SIMPLE GAMES: AN OUTLINE OF THE DESCRIPTIVE THEORY

L. S. Shapley
Mathematics Department
The RAND Corporation

P-2277

April 11, 1961

Presented at Mental Health Research Institute, University of Michigan, Ann Arbor, Michigan, April 20, 1961.

SUMMARY

The elementary properties of "simple" games are set forth. These are multiperson games in which each coalition of players is either completely ineffective or able to win outright. A tabulation of all simple games with four or fewer players is included.

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Introduction

The term "simple" was introduced by von Neumann and Morgenstern [29]* to distinguish a certain class of multiperson games, namely those in which each coalition that might form is either all–powerful or completely ineffectual. These games are especially appropriate to the study of organizations, committees, and legislatures, or more generally of any "political" structure in which power and authority, rather than a monetary type of utility, is the fundamental driving force.

As a class, simple games are relatively independent of many of the restrictive and sometimes controversial assumptions that necessarily underlie the more general theory of games developed in [29] — games in which the players and coalitions receive more or less arbitrary numerical payoffs. In particular, the measurability of "utility," and its commensurability between players, is only of minor importance in the case of simple games. Much of the sophisticated analytical apparatus that has been devised, in [29] and elsewhere, to "solve" or otherwise cope with the more general numerical games, is easier to appreciate and to apply in the context of simple games, and only rarely does the application reduce to a triviality. Finally, a surprising number of the multiperson games found in practice are simple. Thus for several

*Numbers in brackets refer to the bibliography at the end of the paper.

reasons — methodological, pedagogical, and practical, the theory of simple games invites a self–contained, independent treatment.

The present essay is an attempt at a somewhat informal description of a formal system. The mathematical techniques used, mainly those of set theory, are elementary. The theoretical structure here outlined has never been published in extenso although it has been used in numerous research papers and portions of it have found their way into several mathematics textbooks (see [4], [16], [20]). The present account follows generally the exposition used in several series of lectures given by the author at Princeton (1953), Caltech (1956), Stanford (1959), and elsewhere, and draws heavily on a 1954 RAND Research Memorandum with the same title [25].

1. Some particular simple games

To ease into the subject as painlessly as possible, we shall postpone formalities and devote a few paragraphs to a discussion of familiar examples, introducing some of the basic terminology (underlined) in context.

Perhaps the simplest of simple games is the straight majority game M_n, with an odd number, n, of players. An example is the House of Representatives (when voting on ordinary legislation) = M_{437}. In M_n there are 2^n possible coalitions, if we count the null set. They are of two types: winning (more than $n/2$ members) and losing (less than $n/2$ members). Any coalition that contains a winning coalition is also winning, obviously, and any coalition contained in a losing coalition is also losing. Moreover, the complement of every losing coalition is winning.

The last remark is no longer true if n is even, since blocking coalitions (consisting of exactly $n/2$ players) are now possible. Blocking coalitions neither win nor permit their complements to win; thus deadlocks can occur. Games without such coalitions are called strong.

One might be tempted to describe the Senate (on ordinary majority votes) as the nonstrong game M_{100}. However, a little thought shows that the tie–breaking role of the presiding officer makes M_{101} the more accurate model. Any coalition of 51 or more individuals can win, and no others.

Of course, Senate rules allow for a great deal of blocking before issues come to a vote. If we take an extreme case, just for the purpose of illustration, and postulate that each senator by himself is a potential blocking coalition, we obtain an instance of a pure bargaining game, denoted B_{100}, in which all decisions require unanimous consent. In general, B_n is a game with just one winning coalition (the whole set) and $2^n - 1$ losing coalitions, all but one of them blocking. The pure bargaining game is an example of a weak game — that is, a game in which at least one player has veto power.

An even more extreme case, sometimes useful for formal purposes, is the null game on n players, O_n, in which there are no winning coalitions at all. In the formal theory as developed in this paper, however, we shall generally exclude this case.

