A RAND NOTE

ON THE ACCESSIBILITY OF FIXED POINTS

Lloyd Shapley

July 1981

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PREFACE

This Note describes a difficulty that can arise in the computation of equilibrium prices in economics when there are multiple solutions, and proposes a method for dealing with the problem. It was presented at a Seminar on Game Theory and Mathematical Economics at Hagen and St. Augustin in West Germany, October 7-10, 1980, jointly sponsored by the University of Hagen and the University of Bonn. The text will be published in the proceedings of that Seminar. The author's research was supported at The Rand Corporation by the National Science Foundation under Grant SOC 78-04285.

SUMMARY

Path-following algorithms have proved practical for the solution of fixed-point problems arising in economics and game theory. But the value of the methodology is diminished by the possibility of multiple solutions, since there is no guarantee of finding more than one solution even if many paths are traced. This Note describes a way of transforming problems so that the paths are modified while the solutions remain unchanged. As a result, previously inaccessible solutions may become accessible. It is shown that for any piecewise-linear problem in general position there is a solution-preserving transformation that makes all solutions simultaneously accessible.

CONTENTS

I. INTRODUCTION

Path-following procedures have proved effective in solving a variety of problems of the "fixed-point" type, ranging from bimatrix games (Lemke and Howson (1964)) to the determination of economic cores and equilibrium prices (Scarf (1967, 1973)).* The essential idea is to relax one of the defining conditions for a solution, e.g., by replacing one equation by an inequality. If care is taken to avoid degeneracy, a one-dimensional set of "almost-solutions" is thereby created consisting of paths and loops, with the true solutions appearing at the endpoints of the paths. Typically, there will also be a special, external path-end, not a solution, which serves as a convenient starting point. By merely blindly following the path from that starting point, one must come sooner or later to a solution. Moreover, by selecting a different condition to relax one can create additional paths and additional starting points, internal as well as external, and so build up a whole network of connections to aid in the search for other solutions.

Unfortunately, the value of this kind of procedure is severely limited in practice when there is a possibility of nonuniqueness. Though the solver may proudly present his customer with one or more elegantly calculated answers to some practical problem of very high dimension--say, the future equilibrium prices in an economy--he seldom has much to say

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*Many others have contributed to this field; we should like especially to mention the work of Lemke (1965), Kuhn (1968), Eaves (1970, 1971), Wilson (1971), Shapley (1973, 1974), Eaves and Scarf (1976), Smale (1976), Zangwill (1977), Van der Heyden (1979, 1980), and Van der Laan and Talman (1981).

about the other solutions that may happen to exist but are not accessible to the path-following methodology. Indeed, it can be shown rather generally that clumps of isolated solutions, connected only to each other, can readily occur in fixed-point-type problems, giving no warning to the path-follower of their presence. Moreover, there appear to be no characteristic intrinsic features that distinguish between accessible and inaccessible solutions, and hence no reasonable basis for the "customer" to be content with knowing only the accessible solutions.

In this Note we describe a method of "problem manipulation." The general idea is to find ways of transforming the data of a fixed-point problem that leave the solutions unchanged while (hopefully) disrupting the path topology. As a result, previously inaccessible solutions may become accessible. In fact, we shall show that solution-preserving transformations do exist that make <u>all</u> solutions simultaneously accessible. While this represents some progress, the question still remains of finding the right transformation in any given case. In other words, instead of not enough paths to trace, we may have an embarrassment of riches. At the present stage of this investigation we can offer the practitioner little more than a tool for heuristic exploration, for use in cases where he is not satisfied with the original accessible set and has some idea of where other solutions might be found. The next stage should probably involve some actual computational experience with problems of high dimension. In particular, little appears to be known about the rate at which multiple solutions proliferate in practice as the dimension increases.

II. NOTATIONAL PRELIMINARIES

Let $N = \{1, 2, \ldots, n\}$ be a finite set of indices, and let A denote the (n-1)-dimensional simplex

$$(1) \qquad A = \{x \in \mathbb{R}^N_+ : \textstyle\sum_N x_i = 1\}.$$

The vertices of A (i.e., the unit vectors of $\mathbb{R}^N$) are denoted a^i, $i \in N$, and, for each nonempty $S \subset N$, the convex hull of $\{a^i : i \in S\}$ is denoted A_S. Thus, $A_N = A$, $A_{\{i\}} = \{a^i\}$, and in general A_S is an $(|S|-1)$-dimensional face of A.

By a labelling of the simplex A we shall mean an indexed family $L = \{L_i : i \in N\}$ of closed subsets of A, covering A. (Equivalently, we could define a labelling to be the inverse of L, i.e., an upper-hemicontinuous correspondence $\ell : A \to N$, related to L by

$$\ell(x) = \{i \in N : x \in L_i\}; \qquad L_i = \{x \in A : i \in \ell(x)\}.)$$

Note that some of the sets L_i (but none of the sets $\ell(x)$) may be empty. We shall call a labelling proper if

$$\bigcup_{i \in S} L_i \supset A_S$$

holds for all nonempty $S \subset N$.

