A RAND NOTE

Overview of Object-Oriented Microworlds for Learning Mathematics Through Inquiry

David McArthur, Matthew Lewis

RAND

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PREFACE

This Note describes ongoing research at RAND to develop intelligent computer tools for education, funded by the National Science Foundation (Applications of Advanced Technologies Program) under grant MDR-8751515. For the past five years, this research has designed, implemented, and tested various computer-based intelligent tutoring systems and tools. Each of the computer tutors or tools is designed to help students learn different aspects of high school mathematics more rapidly and effectively, using advanced computer software, ideas from artificial intelligence, and state-of-the-art computer hardware.

Earlier documents on this research described the implementation and fielding of an intelligent tutoring system for beginning algebra. This Note discusses the design, implementation, and use of *computer microworlds* that support learning about different kinds of mathematical objects through mathematical discovery or inquiry. This Note should be of interest to researchers in computers and education, and to mathematics teachers and educators.

SUMMARY

This Note discusses *computer microworlds* that have been designed to support the learning of specific mathematical skills involved in discovering, confirming, explaining, and communicating patterns of relationships among variables in mathematical objects. It begins by discussing the software tools themselves and how they promote learning through guided inquiry. The microworlds software is built on a *kernel*, including an object-oriented language for representing mathematical objects, a constraint mechanism for propagating information about values of properties of mathematical objects, and tools for multiple representations of patterns of relationships among mathematical properties (e.g., graphs and tables of values). Specific mathematical microworlds are built on the kernel software and add routines to display and manipulate the mathematical objects that define the particular world.

After discussing the software kernel in general, we report the results of fielding our first microworld—*Polygons*—with high school students. In *Polygons*, students learn through *guided inquiry* about the relationships among polygon properties, including number of sides, interior angle, exterior angle, apothem angle, radius, length of side, apothem, perimeter, and area. We will report some interesting mathematical conjectures young students were able to discover, test, refine, and explain. In addition, because students' skills at learning through inquiry appear limited in several respects, we discuss some of the problems or barriers to learning through discovery that we noticed, and we also describe how software is designed to overcome some of these barriers, and how we propose to extend the microworlds to provide improved learning support.

ACKNOWLEDGMENTS

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I. INTRODUCTION

The past years have heard many calls to reform mathematics education (National Commission on Excellence in Education, 1983; National Council of Teachers of Mathematics, 1989; Resnick, 1987; Thompson, 1988). There is widespread agreement that many traditional skills—for example, mechanical symbol manipulation abilities—are not as important as once envisioned. In part this is so because students can manipulate equations without understanding, because such manipulation is not representative of most mathematical reasoning, and because this kind of mathematics can be done by computer. Mathematics educators have suggested different kinds of skills to take the place of traditional mathematics content in the classroom. Many of the proposals have suggested changes not only to the content of the mathematics curriculum but also to how mathematics is taught. In particular, there is a growing consensus that students should learn through inquiry and through the construction of their own mathematics (National Council of Teachers of Mathematics, 1989; Papert, 1980). However, these suggestions imply substantial unresolved problems for the teaching of mathematics. What specific skills shall we focus on, and what tools can we provide to help students learn, and teachers teach, through inquiry?

In this Note we discuss *computer microworlds* that have been designed to support the learning of specific mathematical skills involved in discovering, confirming, explaining, and communicating patterns of relationships among variables in mathematical objects. We will begin by discussing the software tools themselves and how they promote learning through guided inquiry. Although learning through inquiry is important, most students appear to have limited skills in this area. We next discuss some problems or barriers to learning that we noticed while observing students use the microworlds. We subsequently describe how the software is designed to overcome some of these barriers, and how we propose to extend the microworlds to provide improved learning support. In addition, we also discuss why the activities for using the microworlds are as important as the software itself in determining the overall educational success of the environment. This will bring us back to how our microworlds might best fit into the classroom.

II. LEARNING ABOUT MATHEMATICAL RELATIONSHIPS THROUGH SELF-DIRECTED INQUIRY

Two main goals inform the design of our mathematical microworlds. First, much high school mathematics education focuses on teaching tools that are useful for understanding and analyzing relationships among variables or properties of everyday objects. Unfortunately, teaching the tools often becomes an end in itself (e.g., students spend months learning to solve equations and factor complex expressions). Frequently, these isolated exercises fail to help students make sense of equations; where they come from, why they are useful, and how they fit into other mathematical activities. Our view is that the real payoff of equations and other mathematical tools is their ability to help make powerful inferences or to provide abstract models of relationships. More generally, as Resnick (1987) has noted, we wish to promote the idea of learning mathematics as a set of integrated tools for making sense of mathematical situations.

One major goal therefore has been to provide microworlds in which students can acquire a full range of skills and tools for analyzing relationships among properties of objects. We want to support students while they engage in a broad range of activities, including *discovery, confirmation, explanation,* and *application* of *mathematical models.* Because understanding relationships among variables is key, we do not insist that students necessarily attempt to infer equations or quantitative models of situations; more qualitative characterizations are also encouraged. Instead of starting with a situation that has already been "pre-analyzed" into an equation or other formal representation, students begin with pictorial or "simulation" representations of the objects themselves and infer the nature of the relationships among properties or variables, using whatever tools seem appropriate. Students' activities often take on a "quasi-empirical" or discovery character, where they first choose to explore relationships among one pair of variables, then move on to others. In brief, we want to help students make sense of mathematical abstractions, such as equations, by placing them in a context in which equations arise to make sense of relationships among everyday objects.

A second related theme underlying the design of our microworlds is that students should learn about the process of mathematical discovery itself. There is no rigorous task analysis of mathematical discovery or inquiry comparable to more "well-defined" skills, such as factoring quadratics or performing integration. However, following Lakatos (1976) and Steen (1988), we see discovery as involving the following activities:[1]

- Gathering observations that bear on an issue or topic (an issue is not a conjecture, which must either be true or false, but more of an open question).
- Generating conjectures or hypotheses.
- Confirming a conjecture.
- Disconfirming or refuting a conjecture.
- Refining a conjecture.
- Explaining or proving a conjecture (decomposing a conjecture into "smaller" conjectures that are already regarded as proved).

Mathematical discovery skills have been championed in the past (e.g., Polya, 1962) and more recently (e.g., Schoenfeld, 1985; National Council of Teachers of Mathematics, 1989). Proponents see these discovery skills as intrinsically valuable, representing a key part of mathematical reasoning. Discovery is now also viewed as an important *method* of learning specific mathematical content areas, as well as a *topic* of learning. Papert (1980), among others, has argued that learning through self-guided discovery enables students to actively "construct" their own understanding of mathematical topics, in contrast to didactic methods in which knowledge is communicated to a relatively passive student. Such active construction has been regarded as crucial in promoting a deep understanding of mathematical concepts (e.g., National Council of Teachers of Mathematics, 1989).

Our mathematical microworlds do not support all of the facets of inquiry listed above, nor are selected activities supported in a uniform fashion. In particular, although we have developed intelligent tutoring systems (ITS) for algebra (McArthur, Lewis, Ormseth, Robyn, Stasz, and Voreck, 1989; McArthur, Stasz, and Hotta, 1987; McArthur, Stasz, Hotta, Peter, and Burdorf, 1988), the microworlds do not actively tutor students about inquiry skills. Rather, students learn various aspects of mathematical inquiry by engaging in them, using various microworld tools and activities designed to support and encourage them. Similarly, while an ITS might attempt to tutor mathematical discovery skills separately from skills for learning about relationships among variables of everyday objects, our microworlds attempt to combine the two in a synergistic fashion. However, our algebra ITS software does fulfill several important roles at this time; for example, it supplies the equation manipulation necessary for constraint propagation.

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[1]We will use "discovery" or "inquiry" almost interchangeably in this Note, although a case might be made that inquiry activities include discovery activities as a subset and also include additional activities, such as explanation and proof, that are not strictly part of initial discovery.

III. OVERVIEW OF THE MICROWORLDS

Our microworlds all share a common structure, permitting us to develop different worlds relatively easily. Central to each world is the definition of one or more types of objects. Each object possesses several different properties that are the focus of student learning. The properties are subject to several important constraints: they must have numerical values, and it must be possible to define equations that govern the relationships among properties. The microworlds exploit these constraints to provide powerful tools.

To define a specific microworld, we simply need to stipulate the objects, their properties, equations that relate the properties, and routines for drawing objects. These hook seamlessly into the "kernel" microworld software. The microworld kernel provides tools and routines that apply to all microworlds in a uniform way. For example, equation-manipulation routines take the equations for a microworld, and values for a subset of object properties, then infer values for new properties (constraint propagation); and graphs and tables provide alternate representations of properties of objects.

The microworlds kernel is implemented in the object-oriented languages CLOS (Common Lisp Object System) and CWO (Common Windows Objects), both implemented in Allegro Common Lisp. The object-oriented structure of the microworlds provides important flexibility. All data analysis and organization tools (e.g., graphs and tables) are represented as computational objects, as are the domain objects of each specific microworld. As we will see in the examples below, these objects can be combined or "connected" in powerful ways to support learning and inquiry.

DEMONSTRATION OF THE POLYGONS MICROWORLD

The examples we discuss in this Note come from students using microworlds in a "lab" setting. Typically, one or two students at a time use the software, sharing inquiries. Observers are present to chronicle events, and one is designated to interact with the students (the participant observer). Initially, the observer will act as a mentor, modeling use of the software and demonstrating how to conduct inquiries. As we discuss later, this role fades and transforms over time.