All the games so far mentioned are special cases of $M_{n,k}$, the game where there are n players in all and it takes k or more to win. Thus:

$$M_n = M_{n,(n+1)/2} \qquad \text{(n odd)}$$
$$M_n = M_{n,(n+2)/2} \qquad \text{(n even)}$$
$$B_n = M_{n,n}$$
$$O_n = M_{n,n+1} .$$

If k is greater than n/2 then $M_{n,k}$ represents a perfectly reasonable and realizable voting rule, but if k is less than n/2 we encounter the peculiar possibility of two or more "winning" coalitions forming independently and simultaneously. Games of this kind we shall call improper; they play a role in the theory somewhat analogous to that of "imaginary" numbers in algebra and analysis. Although they may not have a direct interpretation, they are useful in the factorization of larger, proper games, as we shall see presently, and in other ways.

The $M_{n,k}$ games include all the simple games, proper and improper, that are fully symmetric. Of course, there are many other simple games, since the class of winning coalitions can be prescribed more or less at will. An example of some importance is the weighted majority game

$$[k; w_1, w_2, \ldots, w_n],$$

a generalization of $M_{n,k}$, in which the players may have several votes apiece. The winning coalitions here are the sets of players whose "weights" w_i total at least k. A different sort of example of a game that is not fully symmetric can be generated by compounding symmetric games, thus:

$$\text{"Congress"} = M_{101} \times M_{437}$$

(majority in both houses needed to win), or

$$\text{"UN Security Council"} = B_5 \times M_{6,2}$$

(seven out of eleven to win, but five vetoes). The latter is of course a weak game. Note that the factor $M_{6,2}$ is improper, even though the game as a whole is proper. The product operation "x" illustrated here is only one of several methods of building up compound simple games out of smaller components.

2. Formal definitions

We now begin the formal exposition. Let N denote the set of players and $\mathcal{N}$ the set of subsets of N, or "coalitions." Let O denote the empty set of players, $\mathcal{O}$ the empty set of coalitions. Let $\mathcal{S}^+$ denote the set of supersets, $\mathcal{S}^-$ the set of subsets, and $\mathcal{S}^*$ the set of complements, of the elements of $\mathcal{S}$. Let $\bigcap\mathcal{S}$ denote the intersection, and $\bigcup\mathcal{S}$ the union, of the elements of $\mathcal{S}$. (Script letters will be reserved for sets of sets of players, i.e., for subsets of $\mathcal{N}$.)

A game G is an ordered pair:

$$G = (N, \mathcal{W}),$$

where $\mathcal{W}$, the set of winning coalitions, satisfies

(1) $\mathcal{W} = \mathcal{W}^+$

(2) $\mathcal{W} \neq \mathcal{O}$

(3) $\mathcal{W} \neq \mathcal{N}$.

Condition (1) is in accordance with the intuitive meaning of "winning." Condition (2) excludes null games. Condition (3) prevents the empty set from being a winning coalition. Denote the sets of losing and blocking conditions by:

$$\mathcal{L} = \mathcal{N} - \mathcal{W}, \qquad \mathcal{B} = \mathcal{L} \cap \mathcal{L}^*.$$

The set $\mathcal{W}^m$ of minimal winning coalitions is defined by

$$\mathcal{W}^m = \bigcap \text{ of all } \mathcal{S} \text{ such that } \mathcal{S}^+ = \mathcal{W}.$$

We note that

$$\mathcal{L} = \mathcal{L}^-, \quad \mathcal{L}^* = \mathcal{N} - \mathcal{W}^*, \quad \mathcal{B} = \mathcal{B}^*, \text{ and } \mathcal{W}^{m+} = \mathcal{W}.$$

A player $p \in N$ is called a dummy if

$$p \notin \bigcup \mathcal{W}^m .$$

A game is...

proper	if	$\mathcal{W} \cap \mathcal{W}^* = \mathcal{O}$
strong	if	$\mathcal{B} = \mathcal{O}$
weak	if	$(\mathcal{W}) \neq O$
inessential	if	$\mathcal{W} = \{\{p\}\}^+$ for some player $p \in N$.

In the last case, p is called a dictator, and the other players are of course dummies. We note that improper games may be either strong or non–strong, but they are never weak or inessential.