III. THE K-K-M THEOREM

The following theorem of Knaster-Kuratowski-Mazurkiewicz (1926) is basic to an understanding of this whole subject

<u>THEOREM 1.</u> <u>If</u> L <u>is a proper labelling of</u> A, <u>then</u> $\bigcap_{i \in N} L_i \neq \emptyset$.

Let us recall two familiar applications of this theorem.

First, the Brouwer fixed-point theorem. Let F be a continuous map from A into itself, and define $f(x) = x - F(x)$. Note that, by (1),

$$(2) \qquad \sum_{i \in N} f_i(x) = 0, \qquad \text{all } x \in A.$$

Define $L = \{L_i : i \in N\}$ by

$$(3) \qquad L_i = \{x \in A : \max_{j \in N} f_j(x) = f_i(x)\}, \qquad \text{all } i \in N.$$

To see that L is proper, observe that $x \in A_S$ implies $f_i(x) \leq 0$ for all $i \in N$ S, so necessarily $f_i(x) \geq 0$ for at least one $i \in S$. It follows that $\max_S f_j(x) = \max_N f_j(x)$, which puts x in L_i for at least one i in S, as required. Hence L is proper, and by Theorem 1 there is a point $x^* \in A$ that belongs to all of the L_i, $i \in N$. This means that the numbers $f_i(x^*)$ are all equal, and hence, by (2), all equal to zero. Therefore x* is a fixed point of F.

The second application is to the existence of equilibrium prices prices in a trading economy. Let N be the set of marketable commodities, and let $x = (x_1, \ldots, x_n)$ be their prices. Let $f_i(x)$ represent the excess of supply over demand of the i-th commodity as a function of <u>all</u> the prices. Assume that each f_i is continuous and satisfies "desirability":

(4) $$x_i = 0 \Longrightarrow f_i(x) \leq 0, \qquad \text{all } x \in A;$$

and that the f_i collectively satisfy "conservation of money":

(5) $$\sum_{i \in N} x_i f_i(x) = 0, \qquad \text{all } x \in A.$$

Define L from f as at (3). Then (4) implies that L is proper. Theorem 1 again yields the existence of x* such that the $f_i(x^*)$ are all equal to each other and hence, by (5), all equal to zero. Supply therefore equals demand, and we have the desired equilibrium.

IV. SOLUTIONS, ROUTES AND PATHS

Let L[f] denote the labelling that is generated by maximizing over a family of continuous functions $f = (f_1, \ldots, f_n)$, as in (3) above. If these functions are piecewise-linear,* and if the whole family is "in general position," then we shall call L[f] <u>regular polyhedral</u>. For a regular polyhedral labelling it can be shown that, for each $S \subset N$, either the set

$$L(S) = \bigcap_{i \in S} L_i$$

is empty or it consists of finitely many closed, polyhedral components, each one of dimension $n - |S|$. In particular, the set

$$\mathcal{S} = L(N)$$

is finite, and its elements will be called <u>solutions</u>. As will soon appear, if L is both proper and regular polyhedral then the number of solutions is odd, and hence > 0.

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*A <u>piecewise-linear</u> function is one that can be represented as a continuous selection from a finite family of linear functions. We shall stick with piecewise linear functions throughout this paper, though there are certain more or less obvious extensions to wider classes of functions that could be made.

We see no reason to burden this paper with an explicit definition of "in general position."

For each $i \in N$ the set

$$\mathcal{R}_i = L(N \setminus \{i\})$$

will be called <u>route i</u>. In a regular polyhedral labelling, $\mathcal{R}_i$ is (if nonempty) a one-dimensional closed set, each component of which is either an <u>i-path</u> (i.e., an open polygon, having two endpoints) or an <u>i-loop</u> (i.e., a closed polygon, having no endpoints). The following lemma may be verified by considering that the only two ways in which an i-path in general position can be terminated is by running into region L_i or by running into one of the boundary faces of A. In the latter case, if L is proper the face in question can only be $A_{N \setminus \{i\}}$.

