SOME MILITARY APPLICATIONS OF THE THEORY OF GAMES

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1. SUMMARY

Many military problems are concerned with the allocation of forces in space and/or time in a competitive environment. We shall give some examples of the application of the theory of games to such problems. The examples are from three general military fields: strategic air war, tactical air war, and target prediction.

An important problem in strategic air war is the target selection problem — the choice of targets for attack and defense. This problem is formulated both as a finite game and an infinite game. In each case the optimal allocations are described.

In the tactical air war example we view the tactical air war game as consisting of a series of strikes, or moves, each of which consists of simultaneous counter-air, air defense, and close support operations by each side to accomplish a given theater mission. We give some general properties of the optimal allocations.

Finally, the problem of scheduling the launching of missiles is described as an example of target prediction. The optimal launching schedules are described for both sides.

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2. TARGET SELECTION

The target selection problem frequently appears in military situations. In its most general form the problem may be described as follows: Suppose the Attack and Defense each have a fixed quantity of resources to allocate among a series of targets of different values. How should the Attack and Defense select the targets for the allocations?

2.1 Single Target:

First, let us look at the simplest of the target selection problems. The Attack has one unit and the Defense also has one unit to be allocated to one target. Which target should receive the respective allocation?

The game model may be described as follows:

Targets. There are n targets which we label T_1, T_2, ..., T_n. We assume that these targets have values a_1, a_2, ..., a_n, respectively, and are ordered as follows

$$a_1 > a_2 > \cdots > a_n > 0 .$$

Attack. The Attack, Blue, has one attacking unit to allocate to some one of the n targets.

Defense. The Defense, Red, has one unit of defense to allocate to some one of his targets. Further, it is assumed that the unit of defense has a kill probability p—i.e., if an attack is made on a defended target, then with probability p, the Attack fails to attack the target. Hence $1 - p$ is the probability that Blue is successful in attacking this target.

Strategy. A strategy for Blue is a choice of a target for attack. A strategy for Red is a choice of a target for defense. Thus each player has n strategies.

Payoff. We shall assume that if an attack is made on an undefended target, T_k, then the payoff to Blue is the value, a_k, of that target. However, if an attack is made on a defended target, T_i, then the payoff is $(1-p)a_i$, the expected damage to the target. Therefore, the payoff matrix is the following:

Target Selection Payoff

Target Defended

$$
\text{Target Attacked}\quad
\begin{array}{c}
\\ T_1 \\ T_2 \\ \vdots \\ T_n
\end{array}
\begin{array}{c}
\begin{array}{cccc} T_1 & T_2 & \cdots & T_n \end{array} \\
\begin{pmatrix}
(1-p)a_1 & a_1 & \cdots & a_1 \\
a_2 & (1-p)a_2 & \cdots & a_2 \\
\cdots & \cdots & \cdots & \cdots \\
a_n & a_n & \cdots & (1-p)a_n
\end{pmatrix}
\end{array}
$$

2.2 Optimal Selection: From the description of the game it is clear that, in general, neither player has an optimal pure strategy. It is necessary for both players to choose their targets for allocation by means of some randomizing device.

Solving this game, we find that the Attack and Defense should randomize over the same high-valued targets, namely the t highest-valued targets, where

$$\max_{k \leq n} \frac{k-p}{\sum_{i=1}^{k} \frac{1}{a_i}} = \frac{t-p}{\sum_{i=1}^{t} \frac{1}{a_i}} .$$

The optimal strategy for the Attack is to attack target T_i (where $i \leq t$) with probability $\dfrac{1}{a_i \sum_{i=1}^{t} \frac{1}{a_i}}$ and never attack targets $T_{t+1}, T_{t+2}, \ldots, T_n$. Thus the higher the target value, the less likely that target will be attacked.

