AN OPTIMAL CONTROL MODEL OF PRODUCT IMPROVEMENT R&D

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Much of the literature on optimal technical change or optimal inventive activity in economics has dealt with optimal accumulation paths or optimal consumption paths and with long-run steady-state growth.[1] In the management literature, emphasis is given to optimal control of R&D expenditures in order to bring to fruition a complex sequence of specific events.[2]

The problem we will be dealing with in this section has elements of both approaches. The decision is not how much to allocate to such grandiose aggregates as consumption or investment, but simply how much to put into a particular R&D program at any time (a genuine management problem). Yet the problem is formulated in the following way: given a production function for technical change we seek a path which maximizes the present discounted value of the difference between its

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[1]See for example the article by Karl Shell, "Optimal Programs of Capital Accumulation for an Economy in which there is Exogenous Technical Change," or the article by William Nordhaus, "The Optimal Rate and Direction of Technical Change," both in K. Shell, ed., <u>Essays on the Theory of Optimal Economic Growth</u>, M.I.T. Press, Cambridge Massachusetts, 1967.

[2]See, for example, C. W. Hamilton, "Optimal Control of Research and Development Expenditures," Technical Report No. 48, M.I.T. Operations Research Center, 1969.

benefits and costs (definitely a notion from pure economic theory). The model is developed within the context of a program in which an organization attempts to improve the performance of one of its systems subject to an arbitrary fiscal constraint which limits the rate of R&D spending on that system. Such a constraint is meaningful in a world in which limits of this type are set by a political body, or by a higher authority who perhaps has performed a higher level of optimization over several projects. How this higher level of optimization ought to be performed is itself an important and interesting question. Presumably it involves a subjective assessment of the relative plausibility of many possible future states of the world. As such, that question goes far beyond the intention of this paper.

For those who wish to think in more concrete terms, the organization can be thought of as the Air Force engaged in a product-improvement program for one of its jet engines. A fiscal constraint may then have been imposed by the Air Force's chief R&D officer, or by Congress itself. We will use this parable whenever it appears to aid in the presentation of the model.

Formulation of the Problem

Let $P(t)$ be an index of performance of an Air Force system at time t, and let $U(P(t))$ be the utility flow at time t from having performance $P(t)$ available for immediate incorporation into a piece of hardware. $P(t)$ must satisfy the following condition:

(1) $$P(t) \leq \varphi(X(t), t)$$

Table 1

DEFINITIONS OF VARIABLES USED

$P(t)$ = index of performance at time t.

$X(t)$ = cumulative R&D expenditures for performance improvement up to time t.

$E(t)$ = R&D expenditures for performance improvement at time t.

t = time.

$U(\cdot)$ = utility flow at time t.

$\varphi(\cdot)$ = maximum feasible performance at time t.

E_{max} = maximum permissible rate of R&D expenditures for performance improvement at time t.

δ = rate of time discount on expenditures.

ρ^* = rate of time discount on utility.

ρ = $\rho^* - \delta$.

ν = rate of factor augmenting technical change.

$\mu(t)$ = Pontryagin price attached to cumulative R&D expenditures for performance improvement at time t.

$\varepsilon(X)$ = elasticity of marginal utility with respect to cumulative R&D expenditures for performance improvement.

$\varphi(\cdot)$ is the production function for performance improvement, and its principal argument is $X(t)$ the cumulative R&D money spent on performance improvement up to time t; t itself may be an argument of the function if for example there is factor augmenting technical change. Let us assume for now that this production function satisfies modified Inada conditions:[3]

$$(2) \quad \begin{aligned} &(2a) \quad \varphi(0) = P_0 \\ &(2b) \quad \varphi(\infty) = \infty \\ &(2c) \quad \varphi'(0) = \infty \\ &(2d) \quad \varphi'(\infty) = 0 \end{aligned}$$

P_0 is the performance level at the beginning of the product-improvement program. In the case of jet engines, we can take this to be the performance at the time of the MQT (Model Qualification Test).

$E(t)$ is the rate of R&D expenditures at time t, and by definition then, is the rate of change of $X(t)$. $E(t)$ is constrained to be less than E_{max} at any time t.

The problem then is to find a path for $0 \leqq E(t) \leqq E_{max}$ and $P(t) \leqq \varphi(t)$ such that the following integral is maximized:

$$(3) \qquad \int_0^\infty \{U(P(t))\ e^{-\rho^* t} - \lambda E(t)\ e^{-\delta(t)}\}dt$$

where ρ^* and δ are the discount rates for utility and expenditures respectively.

The rate of exchange between utility and expenditures, λ, can be assumed to be unity with complete generality by the appropriate choice of units. The optimal control problem can now be written as:

[3] This production function for product-improvement in jet engines has been statistically estimated at Rand and the estimated form does indeed satisfy these modified Inada conditions.

$$\text{(4)} \qquad \underset{\{E(t),\,P(t)\}}{\text{maximize}} \int_0^\infty \{U(P(t))\, e^{-\rho^* t} - E(t)\, e^{-\delta t}\} dt$$

$$\text{subject to } \dot{X}(t) = E(t)$$

$$\varphi(X(t)) - P(t) \geqq 0$$

$$E_{max} - E(t) \geqq 0$$

$$X(0) = X_0 = 0$$

$$\varphi(0) = P_0$$

$$E(t),\ P(t) \geqq 0$$

Derivation and Analysis of Necessary Conditions

It is clear from the above formulation that it would be inefficient to choose a performance level less than the maximum feasible as given in equation (1). Hence we can replace $P(t)$ in the integral by $\varphi(X)$, and proceed using the Maximum Principle of Pontryagin, et al.

$$\text{(5)} \qquad \underset{\{E(t)\}}{\text{maximize}} \int_0^\infty \{U(\varphi(X))\, e^{-\rho^* t} - E(t)\, e^{-\delta t}\} dt$$

$$\text{subject to (a) } \dot{X}(t) = E(t)$$

$$\text{(b) } E_{max} - E(t) \geqq 0$$

$$\text{(c) } X(0) = 0$$

$$\text{(d) } P(0) = P_0 = \varphi(0)$$

$$\text{(e) } E(t) \geqq 0$$

Let $\mu(t)e^{-\delta t}$ be the Pontryagin price or Hamiltonian multiplier, analogous to the Lagrangian multiplier in static problems. The necessary conditions for a maximum are derived by forming the Hamiltonian, H,

$$\text{(6)} \qquad H = U(\varphi(X))^{-\rho^* t} - E(t)\, e^{-\delta t} + \mu E(t)\, e^{-\delta t}$$

and maximizing H at each point in time with respect to the control variable,[4] E(t). This implies the following rules:

If $\mu(t) > 1$, set $E(t) = E_{max}$

If $\mu(t) = 1$, set $E(t) \varepsilon [0, E_{max}]$

If $\mu(t) > 1$, set $E(t) = 0$.

It is also necessary that $\mu(t)$ satisfy the following differential equations:

$$(7) \qquad \dot{\mu} = \delta\mu - \frac{\partial H^* e^{\delta t}}{\partial X}$$

where H* is the maximized Hamiltonian.

$$(8) \qquad H^*(t) = \max_{E(t)} H(t)$$

Equation (7) can be reduced to a more tractable form:

$$(7') \qquad \dot{\mu} = \delta\mu - U' \varphi'(X) \; e^{(\rho^* - \delta)t}$$

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[4]Assuming the maximization of H offers no obstacle to sufficiency, all we need is the concavity of H in the state variable X to guarantee sufficient conditions for a maximum. We have the following:

$$\frac{\partial H}{\partial X} = U'\varphi' > 0 \text{ if } U' > 0 \text{ and } \varphi' > 0$$

$$\frac{\partial^2 H}{\partial X^2} = U'\varphi'' + (\varphi')^2 \; U'' < 0 \text{ if } \varphi'' < 0 \text{ and } U'' < 0.$$

So concavity of $\varphi(\cdot)$ and $U(\cdot)$ is more than enough to guarantee sufficiency.