When a student begins working with a specific microworld, he or she sees various tools that help create instances of the objects, generate and gather data about objects, see patterns of relationships in the data, and record and test conjectures. Figure 1 shows a

version of our *Polygons* microworld. The *Polygons* properties include number of sides, interior angle, exterior angle, apothem angle, radius, length of side, apothem, perimeter, and area. (Figure 2 shows a polygon with all these properties labeled). As in all microworlds, students usually begin with an issue or goal. Even in a world as simple as *Polygons*, many issues can be examined, from very simple (e.g., how does interior angle vary with the number of sides?), to rather subtle (e.g., if area is held constant, what happens to apothem as the number of sides increases?); and from quantitative (e.g., find an equation that relates exterior angle to number of sides) to qualitative (e.g., how many properties need to be given to determine all others?). Below we discuss how these issues are negotiated. In Fig. 1, the student's goal is to characterize the relationship between N (the number of sides of a polygon) and R (radius), holding P (perimeter) constant. Will R increase or decrease, and how can this change be characterized?

Generally, students begin considering an issue by gathering data. Of course, for relatively simple issues, advanced students may be able to develop conjectures—even proofs—confidently without empirical guesswork. For many issues, however, students will often have little mathematical intuition with which to generate conjectures. Data can often stimulate conjectures, and even students able to generate conjectures usually need the confirmation that data can provide. Among others, Lakatos (1976), Putnam (1975), and Steen (1988) argue that mathematicians, not just students, use "quasi-empirical" methods.

The simplest way to gather data in *Polygons* is to generate instances interactively. The student uses the mouse to click on the "Polygon" button (upper left), and a template giving all the properties of the polygon is displayed (see Fig. 1). Initially, no properties have values; they are typed in by the student. Polygons have two degrees of freedom. When one scale property (radius, length of side, apothem perimeter, and area) and one angle property (number of sides, interior angle, exterior angle, apothem angle) are given, all the others can be determined. Thus, in Fig. 1, after the student has selected a value for P (400) and N (50), the remaining values are computed and displayed by the system. When the student has created a specific polygon, it appears in the pictures window at the lower left. Data concerning an issue may also be generated by typing commands that create polygons. For example, in the commands window at the upper right of Fig. 1, the student has created seven polygons by varying N and holding P constant (400).[1] When students first begin investigating a microworld, they usually create objects one by one. However, they often graduate to the more efficient means afforded by commands.

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[1]Automatic command completion frees students from having to type in names for properties.

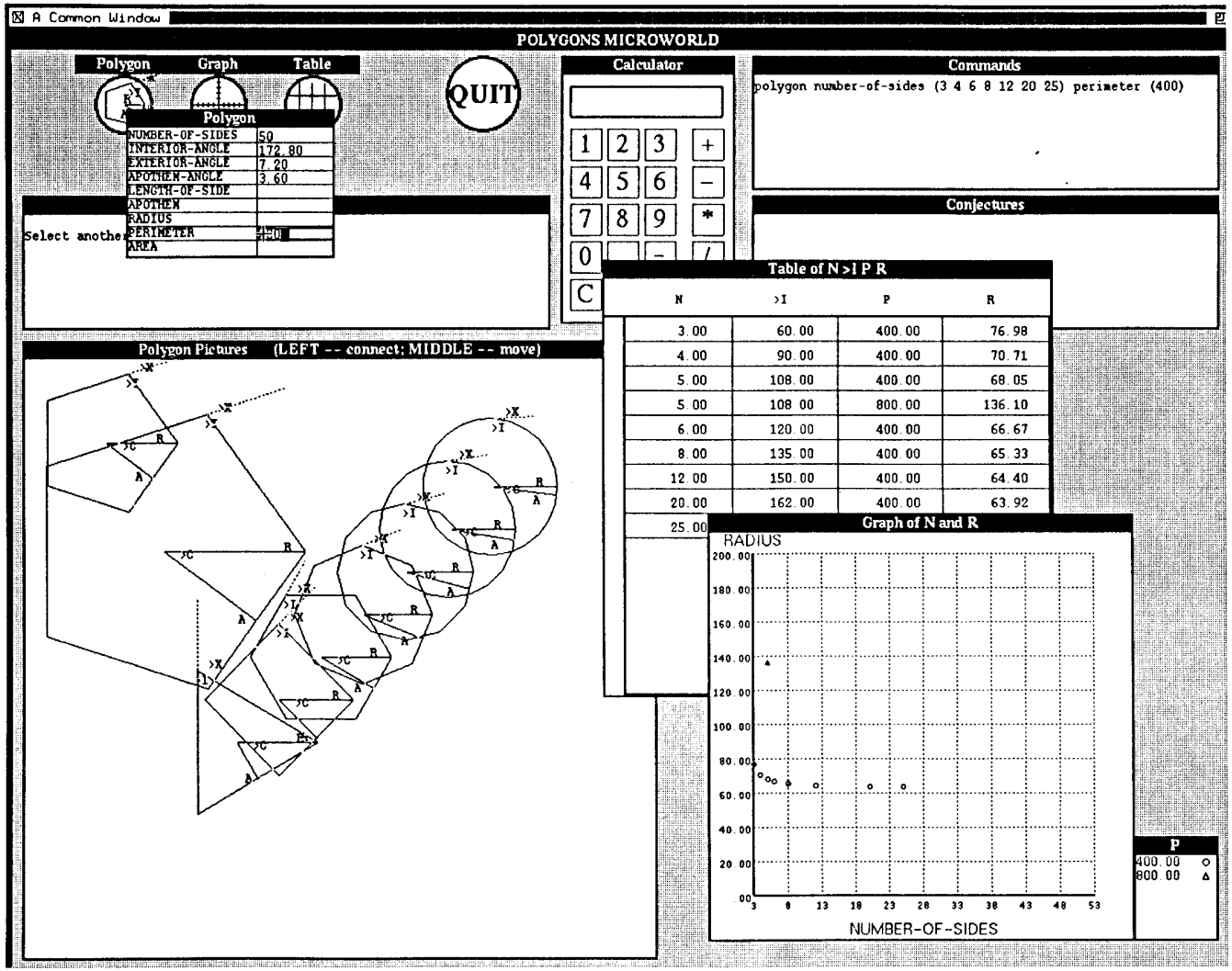
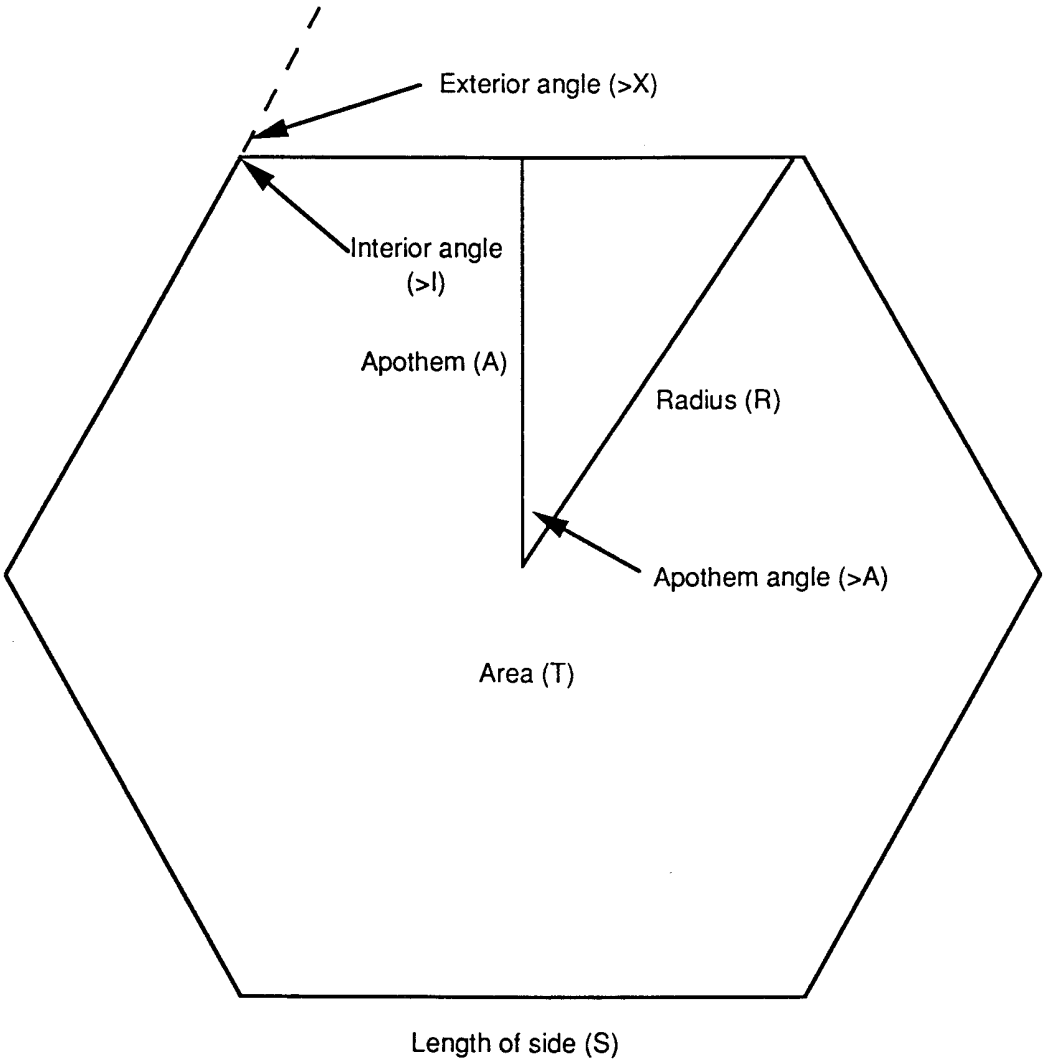


Fig. 1—Investigation of N and R



* Number of sides (N)
* Perimeter (P)

Number of sides	6
Interior angle	120
Apothem angle	30
Length of side	100
Apothem	86.60
Radius	100.0
Perimeter	600.0
Area	25980.76

Fig. 2—Polygon properties

To investigate an issue efficiently, the student needs experimentation skills. In this case, the student needs (at least intuitively) to understand the ideas of selecting a dependent variable (R), of choosing an independent variable (N and P), and of holding one independent variable (P) constant while systematically altering the other (N). Using only pencil and paper tools to explore microworlds, students would probably be limited to varying such easy-to-control properties as N or the length of a side and measuring the values of all others—although some of these, like area, might prove quite difficult to measure accurately. However, because the microworlds provide a general constraint-propagation algorithm, there are no restrictions on which properties the students can choose to vary, and measurements are accurate and instantaneous. The students therefore may play rapidly any kind of "what if" game they can imagine. This freedom gives students considerable power in fashioning inquiries, but it also demands experimentation skills to use the power effectively.