<u>LEMMA 1. If</u> L <u>is a proper, regular polyhedral labelling of</u> A, <u>and if</u> $\mathcal{E}_i$ <u>denotes the set of all endpoints of i-paths, then for each</u> $i \in N$

$$\mathcal{S} \subset \mathcal{E}_i \subset \mathcal{S} \cup A_{N \setminus \{i\}}.$$

<u>COROLLARY.</u> <u>If</u> $i \in N$, $j \in N$, $i \neq j$, <u>then</u>

$$\mathcal{R}_i \cap \mathcal{R}_j = \mathcal{S}.$$

We now extend these routes to the lower-dimensional faces of A. Let Π denote the set of permutations of N, and for each $\pi \in \Pi$, define route π by

$$\mathcal{R}(\pi) = \bigcup_{k=1}^{n} \left[A_{\{\pi 1, \ldots, \pi k\}} \cap L(\{\pi 1, \ldots, \pi(k-1)\}) \right],$$

or, more explicitly,

$$\mathcal{R}(\pi) = \left[A \cap \mathcal{R}_{\pi n} \right] \cup \left[A_{N \setminus \{\pi n\}} \cap L(N \setminus \{\pi n, \pi(n-1)\}) \right] \cup \cdots$$
$$\cdots \cup \left[A_{\{\pi 1, \pi 2\}} \cap L_{\pi 1} \right] \cup \{a^{\pi 1}\} .$$

Thus, for a given $\pi \in \Pi$, route π involves just n faces of A, one of each dimension. If L is regular polyhedral, then $\mathcal{R}(\pi)$ consists of polygonal π-paths and π-loops. If L is proper as well, the endpoints of the portion of $\mathcal{R}(\pi)$ contained in any face will match up exactly with the endpoints in the faces of adjacent dimension, up or down, so that the complete π-paths will not have any endpoints at all in the (relative interiors of) faces of intermediate dimension. In other words, the only endpoints of $\mathcal{R}(\pi)$ are the solutions themselves and the single "starting point" $a^{\pi 1}$:

LEMMA 2. *If L is a proper, regular polyhedral labelling of A, and if $\mathcal{E}(\pi)$ denotes the set of endpoints of π-paths, then*

$$\mathcal{E}(\pi) = \mathbb{S} \cup \{a^{\pi 1}\} .$$

Note that any given π-path (or π-loop, for that matter) can wander freely up and down the scale of face-dimensions. Path-following algorithms of this type are accordingly sometimes called "variable dimension" algorithms.*

For each $\pi \in \Pi$ the π-path having $a^{\pi 1}$ as an endpoint is called the *primary π-path* and the other endpoint of that path is called the *primary π-solution*. The primary solutions for different π's may be distinct, or they may coincide. Indeed, the number of distinct primary solutions may be any number from a to n! (Figure 1.) The possible number of distinct *solutions*, on the other hand, is unbounded, though it is clear from Lemma 2 that it must be odd.

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*Van der Heyden (1980).

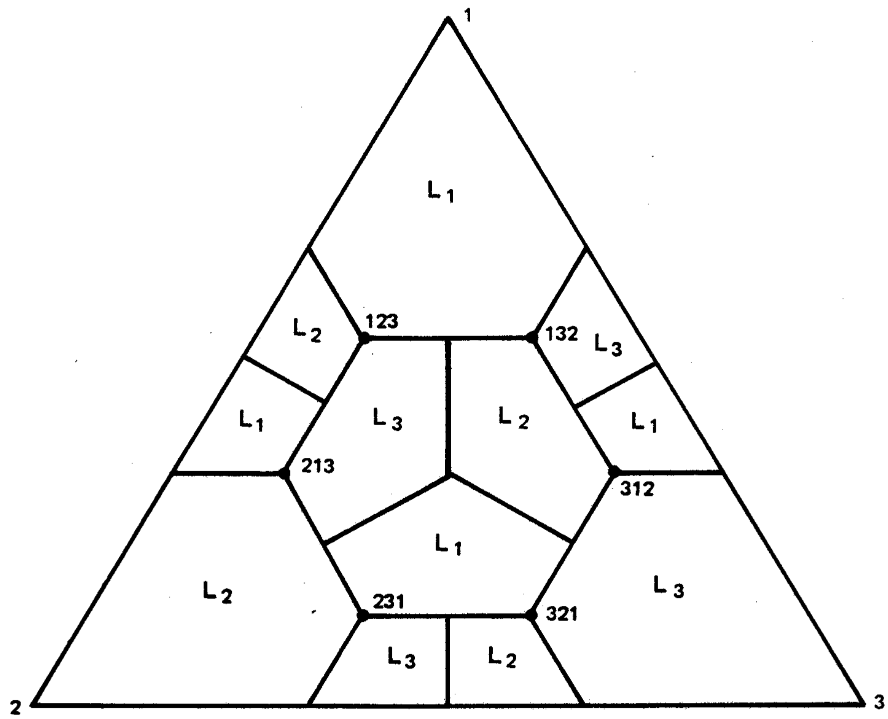

Fig. 1 — Six primary solutions

V. ACCESSIBILITY

Call two elements of $\mathcal{S}$ adjacent if they are the endpoints of some π-path for some $\pi \in \Pi$. Call an element of $\mathcal{S}$ accessible if either it is itself a primary solution, or there is an adjacency chain connecting it to a primary solution. The set of all accessible solutions will be denoted $\mathcal{S}_{acc}$, and the set of all inaccessible solutions $\mathcal{S}_{inacc}$. It is not difficult to see that $\mathcal{S}_{acc}$ is odd and $\mathcal{S}_{inacc}$ is even. (Figure 2.)