The optimal strategy for the Defense is to defend target T_i (where $i \leq t$) with probability

$$\frac{1}{p}\left(1 - \frac{t-p}{a_j \sum_{i=1}^{t} \frac{1}{a_i}}\right) .$$

Thus the target with the higher target-value is more likely to be defended than the low-valued target.

The game value is given by

$$v = \frac{t - p}{\sum_{i=1}^{t} \frac{1}{a_i}} \quad .$$

2.3 Defense of Many Targets: Like most battle situations, the combat between air attack and air defense can be viewed as a zero-sum two-person game: the attacker seeks the greatest possible gains in the form of the destruction of targets, and the defender wishes to make these gains as small as possible.

An important decision of the defender in a battle situation is the distribution of his total defense resources among his targets. An important decision of the attacker is the distribution of his total attacking force among the targets. We shall consider this game in a very simplified form in which we assume only a single possibility of choice for each player, namely, for the attacker the choice of an allocation of his resources among targets, and for the defender the choice of an allocation of his resources among the targets.

We wish to answer such questions as: Shall all the targets be defended? If only some of the targets are to be defended, how shall these be selected? How should the attacker select his targets?

The game model we shall analyze is the following:

Defense. The defender, Red, has D units of defense to distribute among his n targets, which we label $T_1, T_2, \ldots, T_n$. Let us assume that the n targets have values $k_1, k_2, \ldots, k_n$, respectively, and are ordered as follows

$$0 < k_1 < k_2 \cdots < k_n .$$

Attack. The attacker, Blue, has A units of attack to distribute among the n targets. Let us assume the attack is stronger than defense, or $A \geq D$.

Strategy. A strategy for Blue, is an allocation of his attacking force A among the n targets. Thus a strategy for Blue is a set of numbers $x_1, x_2, \ldots, x_n$ such that $x_i \geq 0$, $\sum_{i=1}^{n} x_i = A$.

A strategy for Red is a set of numbers $y_1, y_2, \ldots, y_n$ such that $y_i \geq 0$ and $\sum_{i=1}^{n} y_i = D$. Each y_i represents the number of defensive units allocated to target T_i.

Payoff. Let us assume that one unit of defense can check one unit of attack. Further, let us assume that the amount of damage to any target is proportional to the number of attacking units which outnumber the defense units, the coefficient of proportionality depending on the particular target. Finally, let us assume that the payoff is the sum, over the targets, of the damage to each target.

Thus the payoff to the attack is

$$M(x, y) = \sum_{i=1}^{n} k_i \max(0, x_i - y_i)$$

where $x_i \geq 0$, $y_i \geq 0$, $\sum_{i=\cdot}^{n} x_i = A$, and $\sum_{i=\cdot}^{n} y_i = D$.

2.4 Optimal Attack and Defense: It is apparent that $M(x, y)$ is convex in y, for each x. It is also convex in x, for each y. Therefore Red, the defender, has a pure strategy which is optimal. The attacker has a mixed strategy which is optimal.

It is optimal for the defender to distribute his defensive force D among the high valued targets. It is optimal for the attacker to select one of the high valued targets at random, subject to a given probability distribution, and then allocate his entire attacking force on that target.

To give a more precise description of the optimal strategies for the two players, we introduce the following notation:

$$\frac{1}{h_s} = \sum_{i=s}^{n} \frac{1}{k_i} \qquad s = 1, 2, \ldots, n$$

$$\ell_s = k_s - h_s \left(n - s + 1 - \frac{D}{A}\right) \quad s = 1, 2, \ldots, n$$

m = smallest value of s such that $\ell_s \geq 0$.

In terms of the above definitions, the optimal allocations are as follows:

The attacker's optimal mixed strategy is

(a) Never attack the low valued targets $T_1, T_2, \ldots, T_{m-1}$.

(b) Use the entire attacking force A on a target selected at random, subject to the following probability distribution:

$$\mathrm{pr}\left\{ x_i = A \right\} = \frac{h_m}{k_i} \qquad m \leq i \leq n$$

The defender's optimal pure strategy is

(a) Leave undefended the low-valued targets $T_1, T_2, \ldots, T_{m-1}$.