Gathered data can be represented using several microworld tools. For example, in Fig. 1, the student has represented the created polygons in both a table and graph. Tables can record any number of polygon properties. Graphs only encode the relationship between pairs of properties, although a different symbol may be used to represent distinct values of a third variable (for example, in Fig. 1 the student uses different symbols to plot polygons whose P is 400 and 800). In general, graphs give a selective, visual perspective on the data and are useful in describing relationships when the student has determined the specific properties of interest. On the other hand, tables can provide a broader but less vivid view of data and relationships. They may be more appropriate when students have only a vague idea of the relationships they are trying to characterize. Our limited experience suggests that students are often technically competent using tables and graphs and can interpret their meanings readily. However, the students' understanding of the uses of tables and graphs appears poorly developed. Perhaps because graphs and tables are taught in classrooms as isolated artifacts, students are often unaware of their appropriate roles as *tools* in specific inquiry activities.

Both tables and graphs can be dynamically tailored by the student to simplify the examination of data. Tables can be sorted by the values of any property, permitting easy browsing for data patterns. Graphs can be "zoomed" to focus on a smaller or larger area of interest. As Fig. 3 shows, the student interactively selects a rectangle he or she wants to examine. Figure 4 shows the resulting expanded graph. This figure also shows the student recording a conjecture for the current issue (below the commands window), including both a qualitative description of the relationship and an informal explanation.

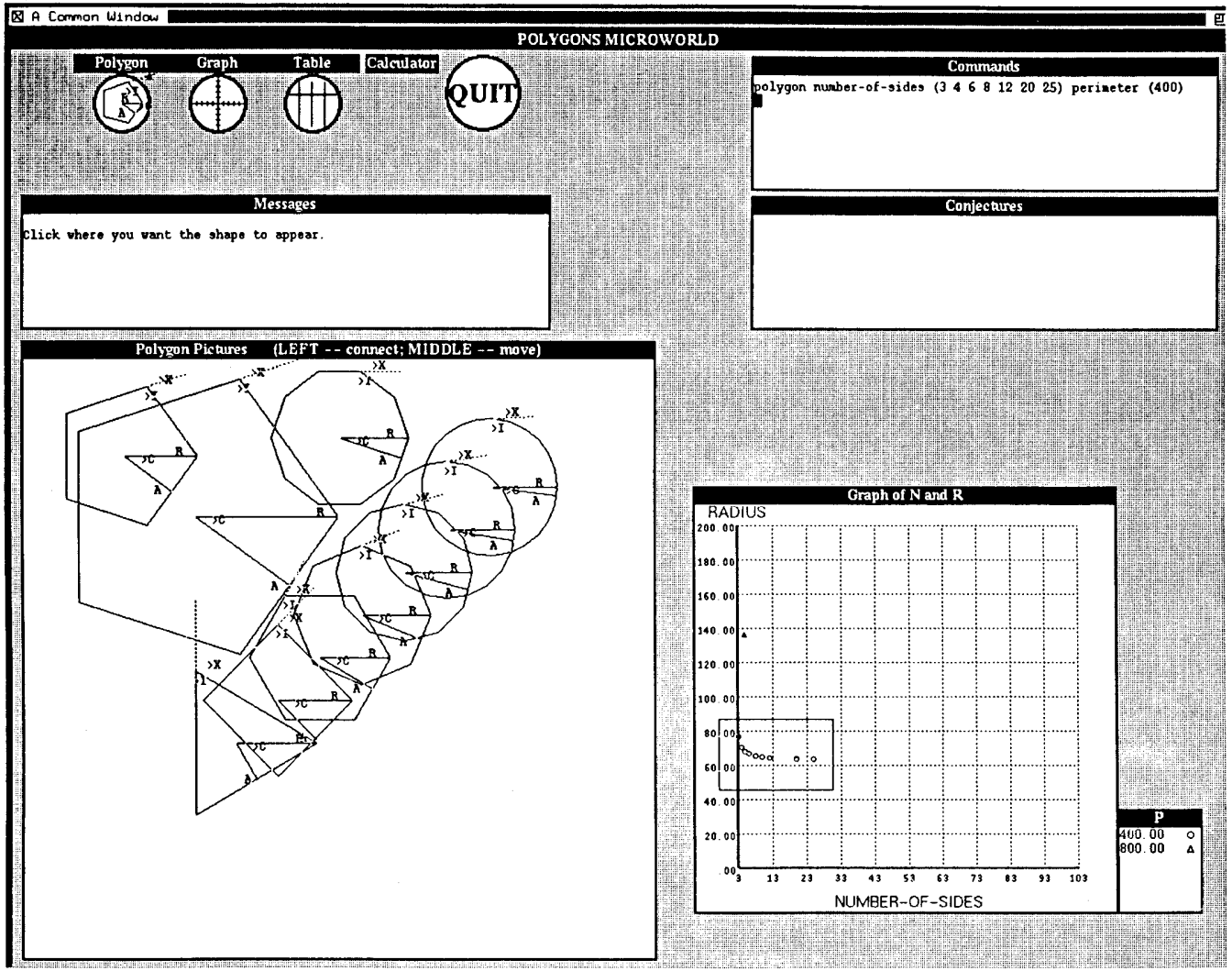


Fig. 3—Zooming in on a graph

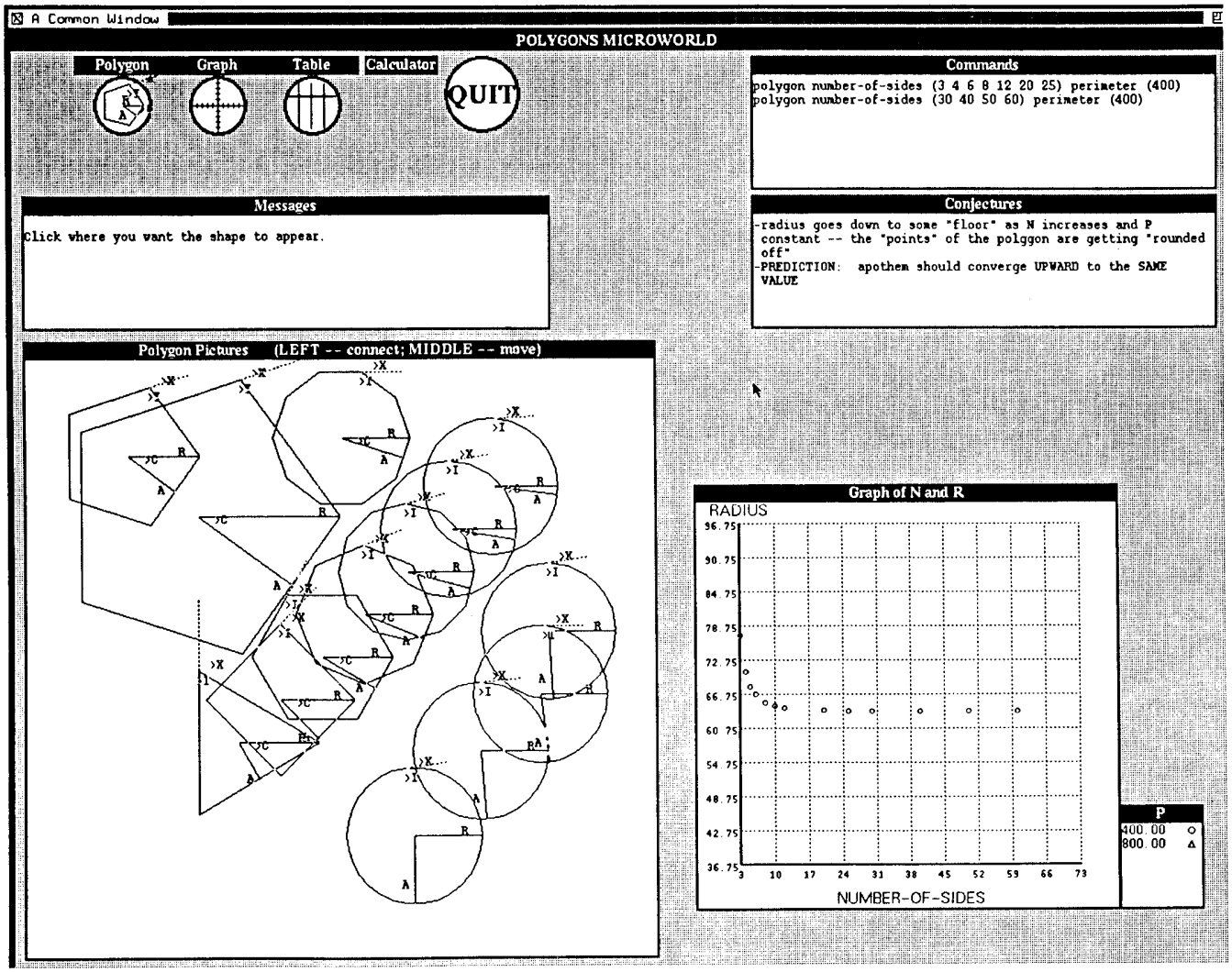


Fig. 4—Completion of graph zooming

Our hope is that the investigation of one issue will trigger interest in new ones. In Fig. 4, the student has suggested a conjecture related to the one just completed: Since R decreases as N increases because the "points" of polygons are being "rounded off" (as the polygons approach a circle), the apothem (A) should increase to the same asymptote (A should be equal to R as N increases to infinity). Testing this conjecture is straightforward, as shown in Fig. 5. The student does not need to gather new data. Rather, the existing polygons simply need be viewed from a different perspective. The student creates this view by generating a new graph for the properties N and A and then "connecting" the graph of N versus R to the graph of N versus A. The new graph then shows that the student's conjecture about N and A is confirmed.