The notion of accessibility is important because of (a) the availability in many applications of very efficient algorithms for tracing the paths, either directly when the problem is sufficiently linear (Lemke (1965), Eaves and Scarf (1976), etc.) or with the aid of suitable discretizations of the simplex when it is not (Scarf (1967), Kuhn (1968), Shapley (1973), Zangwill (1977), etc.), and (b) the nonavailability in many applications of any other good method for finding solutions.

It is possible to associate with each element of $\mathcal{S} \cup \{a^1, \ldots, a^n\}$ an orientation index +1 or -1, defined intrinsically in terms of a certain Jacobian determinant (not depending on π), which has the properties that (a) all "starting points" a^i have the same orientation, and (b) the endpoints of any π-path have opposite orientation.* It

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*Fan (1967), Shapley (1974), Lemke and Grotzinger (1976).

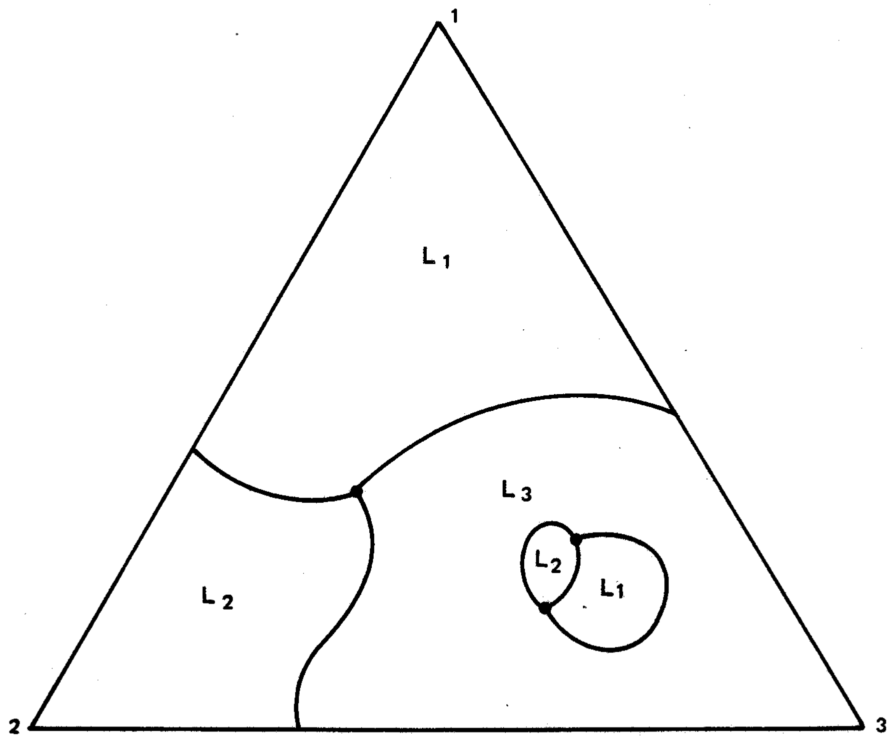

Fig. 2 — Inaccessible solutions

follows that no two primary solutions are adjacent, and that if there are p primary solutions in all then there are at least p - 1 other accessible solutions.

VI. SOLUTION-PRESERVING TRANSFORMATIONS

We begin with a motivating example. Let $f\colon A \to \mathbb{R}^3$ be piecewise-linear and in general position and let $L[f]$ be the associated regular polyhedral labelling. Let $g\colon A \to \mathbb{R}^3$ be obtained from f by the following linear transformation:

$$(6) \qquad \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} .$$

The effect of this transformation is easily described: the first two component functions remain fixed while the third moves closer to the first. Region $L_1[f]$, where the first function is "on top," is not affected by this. But $L_2[f]$ will grow, taking territory away from $L_3[f]$, since the third function can be "pulled down" out of the top position when the first function is on the bottom. The <u>solutions</u> of $L[f]$ are not changed, however, since it is at once verified from (6) that

$$(7) \qquad f_1(x) = f_2(x) = f_3(x) \Leftrightarrow g_1(x) = g_2(x) = g_3(x), \quad \text{all } x \in A.$$

Routes 2 and 3 are also unchanged. But route 1, which is the boundary between L_2 and L_3, will certainly change, and this may cause previously inaccessible solutions to become accessible. (Figure 3.)