That the government should use the same rate of discount on expenditures (costs) and utility (benefits) in a defense type project is questionable. The reasons can be summarized as follows:

(a) the lack of markets in which the demand for public projects like defense can be evaluated,

(b) the oligopolistic nature of the supply side of defense projects, and

(c) the dubious assumption that capital markets for private projects of various risk classes are themselves in equilibrium.

What rate of discount should the government use on expenditures? The government can induce individuals to lend funds voluntarily at the relatively low interest rate on government bonds. To induce individuals to lend more the government may have to raise that rate, but the cost of capital to the government is usually less than the prevailing rate.[5]

The discount rate on utility the government ought to use is the social rate of time preference for that good. The problem is complicated further by the fact that there may be a divergence between private and social rates of time preference, the former being the one

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[5]This is usually attributed to the need on the part of private investors for a risk premium. Should the government also include such a risk premium in its evaluation of benefits? The risk to the government in the type of projects we are dealing with in this chapter (recall our parable of the Air Force investing in a product-improvement program for one of its jet engines) is not that the production function is uncertain (we have assumed it is completely certain) but that the utility function will change, in other words, that states of the worlds will prevail in the future, in which various threats materialize or dematerialize that make defense projects more valuable or less valuable. Indeed in the case of defense the utility of a project depends heavily on the actions of other decisionmakers within the context of a non-zero sum game.

Is it not better then to use an "optimistic-pessimistic-best guess" procedure and to weight the utility outcomes by the subjective probabilities one attaches to these different states than to add on an arbitrary few percent to the minimum social risk discount rate to account for the real uncertainty about the future?

that operates in securities markets for private projects. One may argue that with defense the benefits of freedom and independence that fall upon future generations should be valued nearly as highly as we do.

In addition, there may be "time-time preference," meaning that benefits to be received in the distant future are not "discounted" at the same rate as benefits to be received in the near future. When individuals lend funds to the government, they are usually foregoing consumption in the near future. Defense, on the other hand, seems to be the type of good which has a large component of long-term benefits.

The argument that the market rate of interest is the one the government ought to use to discount costs and benefits is based on the assumption that markets are in equilibrium, in particular that markets for risk exist and that they function in a reasonably competitive manner. Clearly such an assumption is unwarranted. Perhaps we can say that if markets are in equilibrium and a government project is a perfect substitute for a private project then the rate of discount that ought to be used is market's rate of interest for that project. But if a public project is not a perfect substitute for some private project then there is no necessary relationship between the observed market rate (if indeed one such rate can be chosen from the many rates observed) and the discount rate the government ought to use in evaluating that project.

Indeed markets for public goods of the defense type generally do not exist, and even if they did there is good reason to believe that the supply side of such markets would be oligopolistically structured. Furthermore, because of the fundamental problem of public goods one would rarely expect these markets to be in equilibrium.

All this leads to the conclusion that it may be socially optimal to use different discount rates on utility and expenditures for projects of this type. Let $\rho = \rho^* - \delta$ then we can rewrite equation (7') as

$$(7'') \qquad \dot{\mu} = \delta\mu - U'\varphi'(X)\ e^{-\rho t}$$

where ρ can be positive (implying a greater discount rate being applied to utility than to expenditures), zero (implying a common discount rate on both), or negative (implying a greater discount rate being applied to expenditures than to utility). We will handle each of these cases separately beginning with the easiest and becoming progressively more difficulty.

Case 1: $\rho = 0$

The optimal behavior for X(t) and $\mu(t)$ can be represented in a phase diagram in (X, μ) space. Such a diagram is shown in Figure 1. The $\dot{X} = 0$ stationary is actually the region $\{(X, \mu) | X \leqq 0, \mu < 1\}$. All other points strictly above the line $\mu = 1$ have $\dot{X} = E_{max} > 0$, while all points on $\mu = 1$, have $\dot{X} \geqq 0$. The $\dot{\mu} = 0$ stationary can be found by solving the equation

$$(9) \qquad \mu = \frac{U'\varphi'(X)}{\delta}$$

Using the concavity of $U(\cdot)$ and $\varphi(\cdot)$ we have been implicitly assuming, $U'\varphi'(X)/\delta$ is a positive and decreasing function of X.[6] The value of X which equates the discount rate δ to the marginal utility of another dollar of cumulative R&D is denoted X*. Above the $\dot{\mu} = 0$

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[6] $U'''(\cdot)$ and $\varphi'''(\cdot) > 0$ are sufficient to guarantee that the rate of decline is indeed slowing as X gets larger.

stationary $\delta\mu - U'\varphi'(X) > 0$ and hence μ must be increasing over time; similarly below the $\mu = 0$ stationary, μ must be decreasing over time. The question is then: Is there a trajectory beginning at $(0, \mu(0))$ which can satisfy the equations of motion and the transversality condition given as equation (10)?

$$\text{(10)} \qquad \lim_{t\to\infty} \mu(t)\, e^{-\delta t} = 0$$

There is one and only one such trajectory,[7] and it is the one beginning at $(0, \mu^*(0))$ and ending at $(X^*, 1)$. This trajectory is shown in Figure 1. All trajectories beginning above or below $\mu^*(0)$ cannot satisfy the transversality condition (10).[8]

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[7]This follows trivially from the Lipschitzian character of the differential equations (5a) and (7") describing the behavior of X and μ respectively. To check that these equations are indeed Lipschitzian, note that the right-hand sides of both equations are twice continuously differentiable with respect to X, μ, t for all E(t) such that $0 \leq E(t) \leq E_{max}$.

[8]Proof:

(a) Suppose a trajectory begins below $\mu^*(0)$, say at $\hat{\mu}(0)$, then this trajectory meets the $\mu=1$ line at a value $\hat{X}$ and at a time $\hat{T}=\hat{X}/E_{max}$ whereupon X ceases to increase and the value of μ declines according to

$$\text{(i)} \qquad \dot{\mu} - \delta\mu = -U'\varphi'(\hat{X}) = \text{a constant}$$

Hence $\mu(t) = \mu_0 e^{\delta(t-\hat{T})} + U'\varphi'(\hat{X})/\delta$.

We can solve for μ_0 using the condition

$$\text{(ii)} \qquad \mu(\hat{T}) = 1 = \mu_0 + U'\varphi'(\hat{X})/\delta$$

So,

$$\text{(iii)} \qquad \mu(t) = (1-U'\varphi'(\hat{X})/\delta)e^{\delta(t-\hat{T})} + U'\varphi'(\hat{X})/\delta$$

But now since μ must be declining over time in this region, $1 - U'\varphi'(\hat{X})/\delta$ must be negative. It therefore cannot satisfy the transversality condition. Alternatively, since $\hat{X} < X^*$ (see fn. 7),