This example demonstrates a general feature of our microworlds. Graphs and tables are object-oriented in the sense that they represent polygons (in this microworld), not arbitrary data values. They simply present objects from different perspectives. Thus, when we connect one data representation to another we are effectively putting the objects contained in one into the new filter or perspective represented by the other one. In general, any pair of data representations can be connected. When connected, objects in one become shared by the other, and as new objects are added to one representation, they also appear in the other. Representations can be disconnected as desired.

QUANTITATIVE INQUIRIES

In the exploration discussed above, the student's conjectures were qualitative, focusing on whether one variable increases as a direct function of another, and on the shape of that relationship. Depending on the issue and on the student's skills, students may generate more quantitative conjectures. For example, Fig. 6 shows the culmination of a rather long inquiry in which the student quickly established an equation relating N and the apothem angle (>A): $>A = 180/N$.[2] To establish this rule, the student successfully used the strategy of multiplying (>A*N is a constant) and attempted to apply this or a similar strategy to describe the relationship between N and >I. This activity was to some extent motivated by the patterns in the graphs of N versus >I and N versus >A. The student had noticed the patterns were the "reverse" of one another (see the mirror-image graphs in Fig. 6). Hence the student was trying to find a way of "reversing" the operation in $>A = 180/N$. Attempts to generate a constant by multiplication (N*>I) or division (N/>I) were not successful. However, when the student tried >I/>A, he found it yielded N – 2, leading to the equation $>I/>A + 2 = N$.

[2]On screen images in the figures, the apothem angle is referred to using the symbol >C. This has recently been changed to >A, and we use the latter term throughout the Note.