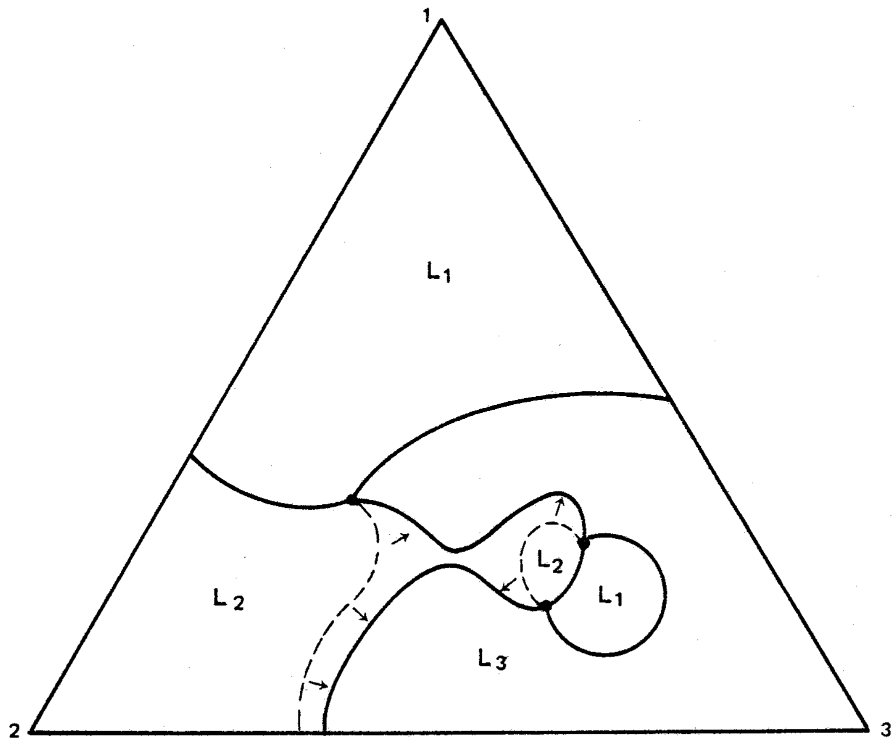

Fig. 3 — Possible effect of the transformation (6)

In a heuristic search procedure, one can imagine using transformations like (6) to "dry up" different regions L_i in turn, in the hope of opening up new portions of the simplex to exploration.

Let us now formulate this kind of transformation in more general terms. Consider a mapping $\alpha : \mathbb{R}^N \to \mathbb{R}^N$, such that

(i) α is piecewise-linear;

(ii) the Jacobian determinant of α is positive almost everywhere, i.e., throughout the interior of each "piece" of linearity;

(iii) there is an increasing bijective function $m: \mathbb{R} \to \mathbb{R}$, such that $\alpha(r, \ldots, r) = (1, \ldots, 1)m(r)$ holds for all $r \in \mathbb{R}$.

Such mappings α may be used to transform functions $f: A \to \mathbb{R}^N$ into functions $\alpha \circ f: A \to \mathbb{R}^N$, according to the rule

$$(\alpha \circ f)(x) = \alpha(f(x)), \qquad \text{all } x \in A.$$

In view of (i) and (ii), α is a bijection, since if there were any uncovered or multiply-covered region in $\mathbb{R}^N$ it would have a "folded" edge, in the neighborhood of which the Jacobian would change sign.* It follows that the inverse of α is well defined, and that it also enjoys properties (i) - (iii) above. Indeed, the set of all such mappings forms a group of transformations, which we shall denote by $\mathcal{G}$.

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*This remark depends upon the <u>finiteness</u> of the family of linear functions from which a piecewise-linear function is a selection.

Property (iii) ensures that $\mathcal{S}$ preserves solutions (cf. (7)). Moreover, each element of $\mathcal{S}$ that is in general position will preserve the regular polyhedral character of L[f] for almost all f. The "properness" of the labelling is not invariant under $\mathcal{S}$, however.

In the case of a *linear* transformation α (e.g., (6)), conditions (ii) and (iii) require that the determinant of the matrix be positive and that the *row sums* be equal. When the row sums are all equal to 1, the transformation can be visualized as replacing each of the functions f_i by a linear convex combination of all the functions, with the different convex combinations being linearly independent.

VII. VARIABLE TRANSFORMATIONS

Consider next a more complex mapping $\beta: (\mathbb{R}^N \times A) \to \mathbb{R}^N$, such that

(iv) for each $x \in A$, $\beta(\cdot, x) \in \mathscr{B}$;

(v) for each $y \in \mathbb{R}^N$, $\beta(y, \cdot)$ is piecewise linear;

(vi) for each x in the (relative) boundary of A, $\beta(\cdot, x)$ is the identity map.