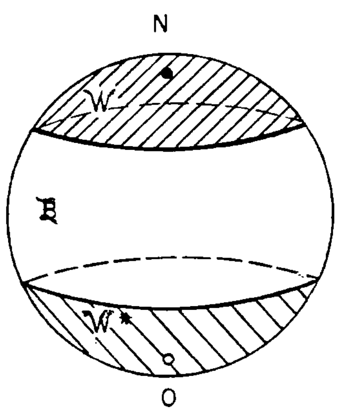
THEOREM. No essential game is both weak and strong.

Proof. Suppose G is weak. Take $p \in (\mathcal{W})$. Suppose G is also strong. Then one of $\{p\}$, $n - \{p\}$ is in $\mathcal{W}$. Since the latter is not possible, we have $\{p\} \in \mathcal{W}$ and $N - \{p\} \in \mathcal{L}$. Hence $\{\{p\}\}^+ \subseteq \mathcal{W}$ and $\{N - \{p\}\}^- \subseteq \mathcal{L}$. This accounts for all coalitions, hence $\mathcal{W} = \{\{p\}\}^+$, making G inessential. QED.

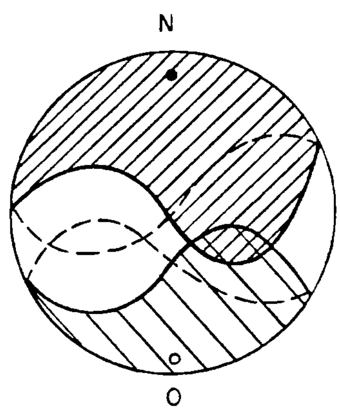
If $\mathcal{W}$ is enlarged (preserving properties (1) and (3)) we shall say that the game in question has been strengthened; if $\mathcal{W}$ is diminished (preserving (1) and (2)) we shall say it has been weakened. It can be verified that repeated strengthening of a game will eventually make it strong, moreover once it becomes strong it remains so. A similar statement holds in the case of repeated weakening.

3. A pictorial representation

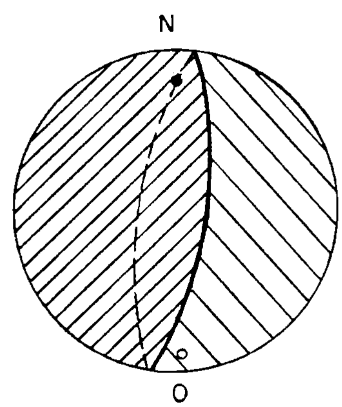
The foregoing algebraic system may be visualized by taking the subsets of N to be points on a sphere, with complementary sets located at diametrically opposite points. Let sets of equal size be located on the same latitude, the size increasing from 0 at the south pole to n at the north pole. Set–inclusion can be represented by links joining each set to its immediate subsets and supersets. (For $n > 3$ these lines cannot be drawn without crossing.) Sets of sets can be represented by regions on the sphere.



An $M_{n,k}$ game



An improper game
(non-strong)



A dictator game
(weak and strong)

For the game $M_{n,k}$, the set $\mathcal{W}$ is now a circular region centered on the north pole, and the set $\mathcal{W}^*$ is the antipodal cap around the south pole. $\mathcal{B}$ is the equatorial zone (possibly empty) between them. More generally, $\mathcal{W}$ is some neighborhood of N; $\mathcal{W}^*$ the corresponding neighborhood of O. In an improper game, $\mathcal{W}$ and $\mathcal{W}^*$ overlap somewhere. In a strong proper game, $\mathcal{W}$ and $\mathcal{W}^*$ can be made to cover the sphere exactly without overlapping. Strengthening a game extends the polar caps; weakening it pulls them back. In a weak game, the $\mathcal{L}$ region extends up almost to the north pole, since there is a one-person blocking coalition.

4. Duality

The <u>dual game</u> G^* of a game $G = (N, \mathcal{W})$ is defined by

$$G^* = (N, \mathcal{L}^*).$$

For example, we have $M^*_{n,k} = M_{n,n-k+1}$. The following properties of the duality relation are fairly obvious, if one bears the graphical representation of the preceding section in mind.

THEOREM. (a) $G = G^{**}$;

(b) G^* is proper if and only if G is strong;

(c) $G = G^*$ if and only if G is strong and proper.

The usefulness of the duality concept depends heavily on the inclusion of improper games in our theory, since, by (b) and (c), the dual of a proper game is either the same game over again, or an improper game.