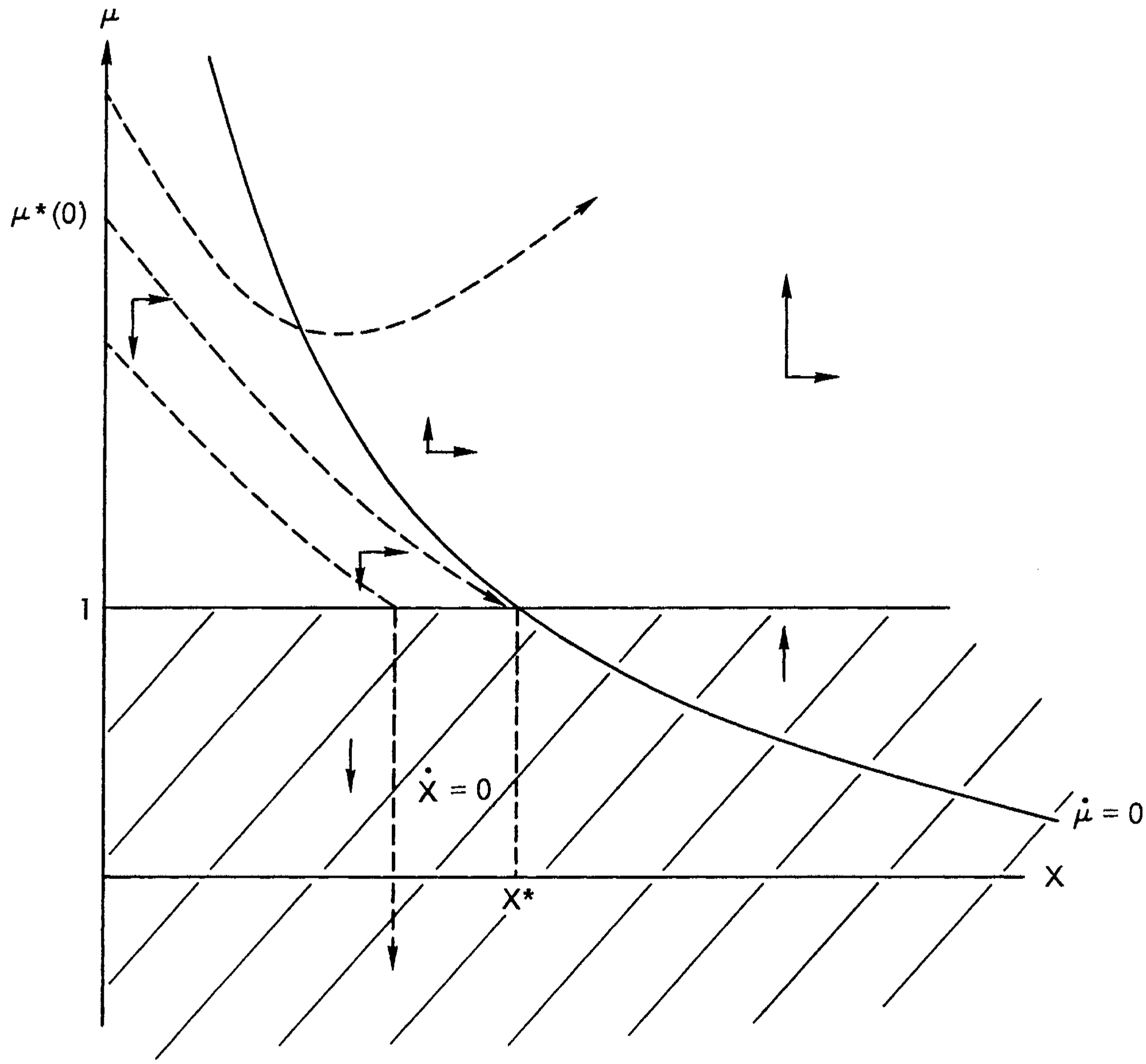

Fig. 1—Optimal trajectory for $\rho = 0$

On the optimal trajectory the transversality condition becomes

(10') $\lim_{t\to\infty} \mu(t)\, e^{-\delta t} = \lim_{t\to\infty} 1 \cdot e^{-\delta t} = 0$

This is consistent with one's intuition: given the concavity of the utility function in X,[9] one might guess that R&D funds should be spent at the maximum allowed rate until some time T* after which fund-

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(iv) $1 - U'\varphi'(X)/\delta < 1 - U'\varphi'(X^*)/\delta = 0$

(b) Consider the trajectory $(X(t), \mu_1^*(t))$ which begins at $\mu^*(0)$ but **which crosses the $\dot{\mu} = 0$ curve with maximum control applied, i.e., $E(t) = E_{max}$. Then**

(v) $\lim_{t\to\infty} \mu_1^*(t)\, e^{-\delta t} \geq 0$

since both terms are non-negative. On this modified "optimal" trajectory, at any time t, $X = E_{max} t$.

Suppose a trajectory begins above $\mu^*(0)$, say at $\tilde{\mu}(0)$, then this trajectory crosses the $\dot{\mu}=0$ curve; both μ and X increase, the latter also according to $X(t) = E_{max} t$. At any time t, the corresponding X values are the same along the two trajectories. Hence

(vi) $\dot{\mu}(t) - \dot{\mu}_1^*(t) = \delta(\mu(t) - \mu_1^*(t))$

Solving the above differential equation, we obtain

(vii) $\mu(t) - \mu_1^*(t) = f_0 e^{\delta t}$

where f_0 is positive since it is equal to $\tilde{\mu}(0) - \mu^*(0)$. Then

(viii) $\lim_{t\to\infty} \mu(t)\, e^{-\delta t} = f_0 + \lim_{t\to\infty} \mu_1^*(t)\, e^{-\delta t}$

> 0

which does not satisfy the transversality condition.

[9]Note this is weaker than the combination of the concavity of $U(\cdot)$ in its argument and the concavity of $\varphi(\cdot)$ in X, but is the same as the concavity of H in the state variable X (see fn. 4).

ing stops altogether. Finding an optimal solution to this dynamic problem is equivalent to solving the simple static problem of maximizing the difference between benefits and costs. The optimal total expenditure is just the one that equates the marginal benefits to marginal costs with the marginal cost schedule cutting the marginal benebenefit schedule from below. This point is precisely X*; the solution to the dynamic problem tells us however that we should endeavor to get to X* as fast as possible.

Summary of Case 1

Given the maximum problem (5) with $\rho^* = \delta$, optimality implies spending at the maximum allowable rate to time T* given by X^*/E_{max} where $U'\varphi'(X^*) = \delta$. After T* spending should be terminated completely. The final level of performance is given by $\varphi(X^*)$.

Case 2: $\rho > 0$

In general, stationary solutions to equation (7') lie in a manifold embedded in (X, μ, t) space. The manifold of solutions to $\dot{\mu}=0$ is shown in Figure 2. The optimal trajectory in this case is characterized by a finite period of maximum R&D expenditures followed by an infinite period of zero expenditures with a declining but positive μ. Figure 2 also shows such a trajectory. The value of X at which R&D expenditures terminate is denoted X**.

It remains to show that the proposed trajectory can comply with the transversality condition of equation (10). Suppose along the proposed optimal trajectory it could be demonstrated that the point $(X^{**}, \mu(t))$ was always below the $\dot{\mu}(t) = 0$ stationary for all $t > 0$, and