(b) Defend the high-valued targets $T_m, T_{m+1}, \ldots, T_n$ by placing

$$A\left\{ 1 - \frac{h_m}{k_i}\left(n - m + 1 - \frac{D}{A}\right)\right\} \quad m \leq i \leq n$$

units at the i-th target.

The value of the game to the attacker is

$$v = A(k_m - \ell_m)$$

At each of the defended targets, the attacker gets the value of the game if that target is attacked by the entire attacking force. At each of the undefended targets, a concentrated attack yields less than the value of the game. If the defender has allocated his defenses optimally, there is no soft spot in his targets.

3. TACTICAL AIR WAR

Both of the preceding models are essentially single stage models — i.e., only one allocation or one strike is made. Many military problems involve multistage processes or planning over time. An example of such a game is a military campaign involving many strikes with each strike consisting of an allocation of resources.

The problem of how to employ tactical air forces during each period of combat will be analyzed as a multi-move game. We wish to determine the optimal allocation of the tactical air forces among the various theater air tasks (counter air, air defense and ground support) in a multistrike tactical air campaign.

Let us assume that at the i-th period of the air operations, Blue has p_i planes and Red has q_i planes to be allocated among the three air tasks. Suppose Blue dispatches x_i planes on counter-air operations and u_i planes on air-defense operations, and the remainder, $p_i - x_i - u_i$ planes, on ground-support operations. Similarly, suppose that on this strike Red allocates y_i planes to counter air, w_i planes to air defense, and the remainder, $q_i - y_i - w_i$ planes, to support his ground forces. For this strike and for any future strikes, the above decisions are made by each side in ignorance of the allocation of the opposing side. It is assumed, however, that each side knows the number of planes that he and his opponent have.

Since Red allocates w_i planes to air defense we can expect a reduction in the number of Blue's planes that get through to counter-air targets. Let the number of Blue attacking planes that penetrate Red's defense be given by

$$\max\,(0,\ x_i - w_i)\quad .$$

The objective of Blue's counter-air operations is to reduce the enemy's air force by dropping bombs on certain targets, and the number of aircraft destroyed will vary with the number of attacking planes that penetrate Red's defenses. Increasing the number of Blue's penetrating planes will diminish the enemy's air force, but cannot reduce it by more than q_i. If we assume that each of Blue's penetrating planes can destroy one plane of the enemy, and that all of Red's aircraft are at risk at the time of a strike, then Blue's i-th counter-air strike will destroy

$$\min\,[q_i,\ \max\,(0,\ x_i - w_i)]$$

Red planes.

The inventory of Red planes at the end of the i-th strike is:

$$q_{i+1} = \max\,[0,\ q_i - \max\,(0,\ x_i - w_i)]\quad .$$

Similarly, the inventory of Blue planes at the end of the i-th strike is

$$p_{i+1} = \max\left[0,\ p_i - \max\left(0,\ y_i - u_i\right)\right] .$$

In order to describe the payoff, let us look at Blue's employment of forces during the campaign. We assume that his objective is to assist the ground forces in the battle area, and the results will vary with the number of planes he allocates to ground-support operations. We assume that it is possible to construct for Blue a payoff function, giving the payoff for each strike of the campaign, in the form of the distance advanced by the ground forces as a function of the number of planes allocated to ground support. This function depends heavily on the characteristics of the ground-support targets – i.e., on the degree of concentration of troops, vehicles, and materiel, and on the fortification of positions.