5. Compound simple games

We now consider some ways of putting games together to make bigger games. We shall consistently assume that the player-sets of all games appearing together in composition are disjoint. In particular, let $N' \cap N'' = 0$, $N' \cup N'' = N$. Then we define the <u>product</u>

$$(N, \mathcal{W}) = (N', \mathcal{W}') \times (N'', \mathcal{W}'')$$

by

$$\mathcal{W} = \{S \mid S \cap N' \in \mathcal{W}' \ \underline{\text{and}}\ S \cap N'' \in \mathcal{W}''\},$$

and the <u>sum</u>

$$(N, \mathcal{W}) = (N', \mathcal{W}') + (N'', \mathcal{W}'')$$

by

$$\mathcal{W} = \{S \mid S \cap N' \in \mathcal{W}' \ \underline{\text{or}}\ S \cap N'' \in \mathcal{W}''\}.$$

To win in the product, a coalition must win in both components; to win in the sum it only must win in one component.

Multiplication and addition satisfy the usual commutative and associative laws, but not the distributive law. Instead, there is a duality principle, which allows us to pass from sums to products.

THEOREM. (a) $(G + H)^* = G^* \times H^*$

(b) $(G \times H)^* = G^* + H^*$.

Some other elementary properties of these operations are the following:

Products. A product is never strong. A product is proper unless both factors are improper. A product is weak if and only if at least one of the factors is weak.

Sums. A sum is generally much stronger than the corresponding product. A sum is strong if and only if at least one term is strong. A sum is always improper and hence never weak.

Weak games. An essential weak game can always be factored:

$$(N, \mathcal{W}) = (W, \{W\}) \times (N-W, \mathcal{W}_W),$$

where W is any nonvoid losing subset of $\bigcap\mathcal{W}$ and $\mathcal{W}_W$ is the set of intersections of elements of $\mathcal{W}$ with $N - W$. The first factor is of course the pure bargaining game on the set W, which can in turn be factored into one–person pure bargaining games if desired.

Dummies. A dummy in a component remains a dummy in the product or sum. A dummy cannot be factored out as the theory now stands, but if null games were admissible we could write:

$$(N, \mathcal{W}) = (D, \mathcal{O}) + (N-D, \mathcal{W}_D) = (D, \mathcal{D}) \times (N-D, \mathcal{W}_D),$$

D being the set of dummies and $\mathcal{D}$ the set of subsets of D.

We have already mentioned "Congress" = $M_{437} \times M_{101}$ and "Security Council" = $B_5 \times M_{6,2}$ as examples of products (see §1 above). Sums, being improper, are harder to find in the natural state. However, situations of the type

$$(G + H) \times K$$

are not uncommon — two bodies, each with power to act in the same sphere, but subject to the ultimate approval of a third group. Sometimes the "x K" part is not apparent in the explicit rules, but takes the form of a slow-moving dynamical process that is sufficient to mitigate the theoretical impropriety of the sum $G + H$. For example, the U.S. Constitution can be amended by the state legislatures, even if Congress is opposed; alternatively, it can be amended by Congress and special conventions in the states, over the opposition of the state legislatures. Here we have the sum of two games — formally improper. However, the "inertia" of the political processes involved is so great that it is unlikely that disjoint winning coalitions will ever form on opposite sides of a real issue.

When we have more than two components, the possibilities for combination are greatly enlarged. A general <u>compound simple game</u>, with m components, is symbolized as follows:

$$H = K[G_1, \ldots, G_m],$$

where K is an arbitrary m-person simple game. The compound game H (sometimes called a "committee composition") has $n_1 + n_2 + \cdots + n_m$ players, if n_i denotes the number of players of G_i. The game K is called the "supergame," or <u>quotient</u>. Each <u>game</u> G_i plays the role of a <u>player</u> in K. Thus, a set of players is winning in H if and only if it includes as subsets winning coalitions in enough of the component games to constitute a winning coalition of K.* For example, a winning coalition in the 9–person compound game

$$M_3[M_3, M_3, M_3]$$

must include at least two players apiece from at least two of the subgames; thus there are 27 minimal winning coalitions, each consisting of four players.