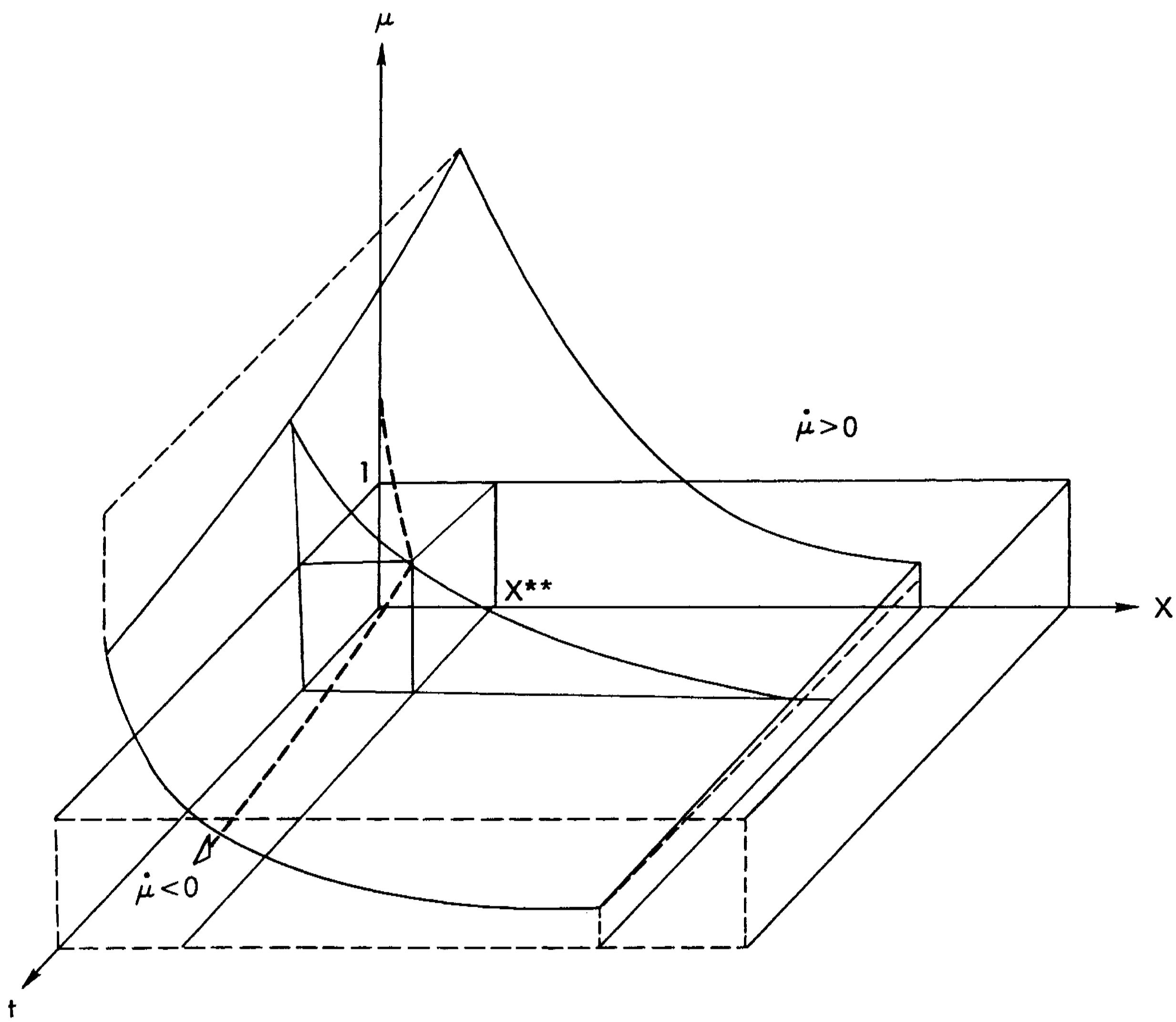

Fig.2—Optimal trajectory for $\rho>0$

$\mu(t)$ was able to approach zero just slowly enough so as to remain positive except in the limit as $t \to \infty$, then the equations of motion and transversality condition could both be satisfied. Now when μ falls to unity, R&D expenditures terminate, and equation (7") becomes

(11) $$\dot{\mu}(t) = \delta\mu(t) - U'\varphi'(X^{**})\, e^{-\rho(t-T^{**})}$$

where $T^{**} = X^{**}/E_{max}$. Since $U'\varphi'(X^{**})$ is just a constant, the solution to (11) is just

(12) $$\mu(t) = \mu_0 e^{\delta t} + \frac{U'\varphi'(X^{**})}{\rho+\delta}\, e^{-\rho(t-T^{**})}$$

Pick $\mu(0)$ such that

(13) $$\mu_0 = [1-U'\varphi'(X^{**})/(\rho+\delta)]e^{-\delta T^{**}} = 0,$$

that is, the point X^{**} is just the one which satisfies $\mu_0|_{\text{opt. traj.}} = 0$, so

(14) $$\mu(t) = \frac{U'\varphi'(X^{**})}{\rho+\delta}\, e^{-\rho(t-T^{**})}$$

Further,

(15) $$\lim_{t\to\infty} \mu(t)\, e^{-\delta t} = \lim_{t\to\infty} \frac{U'\varphi'(X^{**})}{\rho+\delta}\, e^{-\rho(t-T^{**})-\delta t}$$

$$= \lim_{t\to\infty} \frac{U'\varphi'(X^{**})}{\rho^*}\, e^{-\rho^* t+\rho T^{**}}$$

$$= 0$$

Define $\mu^{**}(t)$ to be the value of μ which makes $\dot{\mu}(t) = 0$ for $X = X^{**}$.

Thus

(16) $$\mu^{**}(t) = \frac{U'\varphi'(X^{**})}{\delta}\, e^{-\rho(t-T^{**})}$$

Now since ρ has been assumed to be positive, we have

(17) $$0 < \mu(t) < \mu^{**}(t)$$

This is precisely what was needed to show the optimality of the proposed trajectory of Figure 2.

Summary of Case 2

Given maximum problem (5) with $\rho^* > \delta$ optimality implies spending at the maximum allowable rate to time T** given by $T^{**} = X^{**}/E_{max}$.

After T** spending should be terminated completely. The final level of performance is given by $\varphi(X^{**})$. A fortiori, $T^{**} < T^*$ and $X^{**} < X^*$ for maximum problems having the same utility and production functions.

Case 3: $\rho < 0$

This is the most complex and interesting of the three cases. In certain situations there will be a "turnpike" solution in which the rate of R&D expenditures is positive but less than or equal to the maximum allowable rate; cumulative R&D expenditures may or may not be finite. In other situations the optimal solution is to spend at the maximum allowable rate for all time.

To start with, recall equation (7").

(7") $$\dot{\mu} = \delta\mu - U'\varphi'(X)\, e^{-\rho t}$$

Now however $-\rho > 0$ and $\dot{\mu} = 0$ stationary is constantly shifting upward. Suppose for now the slope of the utility function becomes zero at some finite point $\varphi(\tilde{X})$. Then Figure 3a below depicts the behavior of the $\dot{\mu} = 0$ stationary over time.

The optimal trajectory begins somewhat above the $\mu = 1$ line and moves in accordance to equation (7") and $\dot{X} = E_{max}$ until it just "catches up" to the $\dot{\mu} = 0$ stationary. In Figure 3a this time is shown as $t = t_1$.