The set of all such mappings forms a group $\mathscr{K}$ of transformations on the space of functions $f: A \to \mathbb{R}^N$, defined by

$$(\beta \circ f)(x) = \beta(f(x), x), \qquad \text{all } x \in A, \beta \in \mathscr{K}.$$

Intuitively, we are varying the choice of $\alpha \in \mathscr{B}$ as we move about in the simplex A. As before, $\mathscr{K}$ preserves the solution-set of L[f]; it also preserves "properness," in view of (vi).

THEOREM 2. *If* L[f] *is a proper regular polyhedral labelling of A generated by piecewise-linear functions* $f = (f_1, \ldots, f_n)$ *in general position, then there is an element* β *of* $\mathscr{K}$ *for which*

$$\mathcal{S}(L[f]) = \mathcal{S}(L[\beta \circ f]) = \mathcal{S}_{acc}(L[\beta \circ f]).$$

In other words, $\mathcal{U}$ contains a rich enough array of transformations to make all solutions _simultaneously_ accessible, for any given "general position" piecewise-linear fixed point problem. We do not know whether the same is true for $\mathcal{S}$.

VIII. SKETCH OF THE PROOF

Assume first that $n > 3$. Let y and z be solutions of L[f] that are not already adjacent, and assume that y and z have opposite orientation. Our goal will be to show that a transformation exists in $\mathcal{N}$ that creates a 1-path joining y and z, leaving the rest of the paths in L[f] essentially undisturbed; the proof of Theorem 2 then follows easily.

Step 1. We begin by choosing a point y' "in general position" along the 1-path that issues from y. By definition of $\mathcal{R}_i = L(N\setminus\{i\})$, we have

$$(8) \qquad f_2(y') = f_3(y') = \ldots = f_n(y') > f_1(y').$$

There will be a neighborhood of y' in which the functions f_i are all linear and in which the inequality in (8) continues to hold strictly. Inside this neighborhood, we construct a small convex polyhedron C_y, of dimension $n - 2$, cutting the 1-path at y' and containing y' in its relative interior. Similarly we define z' and C_z on the 1-path coming out of z.

Step 2. Next, we lay out a polygonal path P from y' to z', which will serve as the site of the new 1-path to be created. (Figure 4.) The end-segment of P near y' should lie on the opposite side of C_y from the end of the 1-path coming from y to y', and similarly for the end-segment of

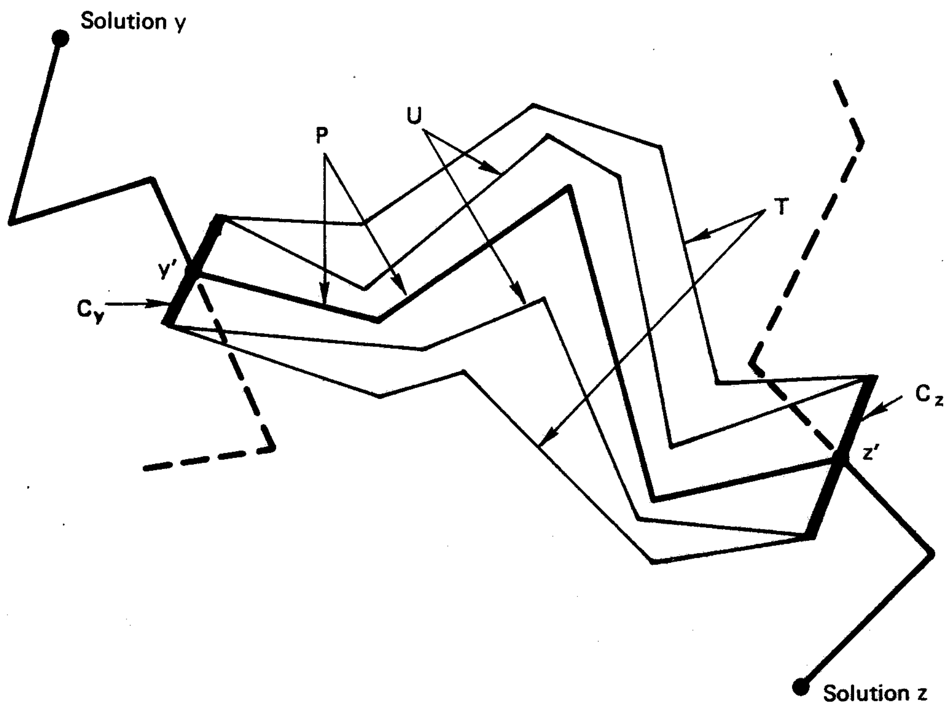

Fig. 4 — Path construction

P near z', so that the trajectory $y \ldots y' \ldots (P) \ldots z' \ldots z$ actually passes <u>through</u> the "caps" C_y and C_z. We also require that P be sufficiently "general" in its placement so that there are at least three different functional values $f_i(x)$ at every point along the way from y' to z':

$$(9) \qquad |\{f_i(x)\}| \geq 3, \qquad \text{all } x \in P \setminus \{y', z'\}.$$

This presents no difficulty if $n > 3$, as we have assumed. (The case $n = 3$ will require a more complicated construction not using (9); see below.) Among other things, (9) guarantees that P does not intersect any existing path of $L[f]$ or go through a solution.