There are some trivial combinations involving the one–person game B_1, as follows:

$$B_1[G] = G$$

$$G[B_1, \ldots, B_1] = G.$$

The two–person games B_2 and B_2^* give us the product and sum operations previously defined:

$$B_2[G_1, G_2] = G_1 \times G_2$$

$$B_2^*[G_1, G_2] = G_1 + G_2.$$

The first new compositions that our general definition provides are the four–person games

$$M_3[B_1, B_1, B_2] \quad \text{and} \quad M_3[B_1, B_1, B_2^*].$$

Numbering the players from 1 to 4, we have $\mathcal{W}^m = \{\overline{12}, \overline{134}, \overline{234}\}$ and $\{\overline{12}, \overline{13}, \overline{14}, \overline{23}, \overline{24}\}$, respectively. We see that the first game is proper

*The theory is readily extended to include quotients K that are general numerical games (not simple), and to include components G_i whose player sets overlap.

and nonstrong, the second strong and improper. In fact, the two games are mutually dual. The general duality relation for compound games is as follows:

$$(K[G_1, \ldots, G_m])^* = K^*[G_1^*, \ldots, G_m^*];$$

it is seen to include the duality principle for sums and products, already stated, as a special case.

The proper/strong/weak properties of compound games, vis–à–vis their components, are easy to work out; we shall not trouble to list them here, except to point out that if all the component games, and the quotient game as well, are strong and proper (i.e., constant–sum), then the compound game is also strong and proper. (An example is the nine–person game $M_3[M_3, M_3, M_3]$ mentioned above.) This means that general compound games, unlike products and sums, can be meaningfully defined within the restricted field of constant–sum simple games considered in von Neumann and Morgenstern's work [29]. This fact is especially interesting because of the intimate connection that exists between the von Neumann–Morgenstern solutions of compound games and those of their components.

A game that cannot be expressed in compound form (except for the trivial "B_1" representations given above) is called <u>indecomposable</u> or <u>prime</u>. The only prime n–person games with $n < 4$ are B_1 and M_3, but there are six prime four–person games (four of them improper) and the number increases rapidly for $n > 4$. Compounds of compounds can be represented in a variety of ways with the aid of the identity:

$$K\left[H_1[G_{11}, \ldots, G_{1m_1}], \ldots, H_p[G_{p1}, \ldots, G_{pm_p}]\right]$$
$$\equiv \left(K[H_1, \ldots, H_p]\right)\left[G_{11}, \ldots, G_{1m_1}, G_{21}, \ldots \ldots, G_{pm_p}\right];$$

in particular, a compound game that is not a sum or product can always be given a prime quotient.

6. Weighted majority games

Often it is possible to specify the winning coalitions of a simple game by attributing "voting strengths" $w_1, w_2, \ldots, w_n$ to the players and finding a number q with the property that

$$S \in \mathcal{W} \text{ if and only if } \sum_{i \in S} w_i \geq q.$$

A game for which this can be done is known as a weighted majority game and is denoted by the symbol

$$[q; w_1, \ldots, w_n].$$

The numbers w_i are called the weights, and q is called the quota. The symbol for a particular game is never unique; in fact, one can always be found in which the quota and weights are all integers. Of course, our ordinary majority games $M_{n,k}$ are weighted majority games:

$$M_{n,k} = [k;\ 1,\ 1,\ \ldots,\ 1].$$

The dual of a weighted majority game can be obtained by replacing q by $w-q+\epsilon$, w being the sum of the w_i and ϵ a small positive number. (If the game is expressed in integers then we can take $\epsilon = 1$.) It is easy to see that every weighted majority game is either strong or proper.

A weighted majority game is called homogeneous if the weights can be assigned in such a way that all the minimal winning coalitions have the same weight. The dual of a homogeneous game is not necessarily homogeneous. In a homogeneous game, or better yet in a game that is both homogeneous and dual homogeneous, the weights of the players give a more accurate guide to the relative strengths of the players than the weights in an inhomogeneous game, where players sometimes must "waste" part of their voting strength even in a minimal winning coalition. The smallest inhomogeneous examples are the four-person improper games:

$$[3;\ 2,\ 2,\ 1,\ 1] \quad \text{and} \quad [4;\ 3,\ 2,\ 2,\ 1].$$

For an example that is proper we need five players — e.g., $[5;\ 2,\ 2,\ 2,\ 1,\ 1]$; for one that is strong and proper we need six — e.g., $[5;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1]$.