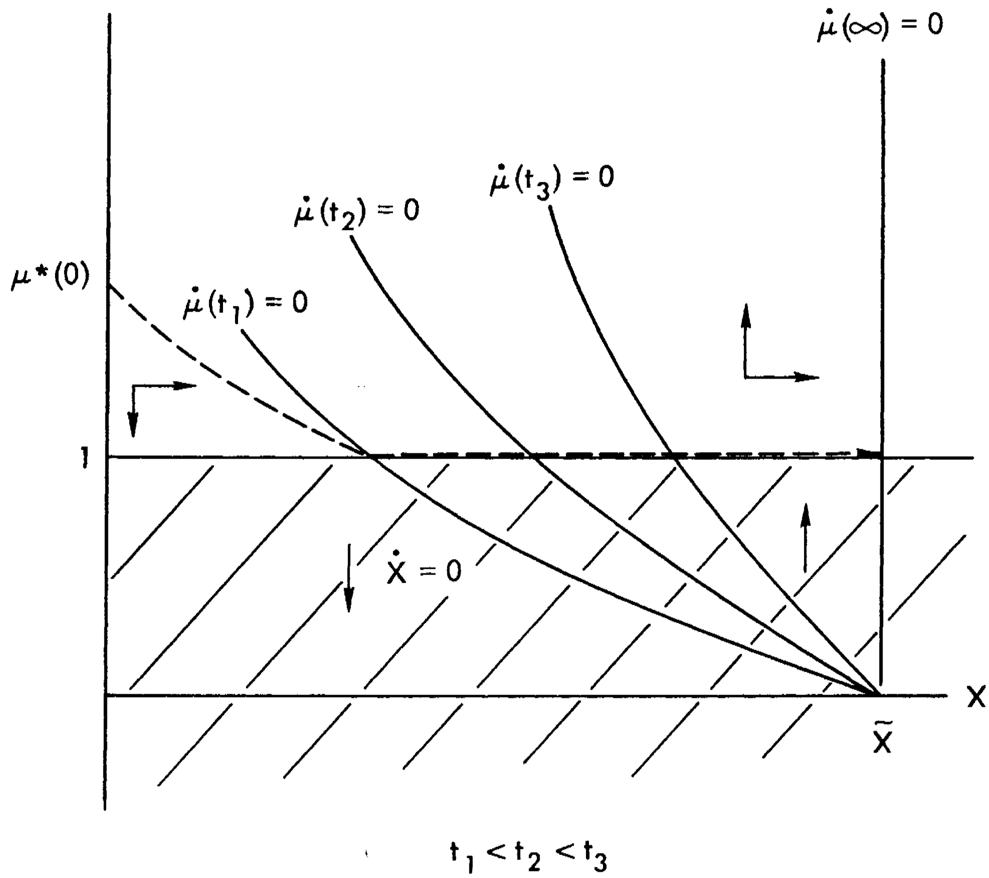

Fig.3a—Optimal trajectory for $\rho<0$ with $U' = 0$ for $X \geq \tilde{X}$

The optimal trajectory then moves along the $\mu = 1$ line at a rate of R&D expenditures so as to remain on the $\mu = 1$ line with $\dot{\mu} = 0$ as cumulative expenditures increase. In other words, optimal expenditures are just enough to keep pace with the apparent rightward movement of the $\dot{\mu} = 0$ stationary at $\mu = 1$. This is the so-called "turnpike" path. Eventually as $t \to \infty$ the $\dot{\mu} = 0$ stationary becomes a vertical line anchored, as are all the $\dot{\mu}(t) = 0$ stationaries, at $X = \tilde{X}$. It is apparent that R&D expenditures should not go beyond $\tilde{X}$ since marginal utility is zero and the integral in equation (5) can only decrease. Yet R&D expenditures do not abruptly terminate as they do in the other cases; here they eventually taper off to zero as the $\dot{\mu} = 0$ stationary reaches the vertical position only in infinite time. The transversality condition is easily satisfied.

(18) $$\lim_{t\to\infty} \mu(t)\, e^{-\delta t} = \lim_{t\to\infty} 1 \cdot e^{-\delta t}$$
$$= 0$$

We can compute the optimal value of $E(t) = \dot{X}(t)$ during the "turnpike" phase. At any time $t > t_1$, $\dot{\mu}(t) = 0$ and $\mu(t) = 1$, and equation (7") becomes

(19) $$0 = \delta - U'\varphi'(X)\, e^{-\rho t}$$

This is an implicit function of X and t, and application of the Implicit Function Theorem yields

(20) $$0 = \rho U'\varphi'(X)\, e^{-\rho t} + (U''(\varphi'(X))^2 + U'\varphi'(X))\, \dot{X}e^{-\rho t}$$

or

(21) $$\dot{X} = \frac{\rho}{[\frac{U''}{U'}\varphi'(X) + \frac{\varphi''(X)}{\varphi'(X)}]}$$

Define the elasticity of marginal utility with respect to cumulative R&D expenditures as

(22) $$\varepsilon = \frac{-d(U'\varphi')}{dX} \frac{X}{U'\varphi'}$$

$$= - [\frac{U''}{U'} \varphi(X) + \frac{\varphi''(X)}{\varphi'(X)}] X$$

So equation (23) becomes

(23) $$\frac{\dot{X}}{X} = \frac{-\rho}{\varepsilon(X)}$$

Thus if the "turnpike" trajectory is to be feasible, we require that

(24) $$\lim_{t\to\infty} \dot{X}(t) = 0$$

In order for equation (24) to hold it is necessary that

(25) $$\lim_{X\to\tilde{X}} \varepsilon(X) = \infty$$

Suppose now we relax the earlier assumption on the utility function. The $\dot{\mu} = 0$ stationary does not reach a limiting position as $t \to \infty$, so along the optimal trajectory cumulative expenditures do not stop at some finite amount. The rate of R&D expenditures on the optimal trajectory may or may not taper off to zero, and there may or may not be a "turnpike" phase in the solution.

Which of these situations holds depends basically on the feasibility condition given in equation (23). Figure 3b shows three possible optimal trajectories. The absolute value of ρ, the size of E_{max} and the elasticity function determined which of the three optimal paths shown is the solution for a particular problem.

Consider path I. The optimal trajectory "catches up" to the $\dot{\mu} = 0$ stationary at time $t = t_1$, and then moves along the $\mu = 1$ line at a rate