If Blue's ground forces now must advance while being subjected to Red's ground-support sorties, Blue's yield in ground-support is reduced by the planes allocated by Red to close-support missions. The net advance of Blue's ground forces during the i-th strike may be expressed by

$$(p_i - x_i - u_i) - (q_i - y_i - w_i) .$$

The payoff for the entire campaign of N strikes is the sum of these net yields for each of the N strikes, or

$$M = \sum_{i=1}^{N} \left[(p_i - x_i - u_i) - (q_i - y_i - w_i)\right] .$$

3.1 Optimal Tactics: We shall give a description of the optimal employment of tactical air forces in terms of the number of strikes and the relative strengths of the two sides.

Campaign ends with ground support. The campaign always ends with a series of strikes on ground support — i.e., during the closing period of the campaign both Red and Blue concentrate all their forces on ground-support missions. In this terminal period both sides have the same optimal tactics, regardless of their initial forces.

Blue (stronger) splits his forces. At all times other than the closing phase of the campaign, Red and Blue have very different optimal tactics. During any of these early strikes, the stronger side, Blue, has a pure strategy. That is, there exists a best allocation of Blue's air force among the three air tasks. The size of the allocation depends on the relative strengths of the two air forces and the number of strikes left in the campaign.

Red (weaker) mixes his tactics and concentrates his forces. The weaker combatant cannot use a single strategy, but must bluff during all the strikes other than those of the terminal phase. Unlike his opponent, the weaker combatant does not have a single allocation that is best. He must use a mixed strategy and gamble for high payoffs. If he is not too weak, then he concentrates his entire force either on counter air or on air defense; but which of these tasks receives the full effort is decided by some chance device. However, if Red is very weak, then he allocates his entire air force to any one of the three air tasks with the particular task again chosen at random. In other words, if a player is very weak relative to the opponent, then he takes a chance on an early payoff. Of course, to be most effective, he must bluff correctly — i.e., the random device should select the tasks with the proper relative frequencies.

Mix and split the same tasks. It is of interest to note that on each strike Red, the weaker side, bluffs with the same tasks that Blue uses in his allocation. Thus if Red is very weak he bluffs with each of the three tasks, and Blue splits his forces among each of the three tasks. However, if Red is moderately weak, he bluffs with two tasks — counter air or air defense — and Blue splits his forces between the same two tasks, counter air and air defense.

4. TARGET PREDICTION

An example of the target prediction problem is the operational problem of scheduling the firing of a missile requiring an exposure time. Suppose that Blue plans to fire one missile before T hours have expired. However, in order to fire the missile, Blue must expose it for t hours, where $t < T$, during which time the missile is vulnerable to attack by Red. Let us assume that Red, not knowing Blue's decision, has only one opportunity to attack. When is the optimal time for Blue to expose his missile?

When is the optimal time for Red to attack?

A strategy for Blue is a choice of time x for Blue to begin exposing the missile for a time t, where, $0 \leq x \leq T - t$. Since Red does not know Blue's choice, a strategy for Red is a choice of time y for Red to attack Blue, where $0 \leq y \leq T$. Blue will fire his missile at time $x + t$ if he has not been attacked by Red during the time of missile exposure. That is, if Red attacks either before or after missile exposure, Blue will be able to fire his missile.

Let the payoff to Blue be 1 if he fires the missile and 0 otherwise. Then in terms of the strategies of the players the payoff is described by the following discontinuous payoff function:

$$M(x, y) = \begin{cases} 1 & \text{if} \quad y \leq x \leq T - t \quad \text{or} \quad x + t \leq y \leq T \\ 0 & \text{otherwise} \end{cases}$$

4.1 Optimal Strategy: Solving this game we find that the game value is a discontinuous function of t. If we define n to be the largest integer contained in $\frac{T}{t}$ then the value of the game is

$$v = 1 - \frac{1}{n} \quad .$$

Blue's optimal strategy is the uniform mixture

$$F(x) = \frac{1}{n} \sum_{j=0}^{n-1} I_{jt}(x) \quad .$$

Red's optimal strategy is also a uniform mixture, but over different points.

$$G(y) = \frac{1}{n} \sum_{j=1}^{n} I_{\frac{j}{n+1}}(y) \quad .$$