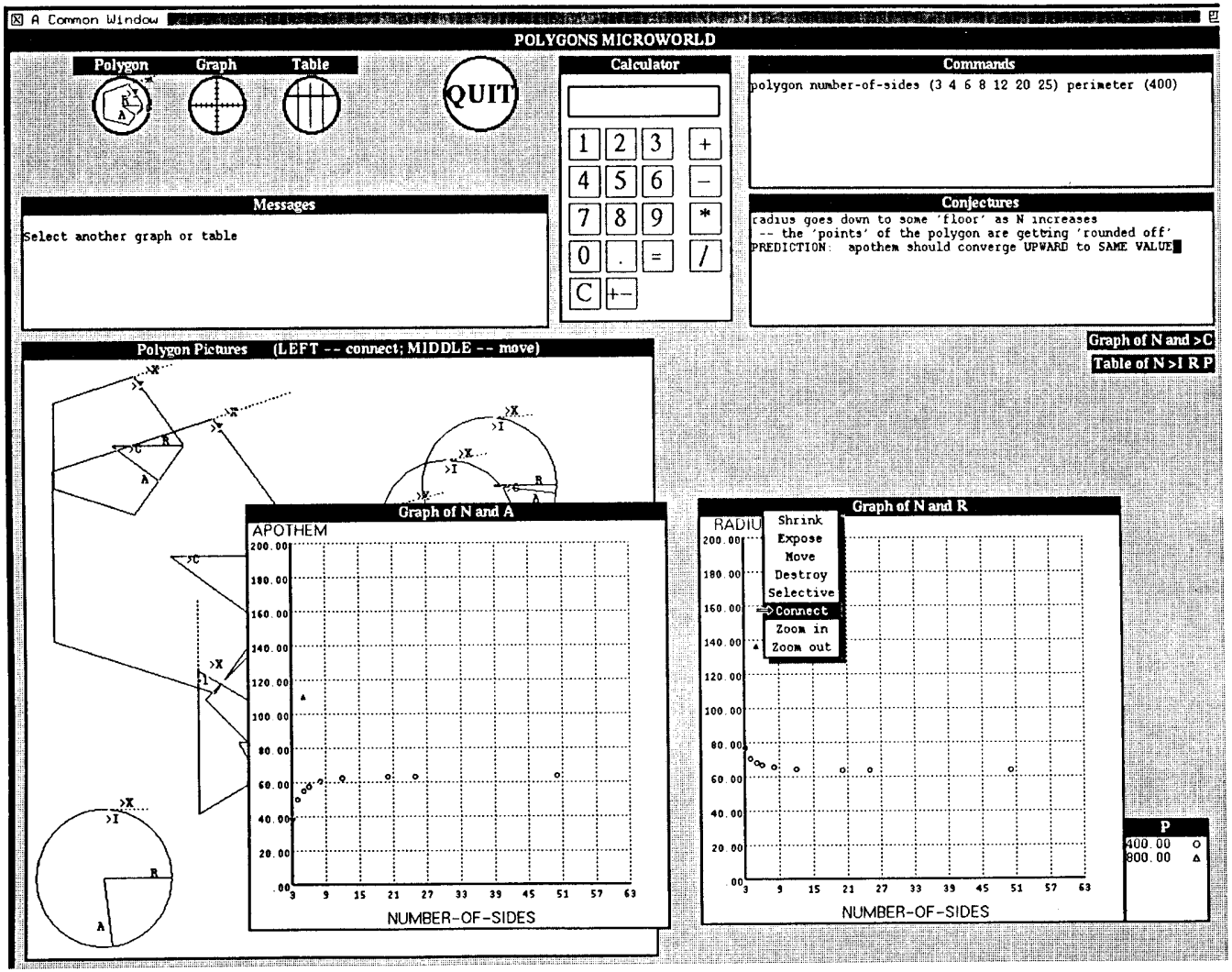


Fig. 5—Connecting different representations

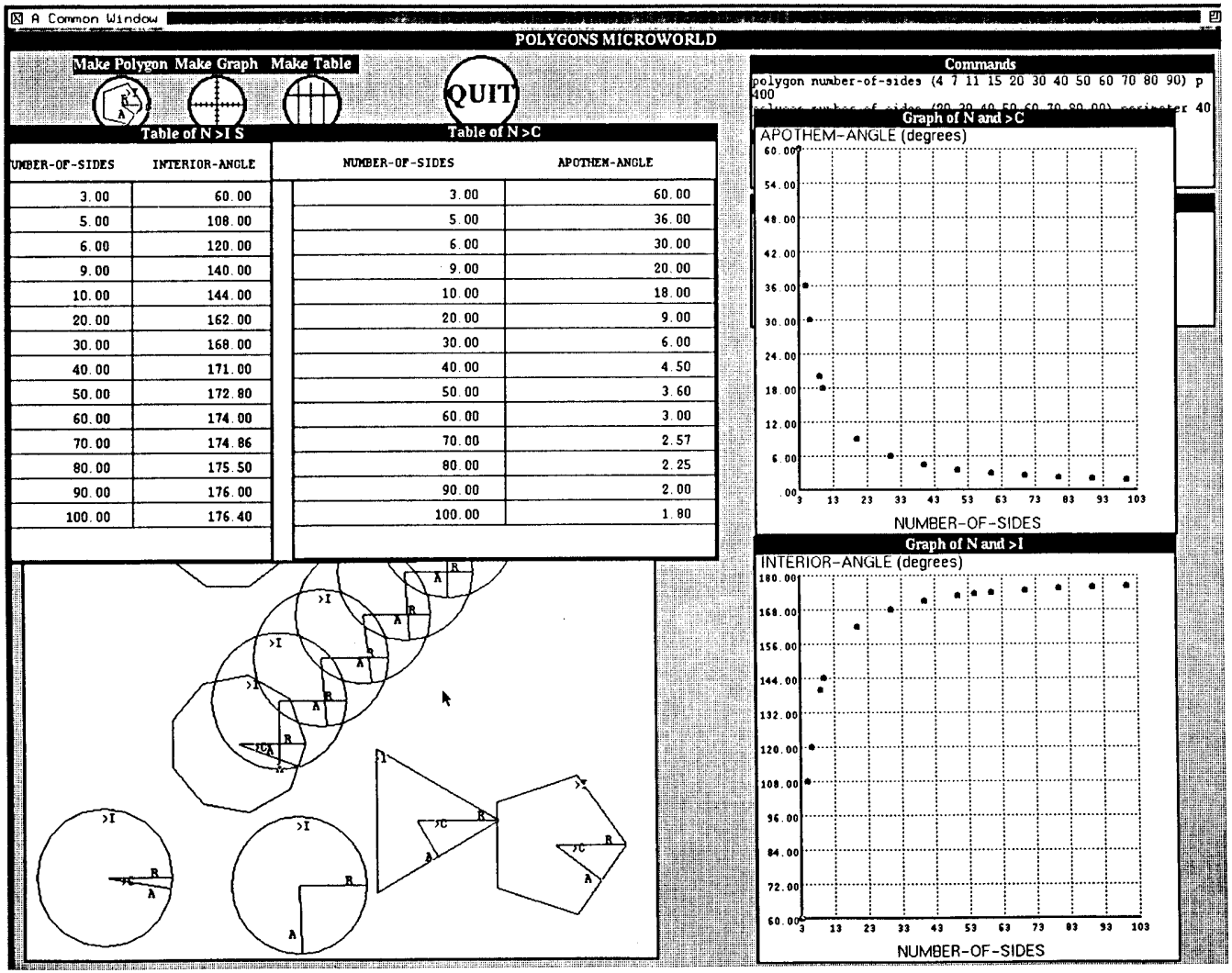


Fig. 6—Screen of investigation of >I and N

There are several points to make about this inquiry. First, although it centered on an equational expression of a relationship, the equation was discovered by the student using some relatively sophisticated techniques; it was not given to the student. Second, notice how one conjecture built upon previously established ones. The student began with the relatively simple issue of characterizing the relationship between N and >A (this was preplanned for the student) and spontaneously used the understanding acquired there in tackling the more complex issue. The skill of building one explanation or conjecture on simpler ones—the *incremental* nature of mathematical inquiry—is common in authentic mathematical practice, although rarely treated in classrooms. Third, although in this inquiry the student formulated an equation, we placed few constraints on the structure of the equation discovered. In fact, the equation the student produced came as a novelty to the researchers observing the session. The "standard" equation is $>I = 180 - 360/N$. Making sense of the novel equation became the topic of an ensuing inquiry that included the student and observers. Fourth, we noted that the student did not spontaneously attempt to make sense of his equations in some respects. In particular, although he checked their validity empirically, the student did not naturally engage in more deductive reasoning that would explain why the equations must be true. Thus, for example, the student could not appreciate why the standard equation, $>I = 180 - 360/N$, might be preferred to forms like $>I/>A + 2 = N$. The former admits of intuitively satisfying explanations along the lines that >I approaches a straight line (180) as N approaches infinity, and the difference between >I and 180 is the exterior angle, 360/N. We will return to this observation in a later section.

Finally, we wish to emphasize that, as in the first example, constructing equations is not the sole goal of inquiry, nor even the main one. Our aim is that students acquire skills in inquiry or discovery and skills in describing relationships among variables. Equations are one way to describe relationships, and they have particular strengths and weaknesses. The main strengths of equations are abstractness and power in concisely encoding a great deal of information about relationships. But this is their weakness as well. Their terseness hides many important implicit or "emergent" features of relationships. Some of these features may be inferred from the equations. However, it is often more natural to derive features from other representations of relationships, as we saw in the first example, where the student developed a more qualitative characterization using graphs and tables. We encourage students to generate relatively discursive, often qualitative, descriptions of relationships and to view equations as just one aspect of a complete description. The acts of establishing, confirming, explaining, and communicating the relationships are primary. Equations, as well as other representations like graphs, are useful tools in these activities.

That equations hide many emergent features is nowhere more apparent than in the study of chaotic and fractal systems (Gleik, 1987). Some systems with profoundly complex behavioral features can be characterized by a few nonlinear equations (Lorenz, 1963). These equations have been used for centuries, but their complex emergent features have become a focus of mathematics only in the last few decades. One important area of research has been to develop new tools (e.g., phase diagrams, fractal pictures) to assist in the characterization of chaotic features that traditional tools like equations are incapable of showing. Thus, in this respect, students' work in generating qualitative descriptions of relationships parallels authentic mathematical activity.

LEARNING DIFFERENT MATHEMATICAL CONCEPTS

In addition to learning skills for acquiring quantitative and qualitative models of mathematical objects, and learning inquiry skills, the microworlds afford other learning opportunities, some of which we envisioned beforehand and some we did not. Most obviously, students can acquire knowledge about the specific objects that populate the world. In the *Polygons* microworld, the students learn some of the "standard rules" for polygons (e.g., $T = 1/2A*P$; $>I = 180 - 360/N$) that constitute most classroom treatments. In the *Triangles* world we are now designing, they will be able to learn basic concepts and identities in trigonometry, and important related laws like Pythagoras. The content of these microworlds often crosses over traditional subject boundaries; ideas from geometry and algebra may merge in a single inquiry.

Although these issues are easy to anticipate, some of the most interesting discussions arise unexpectedly. Figures 7 and 8 show a trace of an investigation of the idea of limits. Two students, who had recently completed sophomore high school geometry together, began with a seed issue we had planted: What happens to T (Area) as N doubles when the length of a side (S) is kept constant? The students conjectured that T should also double. The students first began generating polygons with S = 100. However, for values of N greater than 12, this ran into the practical limitation that the polygons were too big to see on the pictures window (see Fig. 7, Table of N, T, and S). At this point, the students decided to change the investigation in a fortuitous way. To keep the polygons small enough, they decided to halve S as N doubled (without realizing at this time that P would therefore be constant). After several polygons were created and put into the table, the students observed an apparent convergence of T, which was confirmed as polygons of up to 192 sides were generated. They input a conjecture to describe this finding, and one of the observers then asked what would happen as N approached infinity.