The ownership of a corporation by stockholders is a good example of a weighted majority game — w_i being the number of shares held by the i^{th} "player." The U.S. Electoral College is another example, with (currently) 51 players, corresponding to the 50 states and the District of Columbia. A somewhat more surprising example, since the voting strengths are not explicit in the rules, is the U.N. Security Council. The reader will readily verify that the following weights and quota accurately define the voting system, complete with vetoes:

$$B_5 \times M_{6,2} = [27; 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1].$$

This game is both homogeneous and dual homogeneous.

There are many simple games that do not have a weighted majority representation, since it is not possible in general to find a consistent ranking of the relative effectiveness of the different players. For example, $M_3 \times M_3$ has no weighted majority symbol. (Proof: If it did, the weights could be made equal, by symmetry. But some four–person coalitions win and some lose, so there is no number that will serve as the quota.) Indeed, large compound games rarely possess weighted majority representations. However, almost all games with fewer than five players do possess symbols. (See Table 1 in the appendix.)

7. Symmetry

The $M_{n,k}$ games are the only ones that are fully symmetric, in the sense that no group of players can be distinguished from any other group of the same size. If we merely seek indistinguishability of the individual players there are many other possibilities — for example the game

$M_3 \times M_3$ just mentioned, or the cyclic-symmetric five-person game with $\mathcal{W}^m = \{\overline{123}, \overline{234}, \overline{345}, \overline{451}, \overline{512}\}$, etc.

The seven-person game defined by

$$W^m = \{\overline{124}, \overline{235}, \overline{346}, \overline{457}, \overline{561}, \overline{672}, \overline{713}\}$$

is rather remarkable; it is symmetric in pairs of players as well as in individuals. It is the prototype of a series of simple games, based upon the finite projective plane geometries. The players correspond to "points," the minimal winning coalitions to "lines." Every two points determine a line, every two lines meet in a point, and there exists a set of four points no three of which lie on the same line. A blocking coalition is a set that meets every line but contains no line. An n-person finite projective game, as these games are called, exists for each number n of the form $1 + k + k^2$ where k is a prime or a power of a prime. (The games B_1 and M_3 are sometimes included in the series, to correspond to $k = 0$ and 1, respectively.) The seven-person example above ($k = 2$) is strong but the larger ones are not; all are proper (see reference [20]).

8. Solutions

The foregoing account has essentially dealt only with the "rules of the game," and has not touched the question of what might happen if appropriately motivated players were turned loose to operate within these rules. The raising of this question, under various sets of assumptions, has led mathematicians to develop a number of "solution" theories, inquiring into such things as the formation and stability of coalitions, the relative bargaining positions of the players, the stability of different patterns of distribution when there are "spoils" to be divided, the players' expected shares of such spoils, etc. Most of these theories are not aimed exclusively at simple games, but their formulation and application are usually greatly simplified in that context, sometimes even to the point of triviality.

The multiplicity of "solution" concepts is a reflection of the intrinsic indeterminacy of an essential n-person game, viewed in the abstract. Specific outcomes in a specific realization of the abstract game depend heavily on the exogenous conditions of play; a solution theory is useful only to the extent that these conditions fit the assumptions of the theory. In contrast to the more powerful "minimax" theory for inessential games, the n-person solution theories are never completely definitive, even at their best, but merely provide a sort of road map to the irreducible areas of uncertainty. Their value in practice is quite limited, like that of the descriptive theory we have just outlined: not to solve, but to illuminate, the issues faced by the real players.

We intend to give in a companion paper a technical survey of the n-personal solution concepts that are most relevant to the topic of simple games. The reader is also referred to the many items dealing with solutions in the appended bibliography.

9. Outside applications

The mathematical formalism of simple games, as presented in this paper, occasionally proves useful (or at least applicable) in fields outside game theory proper. For example, the connection with finite projective geometry, already mentioned, runs both ways: the game-theoretic analogy has suggested the study of blocking coalitions as purely geometrical entities, and some interesting results in that area have been discovered (see references [10] and [20]).

For an application of simple games to social choice functions, see references [1] and [2].

