 4 — Typical vertices in a partition

At that point they are in a three-way tie with some other voter k, with $n_{ijk} = 3$. Consulting Figure 4 again, we see that when we travel in any direction on a great circle path through a vertex of degree 6, we are on an edge of the partition. But when we hit a vertex of degree 3, the next section of the path we are following is not a partition boundary, and we can continue to ignore that path until another vertex of degree 3 is reached.

Using these observations, we can systematically piece together the regions S_i; in this case they turned out to consist of from five to ten polygons each, ranging from triangles to hexagons.* The areas may then be calculated by spherical trigonometry (Table 3). Figure 5 is an attempt to depict two of the regions on an astronomical sky chart; the heavy curve is the trace of the great circle along which 7 and 8 are tied.

Table 3

AD POWER INDICES

	Justice	Transformation (1)	Transformation (2)
1.	Burger	.065	.064
2.	Brennan	.033	.034
3.	Stewart	.157	.154
4.	White	.129	.124
5.	Marshall	.080	.084
6.	Blackmun	.092	.085
7.	Powell	.346	.356
8.	Rehnquist	.051	.061
9.	Stevens	.047	.039
		1.000	1.001

*Note that the "jigsaw puzzle" has to be solved only once--the linear transformations (1) and (2) change the coordinates of the vertices and hence the areas of the polygons, but not the way the pieces fit together.

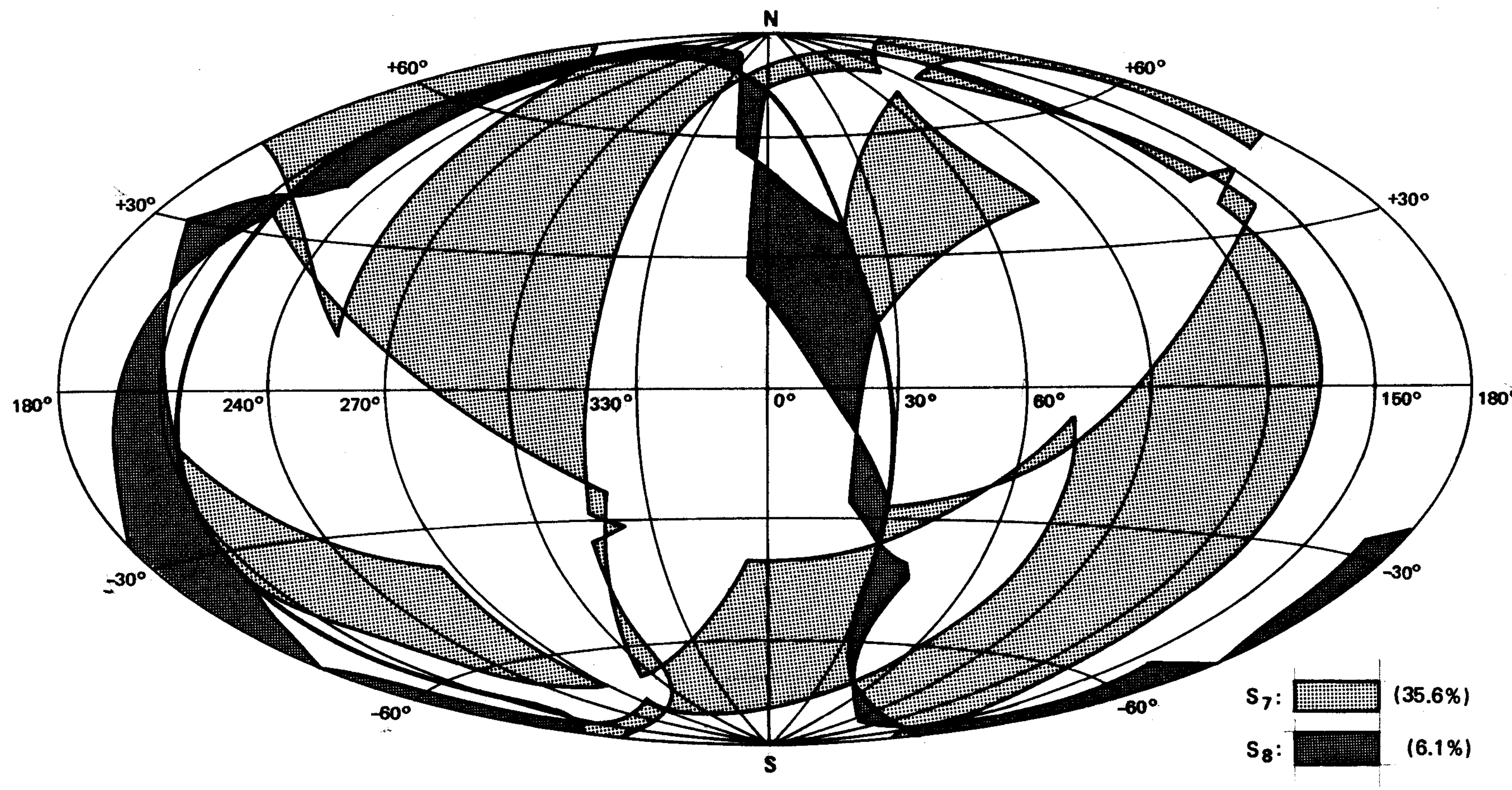

Fig. 5 — Pivotal regions for Justices Powell and Rehnquist.
(Aitoff's equal area projection)

4. Potential Usefulness of the Attitude Dependent Power Index

The methodology developed in this paper is limited in applicability to the case of a small number of voters and so far no more than three dimensions of ideology. For large bodies, such as the U.S. House of Representatives, a somewhat different approach is needed. Either voters with similar ideologies must be merged into blocs (the justification being that the sum of the power indices of voters of nearly equal ideologies is approximately the power index of a single voter casting all of their votes in a bloc), or a density function must be constructed that approximates the discrete distribution of political profiles, and methods of non-atomic game theory applied.

One value of this AD power index might be in assessing the impact of membership changes on political institutions. Of course, one way of assessing the effect of a membership change is to compare the policy outputs of the institution before and after the change. But here the effect of membership change may be confounded with the effect of a change in the external situation with which the political body must deal: for example, energy crises, sudden changes in unemployment rates, etc. If, although the external situation changes, the underlying attitudes in accordance with which voters make their decisions remain the same, and if the distribution of the <u>kinds</u> of issues that arise (although not the actual content of these issues) remains constant over time, then the present power index could be a valuable tool in assessing the effect, for example, of new appointments to the Supreme Court or the effect of an election on the distribution of power in a particular Congressional committee.

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