<u>Step 3</u>. There is also no difficulty in surrounding P with two thin "tubes," T and U, within which (9) continues to hold. Let T and U be (n-1)-dimensional polyhedra having the "caps" C_y and C_z as faces in common, but otherwise let U lie strictly inside T and let P be strictly inside U.

<u>Step 4</u>. Proceeding, construct a piecewise-linear function h on the "inner tube": $h: U \to \mathbb{R}^N$, with the following properties:

(a) $h(x) = f(x)$ for all $x \in C_y \cup C_z$;

(b) $h_1(x) < h_j(x)$ for all $j \neq 1$ and all $x \in U$;

(c) $h_2(x) = \ldots = h_n(x)$ if and only if $x \in P$.

One may regard h as a sort of homotopy, continuously deforming the graph of f restricted to C_y into the graph of f restricted to C_z. From this viewpoint, it is clear that such a function h exists if and only if the two solutions y and z have opposite orientation. Figure 5 illustrates this for n = 4. The clockwise arrangement of regions L_2, L_3, L_4 on the left can be "mated" with the anti-clockwise arrangement on the right but it cannot be "mated" with the reverse arrangement (just as, in one lower dimension, a region that is on the right-hand side of a path cannot suddenly jump to the left-hand side.)*

Step 5. We shall construct $\beta \in \mathcal{N}$ in five stages, making repeated use of the following linear transformation matrix (compare with (6) above), where (i, j) is the coordinate of the entry "θ":

$$M_{ij}(\theta) = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & & & & & & 0 \\ & & \ddots & & & & & \\ 0 & & & 1-\theta & & \theta & & 0 \\ & & & & \ddots & & & \\ 0 & & & & & 1 & & 0 \\ & & & & & & \ddots & \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}, \quad 0 < \theta < 1.$$

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*Orientation theory can be extended from $\mathcal{S}$ to $\mathcal{R}_i$, where it provides an intrinsic sense of direction to every point on a path or loop of a labelling (cf. Shapley (1974)). Depending on its own orientation index, every solution will then have all its paths either pointing "in" or pointing "out".

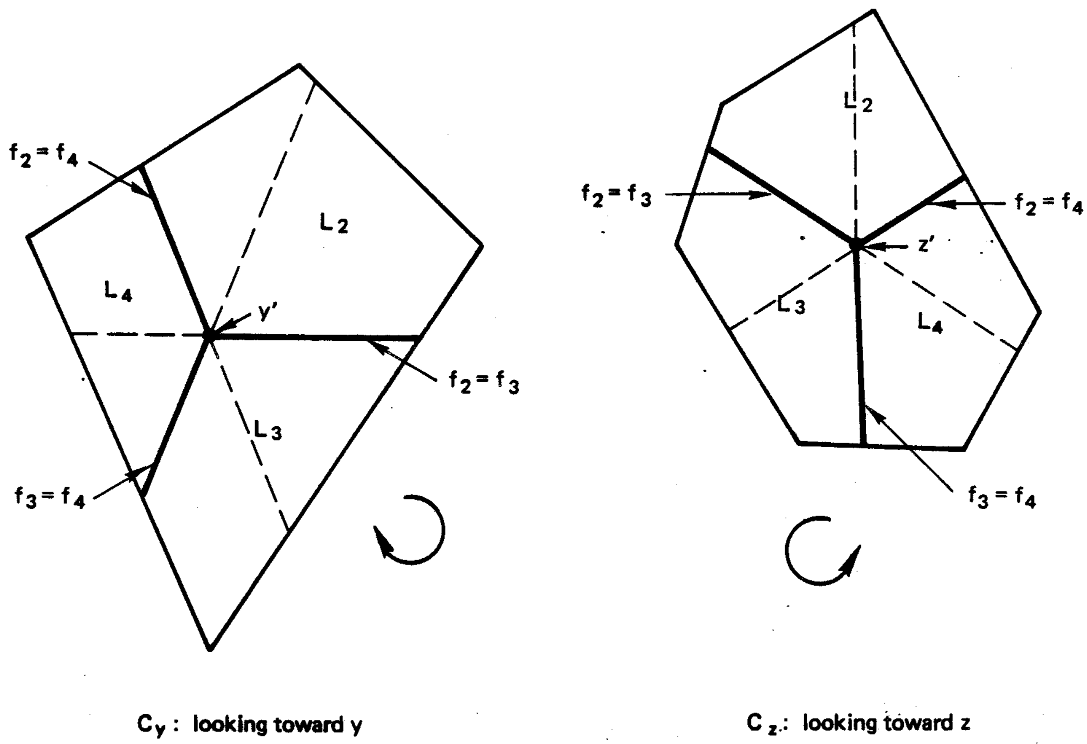

Fig. 5 —Orientations at the caps

As we saw in Sec. VI, the effect of this transformation is to "pull" the value of $f_i(x)$ toward the value of the "anchor" function $f_j(x)$, leaving all else unchanged.