In an application to neural nets, the classes of adjacent neurons which by sending simultaneous or near-simultaneous stimuli can cause a given neuron to fire may be regarded as the winning coalitions in a simple game, characterizing the given neuron's response function.* The familiar "threshhold" model, corresponding to the weighted majority game, is just one example; a rich class of possible response mechanisms can be systematically explored with the aid of the set-theoretical methods that we have brought to bear on simple games. Interconnected neurons can be combined in much the same way as compound simple games are formed, although we cannot ignore the time element so freely in neural nets as we have done in our present account.

A rather similar application can be made to switching theory.

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*Inhibitory stimuli must be treated in a special way if the postulate $\mathcal{W} = \mathcal{W}^+$ is to hold.

In yet another field, the sets of people in an organization that are in a position to exercise control, either over the organization as a whole or over individual members of it, give rise to an ensemble of simple games. These in turn can be used to characterize the structure of command and responsibility within the organization, and provide a rather penetrating tool for analysis. This application to organization theory is close enough to the "game" context for some of the solution concepts mentioned above to be relevant; indeed, an n-person simple game can be regarded as a particular kind of organization, with n bosses and one employee. We intend to expand on this topic in another paper.

APPENDIX

The table on the next page is a complete list of the 28 games with 4 or fewer players. The weighted majority symbol is given where available; in the other cases the $(N, \mathcal{W})$ notation is used. The factors given are all prime, with the exception of B_k and B_k^* for $k > 1$, which decompose into $B_1 \times \cdots \times B_1$ and $B_1 + \cdots + B_1$, respectively. Dual games are listed side by side. Of the improper games, all but the last three are strong, being duals of proper games. The last game listed ("p") has the peculiar property of being isomorphic to its dual, which is the game $(\overline{1234}, \{\overline{13}, \overline{32}, \overline{24}\})$, but not equal to it.

Table 1

SIMPLE GAMES WITH FOUR OR FEWER PLAYERS

(Proper)		(Improper)
a. $[1; 1] = B_1$ (prime)	weak, strong	a^*. (same as a)
b. $[2; 1,1] = B_2$	weak	b^*. $[1; 1,1] = B_2^*$
c. $[3; 1,1,1] = B_3$	weak	c^*. $[1; 1,1,1] = B_3^*$
d. $[3; 2,1,1] = B_1 \times B_2^*$	weak	d^*. $[2; 2,1,1] = B_1 + B_2$
e. $[2; 1,1,1] = M_3$ (prime)	strong	e^*. (same as e)
f. $[4; 1,1,1,1] = B_4$	weak	f^*. $[1; 1,1,1,1] = B_4^*$
g. $[5; 2,2,1,1] = B_2 \times B_2^*$	weak	g^*. $[2; 2,2,1,1] = B_2^* + B_2$
h. $[4; 2,1,1,1] = B_1 \times M_3$	weak	h^*. $[2; 2,1,1,1] = B_1 + M_3$
i. $[3; 1,1,1,1] = M_4$ (prime)		i^*. $[2; 1,1,1,1] = M_{4,2}$ (prime)
j. $[5; 3,2,1,1] = B_1 \times (B_1 + B_2)$	weak	j^*. $[3; 3,2,1,1] = B_1 + (B_1 \times B_2^*)$
k. $[4; 2,2,1,1] = M_3[B_1, B_1, B_2]$		k^*. $[3; 2,2,1,1] = M_3[B_1, B_1, B_2^*]$
l. $[5; 3,2,2,1]$ (prime)		l^*. $[4; 3,2,2,1]$ (prime)
m. $[4; 3,1,1,1] = B_1 \times B_3^*$	weak	m^*. $[3; 3,1,1,1] = B_1 + B_3$
n. $[3; 2,1,1,1]$ (prime)	strong	n^*. (same as n)
o. $(\overline{1234}, \{\overline{12}, \overline{34}\}^+) = B_2 + B_2$		o^*. $(\overline{1234}, \{\overline{13}, \overline{14}, \overline{23}, \overline{24}\}^+) = B_2^* \times B_2^*$
p. $(\overline{1234}, \{\overline{12}, \overline{23}, \overline{34}\}^+)$ (prime)		p^*. (isomorphic to p)

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