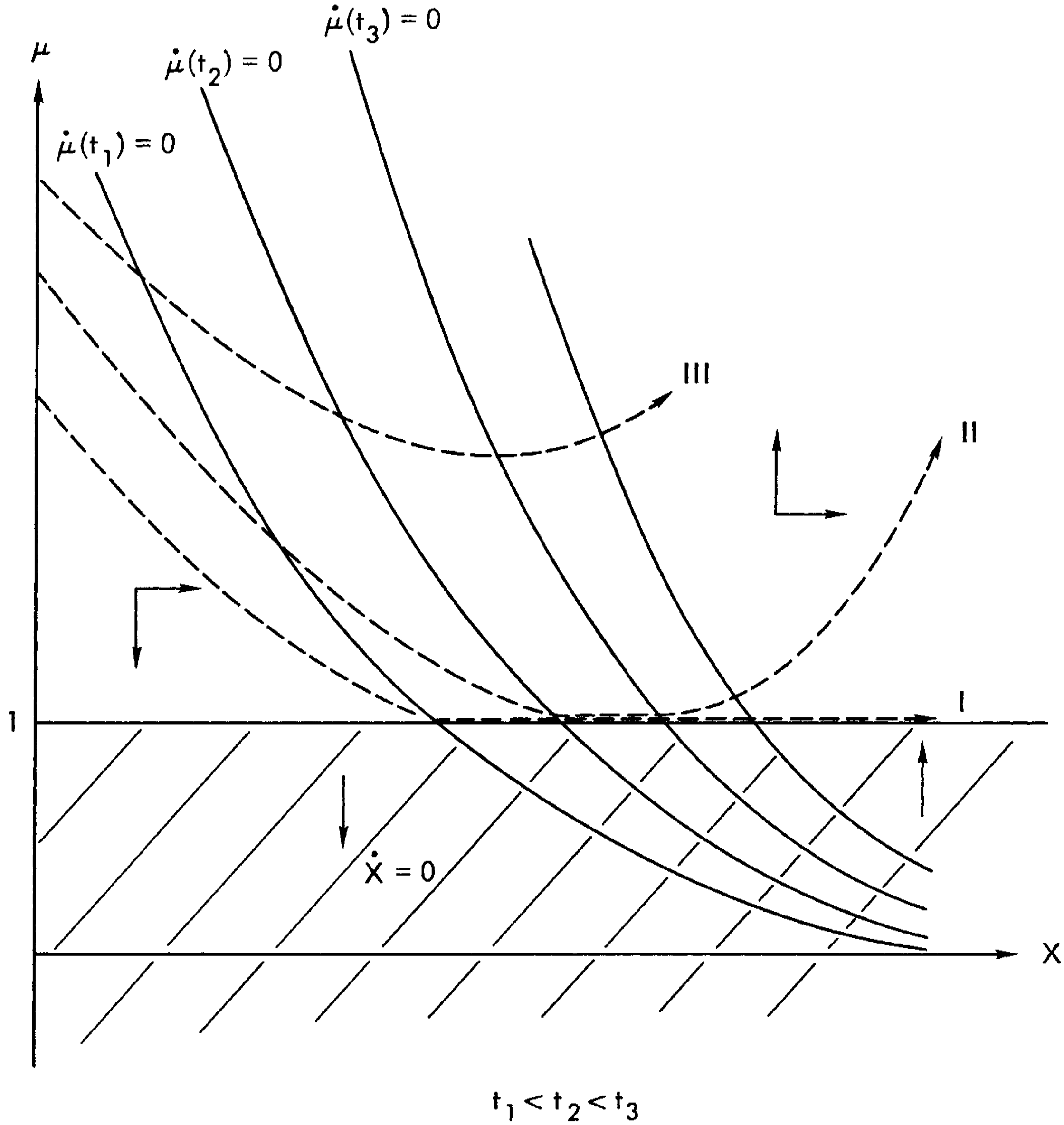

Fig.3b—Optimal trajectories for $\rho<0$ with $U'>0$ everywhere

so as to remain always on the $\dot{\mu} = 0$ stationary. To do this, X must be increasing at the rate given by equation (23) which holds for $t > t_1$. That is,

$$(23') \qquad \dot{X} = \frac{-\rho}{\varepsilon(X)} X \leqq E_{max}$$

In particular, if (23') is satisfied initially and $\varepsilon(X)$ is strictly convex, then

$$(26) \qquad \lim_{t\to\infty} \dot{X}(t) = \lim_{t\to\infty} \frac{-\rho X}{\varepsilon(X)} = 0$$

If $\varepsilon(X)$ is a linear homogeneous function, that is, if $\varepsilon(X) = bX$,[10] then equation (23') becomes

$$(23') \qquad \dot{X} = \frac{-\rho}{b} = \text{a constant}$$

If $-\rho/b < E_{max}$ then a "turnpike" solution is feasible for $t > t_1$, and $X(t) = E_{max}\, t_1 + (-\rho/b)\ (t-t_1)$. In this case the rate of R&D expenditures do not taper off to zero as $t \to \infty$.

It may be possible for equation (23') to be satisfied only for a finite period of time after which X must grow at a rate greater than E_{max} in order to stay on the turnpike. The optimal trajectory must leave the turnpike at this time with μ greater than one and increasing according to equation (7") and X increasing at the rate E_{max}. In Figure 3b this is shown as optimal path II; the "turnpike" phase begins at $t = t_2$ and ends at $t = t_3$. Optimal path III occurs when equation (23') can not be satisfied for any time interval. It remains to show that these paths can satisfy the transversality condition, and this we do below. The argument is similar to Case 1 in which $\rho > 0$; μ, however is increasing instead of decreasing.

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[10] An example of this condition is $U'\varphi'(X) = a^{-(X-\overline{X})}$ for $X > \overline{X}$ and $a > 1$. Then $\varepsilon(X) = X \ln a = bx$.

Let the time when μ starts rising be T*, then equation (7") becomes:

$$(27) \qquad \dot{\mu} = \delta\mu - U'\varphi'(X(t))\ e^{-\rho(t-T^*)}$$

where $X(t) = X(T^*) + E_{max}(t-T^*)$.

Let $\mu^*(t, X(t))$ be the value of μ which makes $\dot{\mu}(t) = 0$ at X(t), then

$$(28) \qquad \mu^*(t,X(t)) < \mu^*(t,X(T^*)) = \frac{U'\varphi'(X(T^*))}{\delta}\ e^{-\rho(t-T^*)}$$

since $\mu(T^*,X(T^*)) = \mu^*(T^*,X(T^*))$ and $\dot{\mu}(t,X(t)) > \dot{\mu}^*(t,X(T^*))$ for $t > T^*$ we must have

$$(29) \qquad \mu(t,X(t)) > \mu^*(t,X(T^*)) > \mu^*(t,X(t)).$$

What this shows is that $\mu(t)$ can increase fast enough to remain always above the $\dot{\mu}$ stationary as both μ and X increase. Furthermore, let $\tilde{\mu}(t)$ be the solution to the related differential equation

$$(30) \qquad \dot{\tilde{\mu}} = \delta\tilde{\mu} - U'\varphi'(X(T^*))\ e^{-\rho(t-T^*)}$$

Hence $\tilde{\mu}(t) = \mu_0 e^{\delta t} + \frac{U'\varphi'(X(T^*))}{\rho+\delta}\ e^{-\rho(t-T^*)}$. Pick $\tilde{\mu}(0)$ (which by necessity must be greater than $\mu(0)$) such that $\mu_0 = [\tilde{\mu}(T^*) - U'\varphi'(X(T^*))/(\rho+\delta)]e^{-\delta T^*} = 0$. Using equations (27) and (30):

$$(31) \qquad \dot{\mu}(t,X(t)) - \dot{\tilde{\mu}}(t) = \delta[\mu(t,X(t)) - \tilde{\mu}(t)] + [U'\varphi'(X(T^*)) - U'\varphi'(X(t))]e^{-\rho(t-T^*)}$$

and by taking limits as $t \to \infty$, we obtain

$$(32) \qquad 0 = \delta \lim_{t\to\infty}[\mu(t,X(t)) - \tilde{\mu}(t)] + \lim_{t\to\infty} [U'\varphi'(X(T^*))-U'\varphi'(X(t))]e^{-\rho(t-T^*)}$$

or

$$(33) \qquad \lim_{t\to\infty} \mu(t,X(t))e^{-\delta t} = \lim_{t\to\infty} \tilde{\mu}(t)e^{-\delta t} - \frac{1}{\delta} \lim_{t\to\infty}[U'\varphi'(X(T^*)) - U'\varphi'(X(T^*) + E_{max}(t-T^*))]\ e^{-\rho(t-T^*)-\delta t} = 0$$

Thus the transversality condition can also be satisfied.

Summary of Case 3

Given maximum problem (5) with $\rho^* < \delta$, optimality is achieved by one of the following programs: (a) spending at the maximum allowable rate for all time $t > 0$; (b) spending at the maximum allowable rate for all but a finite period of time; or (c) spending at the maximum allowable rate for a finite period of time followed by an infinite period with a positive and possibly declining rate of expenditure.

The likelihood of program (a) increases (at the expense of programs (b) and (c)) as (1) the divergence between ρ^* and δ increases; or (2) as E_{max} becomes smaller; or (3) as marginal utility becomes less responsible to increases in X.

Factor Augmenting Technical Change

Let us introduce factor augmenting technical change in the production function for performance improvement by making its argument $X(t)\, e^{\nu t}$ where ν can be either positive or negative. If we assume δ is the common discount rate on utility and expenditures, then maximum problem (5) can be written

$$(34) \qquad \text{maximize} \int_0^\infty \{U(\varphi(Xe^{\nu t})) - E(t)\}\, e^{-\delta t} dt$$

subject to (5a) - (5e).

For practical applications, we may be more concerned with the case $\nu < 0$. This has the interesting interpretation as an exogenous decay in the fruits of R&D expenditures such as an inability to preserve R&D results. This could be due, for example, to rapid turnovers in R&D personnel.

Alternatively the case $\nu > 0$ could correspond to advances in basic research with direct application to the project at hand.