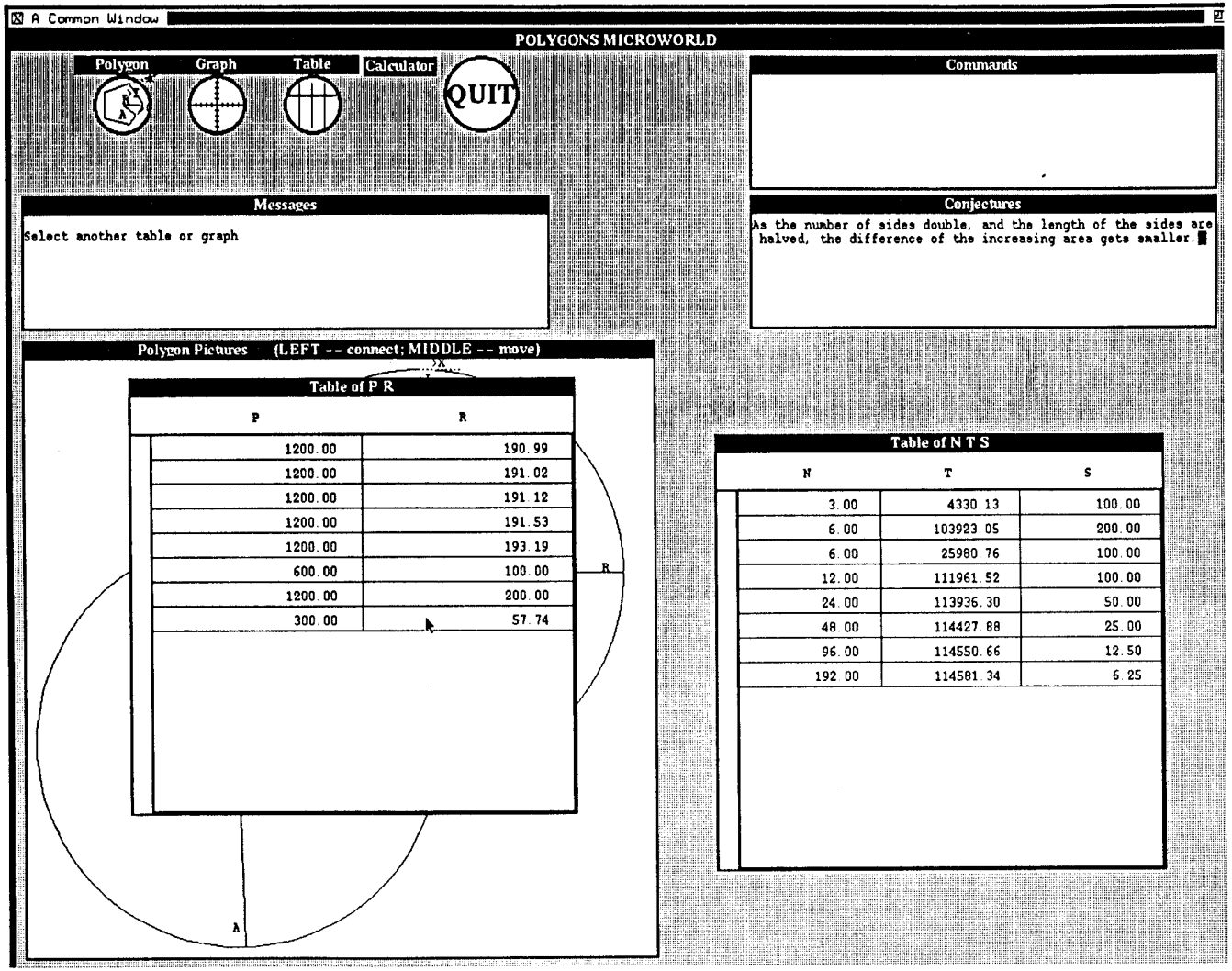


Fig. 7—Investigating limit of area as number of sides increases

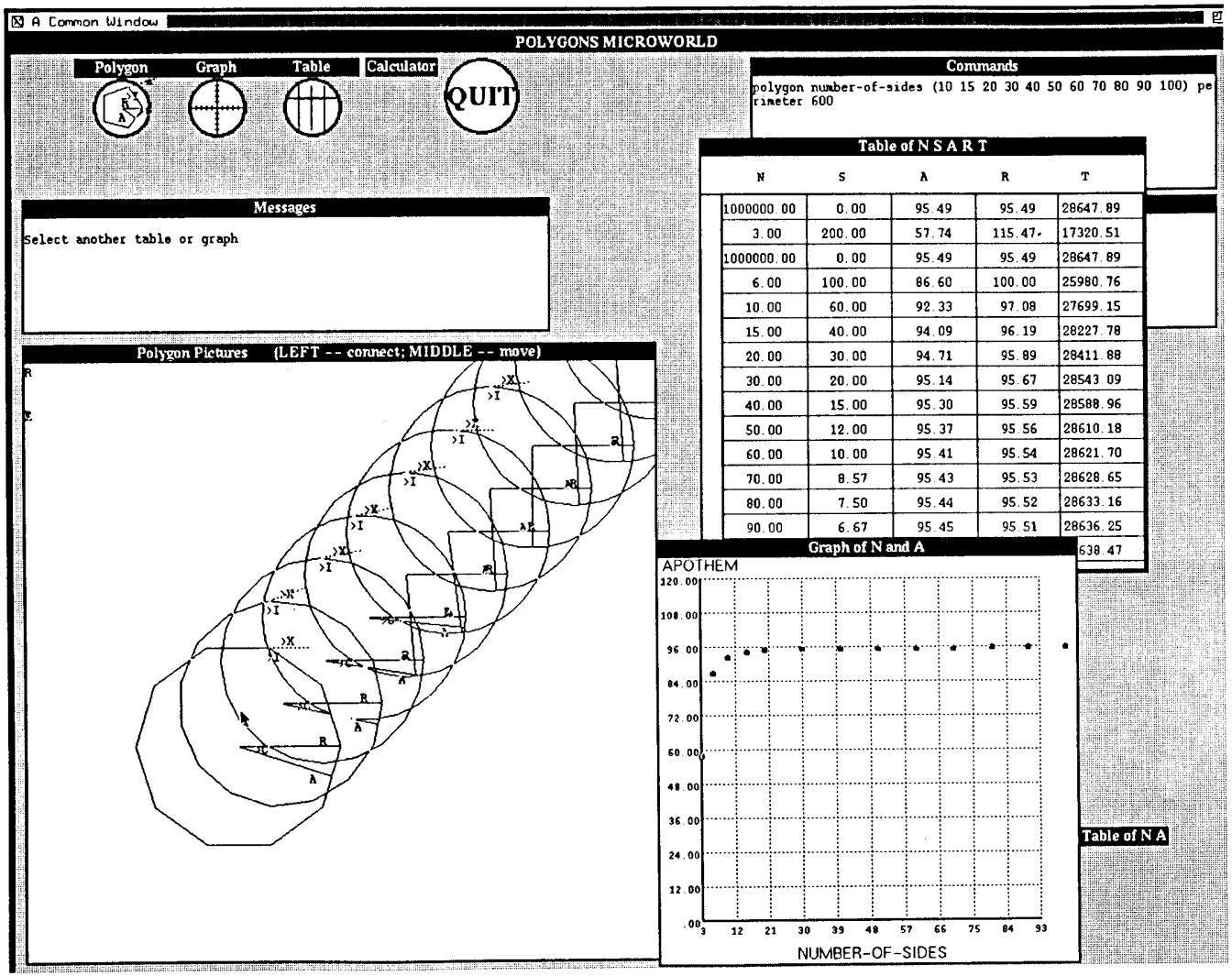


Fig. 8—Investigating how polygons become circles as N increases

It was clear the students were confused by this question. They conjectured that T would continue to increase with no upper bound, or that T might even begin to decrease, although they could not say at what value of N this might happen. Eventually, the students used several tools to help improve their conjectures. First, they suggested that P remained constant for all polygons, and confirmed this conjecture by creating a new table for P and R and connecting it to the existing one (see the left table in Fig. 7). Thus they finally realized T must reach a limit—it is not possible for a polygon to have an arbitrarily large area and a finite perimeter.

One of the students suggested a related issue to follow this one. By inspecting the pictures, she noticed that as N increased the polygon approached a circle. The issue now became to compare the area of the polygons to the area of a circle with the same perimeter (600). Figure 8 shows the product of this investigation. The students first computed the radius of a circle whose circumference was 600, then its area. Next they used commands to generate many polygons with P of 600 and N varying (see the commands window in the upper right of Fig. 8). They confirmed that the areas of the polygons approached that of a circle, and hence that a circle is the most "efficient" regular polygon—it encloses the most area for a given perimeter. Finally, because their table included R and A (apothem—distance from center to a vertex), the students also made and confirmed the following conjectures: (a) that A and R approach a limit as N increases; (b) that the two limits are the same value; (c) that A approaches the limit from "below," whereas R approaches the limit from above (see the graph of N versus A and N versus R in Fig. 5); (d) that the shape of the N versus A curve and N versus R curve are mirror images, reflected in the horizontal line drawn through the limiting value; (e) that the limiting value will be equal to the radius of a circle with the same perimeter; (f) that for circles R = A; and (g) that circles can be viewed as limiting cases of regular polygons as N approaches infinity and as R approaches A. This investigation, dense with interesting issues and much discussion back and forth, took approximately one hour.

In addition to showing how young students can be introduced informally to the concept of limits, this investigation points up several additional features of the use of the microworlds. Perhaps most important, the *Polygons* world has a good density of interesting issues. It is not difficult for us as observers or mentors to find interesting seed issues for students, and the issues are at various levels of complexity. This should enable students at different skill levels to use the microworld. To date we have observed only first- and second-year high school students, but we expect that students two grades lower and at least one grade higher could learn using the microworld.

The relatively high density of interesting issues also promotes student initiative. As the previous example shows, some unforeseen issues can arise *opportunistically*, and we generally encourage the students to pursue these issues if they seem inclined to. In contrast, the microworlds developed by Glaser and colleagues (e.g., Raghavan, Schultz, Glaser, and Schauble, 1989; Shute, Glaser, and Raghavan, 1988) maintain relatively well-defined and predefined objectives for student learning.

In addition to limits, we have observed and engendered opportunistic discussions concerning various mathematical ideas, including:

- Different types of variables or "scales." For example, the way we measure the number of sides of a polygon is not arbitrary, whereas the way we measure the number of degrees in an angle is partly a matter of convention (that there are 360 degrees in a circle).
- Mathematical "basis" and dependent and independent variables. For example, >I and >A are dependent "angle" properties of polygons, >I and S (a "scale" property) are independent, S and T are not independent, and any scale and angle property are a sufficient basis, determining the values of all other properties).
- Constraints that restrict equations when modeling real situations. For example, if the student attempts to set >I to some arbitrary value (e.g., 123), the equation $>I = 180 - 360/N$ yields a value of 6.316 for N. However, the microworld rejects this value for N (and hence >I), and students are naturally led to ask why, given that the equations show nothing wrong with these values.

Although a high density of interesting issues has several benefits for learning, there are costs as well. Issues may be terminated without resolution in favor of others arising opportunistically. In the previous investigation, for example, the students began looking at how area grew as the number of sides doubled, but they dropped that inquiry when they decided to begin halving the length of each side. Thus, the richness of issues demands corresponding skills in managing topics that are set beforehand and those that arise in the course of investigations. As we elaborate below, the students generally lacked these management skills, and they were usually supplied by mentors and observers.

In summary, students have learned effectively using the *Polygons* microworld. They have been able to acquire various mathematical concepts, some that rarely arise in introductory high school classes and some that are usually taught more didactically. In

addition, they show some ability to learn through self-directed construction of knowledge and to learn inquiry skills that would improve their skills at constructing knowledge. But perhaps most gratifying has been their persistence. The few students we have piloted extensively have all learned to use the software rapidly and, once comfortable, have happily used the system for one or two hours at a time.[3] Rarely is there a dull moment in the sessions. The students are usually enthusiastically generating data, offering conjectures, and communicating about ideas with other students or observers.

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[3]This may not indicate the limits of their persistence. We almost always have to terminate sessions for reasons other than student disinterest.

IV. PROBLEMS LEARNING WITH THE POLYGONS MICROWORLD

In spite of students' successes, our initial observations of the *Polygons* microworld have pointed up several key problems that represent potential barriers to student learning. We believe these problems are quite general, and we are gearing the design of the microworlds, as well as the design of activities for using the computer tools, to overcome the problems. In this section we discuss the problems, and in the next section we note several ideas for overcoming them.

MANAGING SEARCH AND INQUIRY

As we noted above, students often lacked skills needed to manage inquiry and to control the search implied by mathematical discovery. To some extent this is not surprising; in their mathematics classes they are rarely, if ever, put in a position where they have to make decisions about which activity to pursue. Skills for managing issues are multi-leveled. At the highest level, students can decide which issue to investigate; at lower levels they need to determine how to investigate it. For example, once they decide they want to understand how area changes as N increases, they still need to engage in several important discovery activities: design an "experiment" (collect data on several polygons, holding S constant, varying N and observing T), use the right data representation tools (tables and graphs for N and T, and possibly S), and interpret the results in terms of the original issue or conjecture. Representations of these kinds of knowledge have been embedded in the LRDC microworlds as "critics" that recognize many specific errors in search management and scientific reasoning (Raghavan, Schultz, Glaser, and Schauble, 1989; Shute, Glaser, and Raghavan, 1988).