A. Removing f_1 from the top. Consider the "stack" of functional values $f_i(x)$ at each $x \in U$. By (9), there are at least three levels. Whenever $f_1(x)$ happens to be on the top we can pull it down below the second level by applying $M_{1j}(\theta)$ using some $f_j(x)$ on the third or lower level as "anchor". The value of θ should be high enough to do the job, and must of course be continuous in x, tapering off to 0 in those parts of U where $f_j(x)$ ceases to be below the second level or where $f_1(x)$ ceases to be on top. It should also be tapered off to 0 in the "buffer zone" T\U. A transformation $\gamma_{(1)}$ is thereby obtained which eliminates f_1 from the top position in a certain portion of U, and which may be defined to be the identity in A\T. Repeating as necessary, after a finite number r of such steps we obtain a transformation $\gamma = \gamma_{(r)} \circ \gamma_{(r-1)} \circ \cdots \circ \gamma_{(1)}$ such that $(\gamma \circ f)_1$ is not maximal anywhere in U. Note that (8) eliminates any trouble at the "caps" C_y and C_z.

B. Putting f_1 on the bottom. In the same way, we can now pull up any of the functions that may happen to lie at or below the level of $(\gamma \circ f)_1$, letting those that lie above $(\gamma \circ f)_1$ serve as "anchors". We thereby obtain δ such that $(\delta \circ f)_1$ is strictly minimal throughout U.

C. Re-aligning the bottom. Next we perform a simple translation, raising all functions together by the amount

$$h_1(x) - (\delta \circ f)_1(x)$$

at each $x \in U$, and tapering off to 0 in $T \setminus U$ as before. This yields a transformation ε such that $(\varepsilon \circ f)_1 = h_1$ throughout U, with all other $(\varepsilon \circ f)_j$ still lying above $(\varepsilon \circ f)_1$. Outside T, ε is still the identity transformation.

D. Spreading the stack. At each $x \in U$ we multiply the distance of all the functions above $(\varepsilon \circ f)_1$ by a factor $c(x)$, sufficiently large to lift each function to or above its prescribed value $h_j(x)$ throughout U. Taper $c(x)$ continuously down to 1 at the caps C_y, C_z, as well as in the buffer zone $T \setminus U$. The resulting transformation η has the property that $(\eta \circ f)_i \geq h_i$ in U for all i, with equality for $i = 1$.

E. Finishing the job. Finally, by the now-familiar technique, we pull down each of the functions $(\eta \circ f)_j$, $j > 1$ to its prescribed value h_j, using $(\eta \circ f)_1$ as "anchor." Call the resulting transformation ζ; it has all the essential properties of the members of $\mathcal{N}$, including continuity, except that piecewise linearity may have been lost along the way. However, the construction has been "robust" throughout, so we can take $\beta \in \mathcal{N}$ to be a piecewise-linear approximation to ζ in such a way that $L[\beta \circ f]$ agrees exactly with $L[f]$ outside the "outer tube" T, and yet has a 1-path joining the solutions y and z along the course y ... y′ ... (P) ... z′ ... z. The only other effect of β on the path network will be the realignment of the unused portions of the original 1-paths out of y and z; they will now be connected somehow to each other by a section of 1-path that will necessarily have been created in the "buffer zone" $T\setminus U$ by our manipulations. (It is possible also that some loops will have been created in $T\setminus U$, but no additional paths.)

Once we agree that any two solutions of opposite orientation can be made adjacent by this construction, there is no further real difficulty in interconnecting all solutions simultaneously, even when $n = 3$. Since at least one solution is always accessible, they all become accessible.

The argument for $n = 3$ requires a more complicated transformation that rotates $\mathbb{R}^3$ about its diagonal in order to put f_1 on the bottom throughout U. This rotation permutes the labels 1, 2, 3 of the f_i, so that when one end of the tube is reached, say C_z, the labels may not match across the cap. It is then necessary to extend the tubular neighborhoods T and U to include the solution z itself. This having been done, the new 1-path runs its prescribed course y ... y′ ... (P) ... z′ ... z, just as before. Moreover, the previous 2-path and 3-path leading to the solution z, though modified by their passage through the "buffer zone," still necessarily reach their destination, since they have nowhere else to go. We omit further details.

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