The rules for choosing E(t) remain unchanged but the differential equation describing the behavior of μ becomes:

$$(35) \qquad \dot{\mu} = \delta\mu - U'\varphi'(Xe^{\nu t})\, e^{\nu t}$$

Define Y to be cumulative augmented R&D expenditures, that is, $Y = Xe^{\nu t}$ then equation (35) has precisely the same form as equation (7") with ν taking the place of $-\rho$. We can take advantage of this similarity by analyzing this problem in (Y,μ) phase space instead of (X,μ) phase space. The transformation from X to Y gives rise to a $\dot{Y} = 0$ stationary obtained by solving

$$(36) \qquad \dot{Y} = \dot{X}e^{\nu t} + \nu Y$$

$$= E(t)e^{\nu t} + \nu Y$$

If $\mu(t) > 1$, $Y = \frac{E_{max}e^{\nu t}}{-\nu}$

If $\mu(t) = 1$, $Y = \frac{E(t)e^{\nu t}}{-\nu}$, $0 \leq E(t) \leq E_{max}$

If $\mu(t) < 1$, $Y = 0$

Again we must distinguish two cases corresponding to $\nu \gtrless 0$. Only if $\nu < 0$ will part of the $\dot{Y} = 0$ stationary be in the positive quadrant of the phase diagram in (Y,μ) space. We will handle each of these cases separately.

<u>Case 1: $\nu < 0$</u>

The optimal program in this case is qualitatively similar to Case 2 $(\rho > 0)$ of the previous section. Optimality is achieved by spending at the maximum allowable rate till some time T* after which expenditures stop altogether. Cumulative R&D expenditures remain fixed but cumulative

R&D expenditures augmented by the factor $e^{\nu t}$ decline to zero as $t \to \infty$. Figure 4 shows the $\dot{\mu} = 0$ and $\dot{Y} = 0$ stationaries in (Y, μ, t) phase space. Along the optimal trajectory μ also approaches zero at $t \to \infty$, thus satisfying the transversality condition.[11]

Case 2: $\nu > 0$

In this case the entire positive quadrant of the phase diagram in (Y,μ) space is a region of $\dot{Y} > 0$. The set of optimal programs is qualitatively similar to the set of optimal programs in Case 3 ($\rho < 0$) of the previous section. But because Y has an exogenously exponentially growing term it is more likely that a "turnpike" solution can be found in which the control is gradually curtailed. In combination factor augmenting technical change and different discount rates may work in the same direction or they may oppose each other. The final solution will,

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[11] The brief sketch of the proof would be as follows:

Let $g(Y,t) = U'\varphi'(Y)e^{\nu(t-T^*)}$, then for $t > T^*$

(i) $\frac{dg}{dt} = (1-\varepsilon_1)\ \nu g$ where

(ii) $\varepsilon_1 = -\frac{dU'\varphi'(Y)}{dY}\ \frac{Y}{U'\varphi'(Y)}$

ε_1 is the elasticity of marginal utility with respect to cumulative augmented R&D expenditures. Note that by assumptions already made $\lim_{Y\to 0+} \varepsilon_1(Y) = 0$.

Now by equation (i), $\frac{dg}{dt}$ is bounded by νg and

(iii) $\dot{\mu} = \delta\mu - U'\varphi'(Y)e^{\nu(t-T^*)} \leq \delta\mu - g^*_0\ e^{\nu(t-T^*)}$

where $\mu(T^*) = g^*_0/\delta = 1$ and $\dot{\mu}(T^*) = 0$. Thus for $t > T^*$, $\dot{\mu}(t) < 0$ and $\mu(t) \geq \mu_0 e^{\delta t} + (g^*_0/(-\nu+\delta))e^{\nu(t-T^*)}$. We can set $\mu_0 \equiv \theta$ without violating any previous condition, then $\lim_{t\to\infty} \mu(t) = 0$.

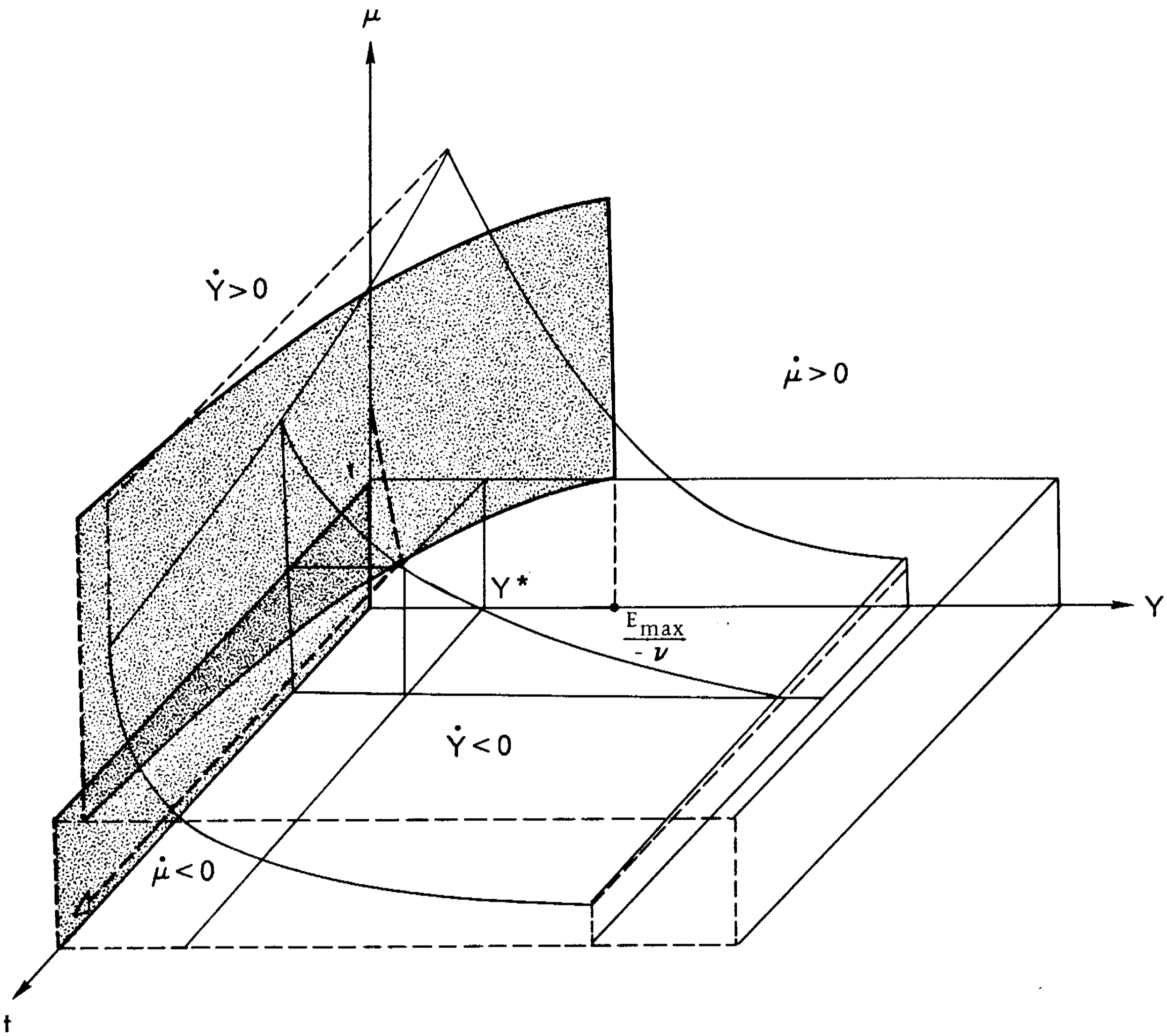

Fig. 4—Optimal trajectory for $\nu<0$

of course, depend on the relative strengths of these two effects.