Overall, the students evidenced reasonable middle- and lower-level skills in designing and interpreting experiments, perhaps learned in science classes. However, on more than a few occasions we observed inappropriate behavior; for example, when asked what kind of polygon maximizes area for a given perimeter, one student began generating polygons with S held constant and N varying. Yet this type of mistake was relatively rare. On the other hand, with a few exceptions, students were generally reluctant to make their own high-level decisions about which issues to pursue. Most of the time, the students looked to one of the participant observers to suggest new directions for inquiry or at least to confirm a student's idea. If they did decide to pursue an issue, it was usually at the expense of the

current one, as in the above example. In particular, they never simply noted an interesting issue for possible future consideration (using either pencil and paper or the conjectures window) while continuing with the current inquiry. As a result, the students' inquiries sometimes had a "thrashing" quality, in which one issue would take center stage, another would arise to replace it, and later the first issue would resurface, usually through an observer's suggestion. Similar behavior was observed in our studies of discovery-based tutoring with human tutors (Lewis, McArthur, Stasz, and Zmuidzinas, 1990).

DATA-DOMINATED OR "HACKERLY" REASONING

A second general feature of students' performance was its heavy "empirical" bias. This feature has several related manifestations. For example, in our first session with one student, the issue was to characterize the relationship between N and >I. The student began reasonably enough by generating several polygons in which N systematically varied. These were placed in a table, where the student then looked for patterns. However, instead of noticing the overall way in which >I increased with N, the student fixated on spurious features of the numbers, for example, how >I was "odd" or irregular when N was not even.[1]

More generally, students often looked at any pattern, however accidental, as possible evidence of a relationship. Put in simple information-processing terms, they would often not subject the "generate" part of their "generate-and-test" cycle for examining conjectures to sufficient common-sense constraints. For example, the same student who initially looked at spurious patterns in the numbers for N and >I also "mindlessly" (Salomon, 1985) combined values for >A and N in various ways when looking for a rule that would relate these two properties. There was no attempt to reflect, for example, on whether subtracting values of >A and N made sense in terms of the student's informal knowledge of the physical properties these number pairs represented. Rather, in this example and others, there was a strong tendency to abstract away from the physical situation and the constraints that intuitive knowledge of the situation could provide. The students would instead often focus on making sense of the numbers themselves.

There is a strong relationship between this kind of heavily data-dominated inquiry and what used to be called "hacking" (before the term became synonymous with illegal entry into a computer system). As Weizenbaum (1976) so vividly described, hackers often

[1]This isn't entirely nonsensical, because *180/N* is evenly divisible by several small even numbers. However, it misses any general features of the relationship between N and >I.

solve software problems by trying out many "fixes" until their code behaves as desired. What makes their performance "hacking" is that instead of beginning by *reflection* to understand the underlying cause of a failure, they simply try out a variety of plausible patches, relying on the speed of the machine to quickly test the fix and to help them decide on another if it too fails. Like the hacker, our students frequently generated many conjectures about the nature of relationships and used the microworld tools to test them rapidly, in preference to reflective thinking about the semantics of the objects being investigated.

We are of two opinions about this behavior. On the one hand, we caution against calling it simply "mindless." From a more positive view, it works (after all, the student above eventually stumbled across *>I/>A*, which yielded a fascinating invariant), and consistent with Lave's (1988) observations of "just plain folks," it can be seen as a natural adaptation to the kind of problem-solving style afforded by the environment. Computers *do* make hacking an effective way of resolving dilemmas. On the other hand, our goal is for students to learn something about reasoning with mathematical objects and properties of objects, not (just) about how to manipulate numbers. To the extent they can reduce the former to the latter, they have subverted our goals.

FAILURE TO EXPLAIN EMPIRICALLY GENERATED AND CONFIRMED CONJECTURES

The third general limitation in students' reasoning was that unless we expressly asked them to, they never attempted to *explain* their conjectures. We would often see them engage properly in the scientific practice of confirming their discoveries; for example, if they found that *>I = 180 – 360/N* for a few polygons, they would be sure to generate a few more, to make sure the equation was not just accidentally true of one or two polygons. But the idea of explaining why such an equation is *always true* never spontaneously occurred to students.[2] By "explain a conjecture" we mean something both more general and less well-defined than "proof." First, explanations of a conjecture might involve an interpretation of the formal result in terms of parts of the object being represented; for example, a student could explain *T = 1/2A*P* by observing that a polygon is made up of a set of triangles each with altitude A and with a total base length of P. Second, explanation can involve a reduction of a given conjecture to "smaller" conjectures that have already been explained. Formal proof is an explanation of this kind, but we were also interested in more "verbal decompositions." For example, a reasonable verbal description that explains *>I = 180 – 360/N* by decomposition might be to point out that *>X = 360/N* and *>I = 180 – >X* have

already been conjectured, verified, and explained. Neither of these activities was natural to the students, although a few verbal explanations were attempted on request.

Generally, then, there is evidence that the students' models of mathematical inquiry—the activities of generating, confirming, refining, explaining, and communicating conjectures, discussed briefly at the beginning of the Note—are incomplete and inaccurate. Based on our limited data, students' models do have some local structure. Students are comfortable with the idea of generating a conjecture, designing a simple experiment to get data to test the conjecture, and interpreting the data. However, they have relatively few skills for building one local inquiry (conjecture) on another, or for deciding what new issue to move to when another is complete. They seem to be aware that this more global decisionmaking is important, but they usually turn to mentors or observers to make these choices. Thus, this more global thinking is perhaps part of their models of mathematical inquiry, although not one for which they have many skills. Finally, the idea of acquiring a deep explanation of why a verified conjecture is true seems to be a very weak part of their models, if present at all. Instead of engaging in explanations, students seem to just want to go on to a new issue. Perhaps this is because students do not see deep explanations or proofs as really adding to their knowledge. Once a conjecture is empirically verified, they appear to think that new learning is only possible by moving on to verify new conjectures. In a way they are right—proof is a rather unique and strange part of the mathematical culture. Although physical sciences include all of the empirical aspects of mathematical inquiry, they possess no real analogue of proof.

However, we believe this unique role of explanation is essential for students to learn. We are not particularly interested in fine-tuning students' formal proof skills. Rather, we want to promote their view of mathematics as a web of interconnected conjectures, where explanations connect or "reduce" one conjecture to another and, conversely, where inquiries can be structured by building more complex conjectures from simpler ones (building blocks) that have already been tested and explained. Students who lack this understanding of mathematics have shown some basic misconceptions about the nature of the microworld inquiries. For example, the student who in Fig. 6 discovered and verified $>I/>A + 2 = N$ regarded his rule much as an ornithologist might view the discovery of a new species of bird; it was not seen as connected to the "standard" rule $>I = 180 - 360/N$—something that could have been inferred through reflection—but as a separate piece of knowledge, comprehensible only through empirical discovery.

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[2]To date, no student has used the microworlds more than six hours. Perhaps with more experience, and with more coaching (as we note below), their behavior may change.

In short, our students' understanding of their inquiry and the knowledge they could arrive at through inquiry was overly empirical both in the sense of being "hackerly" and in lacking explanations. Although we believe the empirical aspects of the mathematical enterprise have often been unfairly ignored (see e.g., Lakatos, 1976; Putnam 1975; and Steen, 1988), we also believe in the importance of combining it with a more deductive style of mathematical reasoning. The deductive or reflective thinking we wish to promote, however, focuses less on symbols and more on informal mathematical explanation and on the general idea of seeing conjectures as connected in terms of justification (reducing more complex conjectures to simpler ones) and construction (building more complex conjectures from simpler ones).

V. SUPPORTS AND ACTIVITIES FOR LEARNING WITH THE MICROWORLDS

Various aspects of the microworlds software have been crafted to overcome problems noted above, to generally enhance the learning value of the environments, and to encourage productive discussions. In this section we discuss several aspects of the microworlds' design that represent solutions to these problems. We discuss two kinds of solutions: aspects of the software itself and aspects of how the software is used. In addition to discussing existing features of the microworlds, we also describe ideas for new tools, based on our pilot results with *Polygons*. One common theme that runs through the different solutions is that effective learning tools for inquiry act to *magnify* and *transform* students' discovery skills. They thread a narrow path between the dual difficulties of not constraining students' search enough, resulting in floundering and inefficient student learning, and constraining it too much, resulting in a loss of student initiative.

COMPUTER TOOL SUPPORTS FOR LEARNING

Clerical Assistance

One of the most obvious yet most useful roles of the microworlds kernel is to provide relatively low-level clerical support. To understand many qualitative and quantitative conjectures, it is necessary to be able to generate a large amount of data quickly, potentially permitting students to see emergent patterns. In *Polygons*, students quickly create and measure many objects. Instead of taking several minutes to record one polygon, the students generate many in one minute, without any danger of measurement error. Similarly, students are freed from the clerical details of neatly recording measurements in tables or graphs to help make patterns apparent. Using these tools students can effectively reduce the size of search spaces by increasing the speed with which they navigate them. They may still flounder, but the time taken in thrashing is cut dramatically.

The above examples show how the microworlds can improve students' performance quantitatively, speeding up the accomplishment of relatively insignificant tasks. However, potential qualitative changes in students' reasoning are implicit here too. Most simply, instead of concentrating on *how* to use data-representing tools, students can focus on more important decisions about *what* tool to use and *when* to use it in their investigations. As we noted above, the students we have observed are not highly skilled in these decisions. In addition to accelerating certain activities and permitting students to focus on others, the

microworlds also enable students to do things that pencil and paper cannot accomplish. Chief among these is the constraint-propagation mechanism that permits students to construct objects by specifying any logically sufficient set of property values. For example, in *Polygons*, students may examine a set of polygons with a constant area and varying number of sides. Such a set would be virtually impossible to construct without the computer. Thus, as Pea (1987) has noted, in several ways the microworlds tools transform the skills students practice and learn in inquiry, as well as magnify them.