Some Comments on the Cases so Far:

1. One question we are tempted to ask is how is this material useful. First, it can suggest within the very modest level of detail of the model how R&D programs ought to be run. As such the model is a normative one.

Second, we may wish to compare the predictions of the model with some observed behavior. For example, it is instructive to ask: given the types of solutions of the model, what assumptions about parameters and behavioral rules give the solution that best reflects the actual behavior in Air Force product-improvement programs? The case that seems most applicable is Case 3a in which the marginal utility after some level of performance is precisely zero and utility is discounted over time at a rate less than expenditures.

The result is that the rate of expenditures, while beginning at the maximum allowable level, gradually tapers off to zero. Although this tapering off takes infinite time, something we presumably do not observe in the real world, it is possible to explain the apparent truncation of expenditures by taking into account the fact that after a certain amount of time, the optimal level of expenditures is small in comparison to the real world cost of arranging for funding with the fiscal authorities. That is to say, the effect of bargaining costs associated with procuring funds is to cause an abrupt termination of expenditures even though the optimal program in the absence of such bargaining costs calls for continuing the program, albeit at a low level.

The assumption of Case 3a that $U' = 0$ after some level of performance $\tilde{P}$ can logically arise from a belief that either the enemy's technology can be completely dealt with using only $\tilde{P}$, or that should a capability greater than $\tilde{P}$ be needed an entirely different way of countering the threat would become cheaper.

The other assumption of Case 3a that $\rho^* < \delta$ seems reasonable in light of the observed behavior of the military organizations. (We have to rely on these observations since there are few, if any, statements on this subject.) Typically they exhibit a high discount rate in their desire for resources, and at the same time they seem to encourage the use of a low discount rate in the evaluation of their output, i.e., defense or more specifically, deterrence.

We can carry this line of reasoning further. Because our political process determines the size of our defense budgets rather than a market mechanism, within the DoD as a whole and within each military service, there is a shadow price associated with discretionary capital expenditures which acts as an opportunity cost of capital. This shadow price on capital expenditures in the military may bear no relation to the opportunity cost of capital in the "outside world." It could be that this opportunity cost, δ^*, is used by the military services to discount expenditures in maximum problem (5) instead of the socially optimal discount rate δ.

Furthermore what δ^* is would presumably depend on the overall size of the defense budget (or each Service's budget): the more austere that budget, the higher δ^*. In a world of austere budgets most likely fewer projects would be funded, but for those that were funded (keeping in mind that we are talking about projects which can be correctly described by the model, e.g., product-improvement projects), following the optimal program would mean carrying the project for a longer period than one would optimally under a less austere budget.

Indeed if the military's "shadow" opportunity cost on capital expenditures is greater than the socially optimal discount rate on expenditures, δ, then projects in the military will either be carried too long, or too much money will be spent on them, or both.

2. One of the implicit assumptions made in the model is that the rate of expenditure is free to vary over the range zero to E_{max} without altering the production function for product improvement. In fact the rate of expenditures can make discontinuous jumps in that range effortlessly. This assumption is acceptable only if it is believed that the costs-of-adjustment associated with charges in E(t) are negligible in the range zero to E_{max}. Put another way, our assumption is reasonable if we believe that once a managerial base has been established it is relatively easy to make adjustments within the capacity of that base but that adjustments beyond the capacity of the managerial base can only be made at a cost.

Suppose at various times known in advance, E_{max} can be changed but with a cost-of-adjustment dependent on the size of the change being incurred. It is clear that E_{max} would never be lowered voluntarily since that would not add options and would involve a positive cost. Hence, E_{max} would only be increased, a larger E_{max} making a larger program with more personnel and more specialization possible. The cost-of-adjustment to a higher E_{max} arises from the necessity to expand the managerial base and to improve intra-project communication and coordination stemming from the greater number of personnel and increased degree of specialization.

The question is then how should E_{max} be controlled to yield a higher value of the objective function in maximum problem (5). The answer can be found by solving a combination continuous-discreet dynamic programming problem.

For simplicity let $\rho^* = \delta$ and define

$$V(\tau, E_{max}) = \max \int_{\tau}^{\infty} \{U(\varphi(X)) - E(t)\}e^{-\delta t}dt \tag{37}$$

subject to

(a) $\dot{X}(t) = E(t)$

(b) $E_{max} - E(t) \geqq 0$

(c) $X(\tau) = X_{\tau}$

(d) $E(t) \geqq 0$

Let $C(E^*_{max} - E_{max})e^{-\delta\tau}$ be the present discounted value of the costs-of-adjustment from E_{max} to E^*_{max}. If there is just one time τ at which a decision to change E_{max} can be made, then we need only compare $V(0, E_{max})$ with $\sup_{\{E^*_{max}\}} [V(\tau, E^*_{max}) - C(E^*_{max} - E_{max})e^{-\delta\tau}]$. The optimal policy is to stick with E_{max} if the former expression is greater or the same and to switch to some E^*_{max} if the latter is greater.

If there are several such decision points $\tau_{-n}, \tau_{-n+1}, \ldots, \tau_{-1}, \tau_0$, let

$$f(\tau_i) = \max\,[f(\tau_{i-1}), \sup_{\{E^*_{max}\}} [V(\tau_i, E^*_{max}) - C(E^*_{max} - E^{\tau_i}_{max})e^{\delta\tau}]], \tag{38}$$

$i = -n, -n+1, \ldots, -1, 0$, where $f(\tau_{-n-1}) = V(0, E^0_{max})$ is given since E^0_{max} is given. Using this formulation the optimal program can be found by the traditional techniques of dynamic programming.

If $V(\tau, E_{max})$ is concave with respect to E_{max}[12] and C is strictly convex with respect to its argument, then changes in the optimal value of E_{max}, if any, will be finite.

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[12]Recall $\pi(t) = -1 + \mu(t)$ is the loss at time t due to constraint (5b). The total loss associated with a finite E_{max} is given by:

(i) $\int_0^{T^*} \pi(t)dt$ where $T^* = X^*/E_{max}$.

Differentiating with respect to E_{max},

(ii) $$\frac{d}{dE_{max}} \int_0^{T^*} \pi(t)dt = \frac{-X^*}{(E_{max})^2} \pi(T^*) + \int_0^{T^*} \frac{\partial \pi}{\partial E_{max}} dt < 0$$

since $\pi(T^*) = 0$ and $\frac{\partial \pi}{\partial E_{max}} < 0$ for $t \varepsilon [0,T^*]$.

Hence the value of V must increase as E_{max} increases.

Similarly the second derivative

(iii) $\frac{d^2}{dE^2_{max}} \int_0^{T^*} \pi(t)dt = \int_0^{T^*} \frac{\partial^2 \pi}{\partial E^2_{max}} dt$ will be positive if $\frac{\partial^2 \pi}{\partial E^2_{max}}$ can be shown to be positive for $t \varepsilon [0,T^*]$. This would complete the proof of the concavity of V.

REFERENCES

Arvis, Rutherford, Discrete Dynamic Programming, Blaisdell Publishing Company, New York, 1964.

Hamilton, C. W., "Optimal Control of Research and Development Expenditures," Technical Report No. 48, Operations Research Center, Massachusetts Institute of Technology, 1969.

Pontryagin, L. S., et al., The Mathematical Theory of Optimal Processes (translated by K. W. Trirogoff), Interscience Publishers, Inc., New York, 1962.

Rothschild, Michael, "On the Cost of Adjustment," Harvard Institute of Economic Research Discussion Paper No. 138, Harvard University, 1970.

Shell, Karl (ed.), Essays on the Theory of Optimal Economic Growth, The M.I.T. Press, Cambridge, Mass., 1967.