Layered Complexity and Graphical Presentation of World Models

One reason students have difficulty managing inquiry searches is that the space of choices—issues and method for investigating them—is very large, even for relatively simple topics like *Polygons*. Many techniques can effectively cut down the size of search spaces. Clerical support and active coaching (see, e.g., Raghavan, Schultz, Glaser, and Schauble, 1989; Shute, Glaser, and Raghavan, 1988) can make search faster; an alternative is to make the space itself smaller. In the microworlds, we reduce the search space by making the visible properties—which define the dimensions of the space—subject to user control. For example, when introducing *Polygons*, we often begin by exposing only the number of sides, interior angle, length of side, and perimeter. Other properties are not visible in the template where objects are created (see Fig. 1) or in menus that students use to define graphs and tables. Effectively, this constrains the search space of interesting conjectures. Later in the session, we add the remaining properties.

The selective hiding and incremental revealing of microworlds properties is related to the ideas of layered complexity (Kaput, 1990) and model progression (White and Frederiksen, 1986). More generally, it is an example of the need to carefully craft the graphical *world model* that a microworld presents to users. At least since STEAMER (Hollan, Hutchins, and Weitzman, 1984), developers of simulations and microworlds for education have been aware that good graphical models must be both more and less than realistic pictures of the worlds they model. As with our layered complexity, such models must be able to hide features that are unimportant and to redefine dynamically what features are important. Similarly, good world models often represent explicitly features that are normally implicit. For example, White and Horowitz (1987) describe how a "data cross," which explicitly represents invisible features like velocity, helps students understand physical laws in simulated worlds.

Direct Manipulation

Direct manipulation interfaces are another relatively new tool for developing educational environments (Smith, 1986; Smith, 1987). Using a direct manipulation interface to an artificial world, a user creates, modifies, and arranges objects directly rather than mediated by menus and commands. Currently, our microworlds are being reimplemented to use direct manipulation more extensively. For example, in a *Triangles* world we are already planning, students will create objects by drawing shapes and will "rubber-band" (stretch) them to change values of various properties. "Data boxes" that give numeric values for properties will also be provided. They will be used either passively (to display property values when objects are directly manipulated) or actively (to edit property values that then get reflected in visual changes to the object). Direct manipulation interfaces not only simplify use of the microworlds, but should also reduce the mediation between students' reasoning and mathematical objects. As we noted above, students have a tendency to focus on numbers, not the properties they represent. Direct manipulation interfaces should help refocus their pattern detection on objects.

Explicit Representations of Conjectures, Explanations, and Mathematical Connections

As noted above, students generally fail to explain verified conjectures. More generally, they approach each inquiry separately; their model of inquiry does not see mathematics as a web of interconnected conjectures, where explanations connect or "reduce" one conjecture to another and, conversely, where inquiries can be structured by building more complex conjectures from simpler ones (building blocks). We are now in the process of designing tools to encourage explanation and this relational view of mathematics in several ways.

Our main technique will be to *reify* explanations and conjectures. Currently, students can type in conjectures (see the "Conjectures Window" in the figures), but they are just passive pieces of prose. In the future, students will complete hypertext "cards" for each conjecture. On each conjecture card, students may place information about data used to generate or confirm the conjecture. In addition, hypertext links will be used to connect conjectures explicitly. Such links will represent "reduction" or "construction" relationships between conjectures, hence they will explicitly encode "visual" explanations.

A Networked Library of Issues and Audit Trails of Conjectures and Explanations

A related new idea to help students manage inquiry will be to provide a library of issues. The library will be structured as a visual lattice (with simpler issues directly connected to slightly more complex ones), and students will browse the library using a mouse to select new issues. By the position of the issue they pick, relative to ones they have already investigated successfully, the students should have a good idea of whether the issue might be novel and difficult (distant from investigated issues in the lattice) or relatively familiar and easier (near to investigated issues). Thus, again, spatial representations may constrain the search space and inform a student's inquiry. Unlike other generic parts of the microworld kernel, a library will have to be constructed for each microworld. However, we can at least provide simple tools that ease the construction of the lattice, and general off-the-shelf graphical routines for its display and interactive browsing are also available. Additionally, libraries need not be complete (or even very extensive) when first built. Indeed, we hope students will find their own issues to add.

The library of issues should be useful by itself but might be particularly valuable when combined with the conjecture hypertext cards. Successful students will typically generate one or two conjectures for each issue; for example, if the issue in *Polygons* is "How does >I change with N?" a student might create a (quantitative) conjecture card stating "$>I = 180 - 360/N$," along with annotations. In addition, to explain the conjecture the student might have linked this card to other cards for "$>I + >X = 180$" and "$>X = 360/N$." This cluster of cards represents an *audit trail* summarizing the student's reasoning for the issue.

To make audit trails useful, students will be able to attach audit trails of their reasoning to issues in the library of issues, and the library will be shared by all students. This *networked* library of issues, conjectures, and explanations may serve several purposes. It should provide a convenient way for the teacher or observer to assess students' progress. In addition, attaching a conjecture and its support to an issue can help to remind students of their reasoning. However, we expect explicit reasoning summaries to be even more useful to other students. For example, novice students will be able to obtain models of good (and bad) inquiry activities; students at the same level will be able to share inquiries; and more advanced students will be able to "electronically mentor" by directing less advanced students to new issues and explanations and by offering their own issues as challenges. More generally, we hope that by requiring students to make conjectures and explanations explicit, and by sharing reasoning and justifications with other students, we can encourage a culture

of critical constructive thinking. TERC's experience in networking the computer environment *Connectany* (Barkley, 1990) indicates that such a hope is not unreasonable.

COMPUTER USE SUPPORTS FOR LEARNING

In addition to software tools, to make the microworlds effectively support inquiry learning, we believe it is also important to craft the activities in which they are used. We do not assume that students will learn through direct communication of knowledge from the microworlds. Rather, we want to provide tools that give students opportunities to learn through their own reflection and, as an extension, through conversations with colleagues as they use the microworlds. Thus, it becomes important to design activities that will lead to interesting kinds of reflection and conversation.

One technique we will continue to use, and also to expand upon, is to give novice users "seed" issues (like investigating how interior angle increases as N goes up). We typically ask students to work in pairs, give them one or two relatively straightforward seed issues, and require them to keep track of any other interesting issues or ideas that arise during this initially focused investigation. This organization encourages explorations that are somewhat controlled but that also can take unanticipated turns as students gain confidence. For example, the investigations discussed in Figs. 7 and 8 quickly turned from seed issues to opportunistic ones.

The use of seed issues imparts an implicit sequencing to inquiries that can make difficult issues accessible in steps. For example, the student in Fig. 6 undertook a rather circuitous investigation of the relationship of >I and N. Although he discovered an interesting conjecture, in retrospect we might have guided his inquiry more effectively had we first started with the seed issue of investigating the relationship of >X and N, then followed it with >I and N. With later pairs of students, this sequencing worked to guide them easily from $>X = 360/N$ to $>I = 180 - 360/N$.

These examples show that sequencing of seed issues, like layering complexity, can incrementally increase students' search space as their expertise grows. In addition to providing a search management technique, it can also help focus students on explanation and on making connections between conjectures. For example, when we sequence the issue of discovering a relationship between >X and N before investigating >I and N, students appear more likely to explain $>I = 180 - 360/N$ in terms of $>X = 360/N$ and thus to see connections (composition and decomposition) between these two conjectures. In short, the connections we hope to reify using the library of issues and conjectures are also the ones we hope to draw students' attention to by sequencing seed issues.

Our preliminary studies with *Polygons* have convinced us that students often need more active support than that afforded by the simple sequencing of issues. Even after issues are selected, students need models of good inquiry skills with which to investigate issues. Accordingly, we are experimenting with different kinds of *roles* when the microworlds are used by multiple students (and participant observers). Informal observations suggest that roles and pairings should change with experience. Briefly, it has worked best to pair neophytes with *mentors* (usually observers, since few students have reached mentor level). Mentors can acquaint novices with the technical details of using the microworlds and also model skills for deciding on issues and tools—hence providing additional support in managing searches through large spaces. Students play the role of research assistant.

When students acquire some technical facility with the microworlds, *co-investigator* roles seem useful. Here, pairs of students divide the labor of an inquiry relatively equally, and both students engage in at least some higher-level decisionmaking, graduating from basic apprenticeship. We do not believe it is critical that students be equally experienced; it is more important that the less experienced investigator not view the more experienced player as the sole source of decisions and knowledge. This counters most students' classroom experience, which gives them a world view of mathematics in which knowledge is not subject to discovery but rather is received from teachers.

Finally, we also believe competitive and *devil's advocate* roles are useful. As noted previously, students who have used *Polygons* for several hours often reach a steady state in which conjectures are inductively confirmed but not deductively explained. To encourage explanations, participant observers have begun adopting a critical role, asking students to justify their confirmed conjectures and then modeling explanations if students are unable to supply them. We hope to extend this role to situations where several pairs of students investigate similar issues, networked together by the shared library of issues, conjectures, and explanations. Here, the devil's advocate role would be played by the different "competing" pairs of students, not by an observer.

The above examples illustrate just a few interesting uses of the microworlds. In general, their goal is to help students manage inquiries and to shape students' ideas about the nature of mathematical inquiry and knowledge. The examples also underscore that how the microworlds are used may be as important in determining their overall educational value as the software itself. The microworlds may provide the opportunity for new learning experiences, but only through carefully crafted activities and uses—ones that guide students while still permitting them to control the initiative—can we hope to change the culture of mathematical inquiry in the classroom.

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