VARIATIONAL PROBLEMS WITH STATE VARIABLE INEQUALITY CONSTRAINTS

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The RAND Corporation, Santa Monica, California

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The material in this paper is similar to that contained in the author's doctoral dissertation in Applied Mathematics submitted to the Division of Engineering and Applied Physics at Harvard University. Earlier versions of some of this material have appeared in RAND publications P-1464, P-2357, and P-2374.

This revision to P-2605 incorporates some changes in the emphasis due to results obtained in the past year. The reader is especially referred to a number of new footnotes pointing out these changes and to the more recent reference items (e.g., Refs. 3, 5, and 9).

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VARIATIONAL PROBLEMS WITH STATE VARIABLE INEQUALITY CONSTRAINTS

SYNOPSIS

Since our primary interest in this Paper is the development of analytic and computational results applicable to the optimization of missile or airplane trajectories, we shall restrict our attention to problems involving one independent variable. We shall, except in the introductory chapter, consider a problem requiring the determination of a control or decision function that, in conjunction with a set of differential equations of motion dependent upon the control, yields a maximal or minimal value of an objective function evaluated at an unspecified future time T, at which time certain specified final conditions are satisfied. This general problem is called the "Problem of Mayer" and is one of three completely equivalent formulations of any one-dimensional variational problem.

We shall derive new results concerning the characterization of the optimal solution of a variational problem in which the variables involved are restricted, by an inequality constraint, to lie only in a specified region of space. The computational aspects of this problem lead, in the concluding chapters, to a rather thorough investigation of techniques of numerical solution of unconstrained optimal trajectory problems.

In Chapter 1 we shall introduce the reader to our rather unconventional point of view, called dynamic programming by Bellman,[1] by studying the familiar "Lagrange" statement of a variational problem. This genre of problem, equivalent by variable transformation to the Mayer problem, but occupying a favored position in the historical evolution of the calculus of variations, considers a definite integral in which the integrand depends on an unspecified function and its derivative. This function is to be chosen so as to render the value of the integral either maximal or minimal. We shall reproduce the familiar Euler-Lagrange equation and other conditions necessary for optimality by means of a new derivation.

In Chapter 2 we shall turn our attention to the Problem of Mayer that will occupy us through the remaining chapters, and, using the dynamic programming viewpoint, derive the fundamental result, analogous to the Euler-Lagrange equation, called the Multiplier Rule.

Two types of inequality constraints will be considered in Chapters 3 and 4. In Chapter 3 we briefly derive a simple known result characterizing the optimal solution of a problem involving a constrained decision or control variable. Here the trajectory may traverse any region of space so long as the variable under our control--for example, the acceleration or angle of attack--remains suitably bounded.

In Chapter 4 we derive new and more complicated results for the problem where the solution is restricted to a particular region of space by an inequality constraint not explicitly involving the decision variable. We call such a restriction a state variable inequality constraint (some authors call this a phase space constraint).

We then turn to computational aspects of variational problems in an attempt to implement our results. In Chapter 6, after discussing and dismissing in Chapter 5 the conventional approach to numerical solution, we use dynamic programming to develop a relatively new and successful computational technique for solving unconstrained problems. This gradient technique can be, and has been, derived from other viewpoints and is known to a few practitioners of the numerical art, such as Kelley and Bryson, but our derivation of the relevant results is new.

Chapter 7 discusses and illustrates the application of the numerical scheme to an unconstrained problem.

Finally, in Chapter 8, we bring the above numerical technique to bear upon the analytic conditions derived in Chapter 4 for state variable inequality constrained problems. This yields a cumbersome but workable computational algorithm which we illustrate with a simple example. We consider here a problem containing one constraint, not dependent upon time. Problems with several, or time dependent, constraints are more difficult.

Chapter 1

THE PROBLEM OF LAGRANGE

1.1 INTRODUCTION

In traditional variational parlance, the problem of Lagrange involves the maximization or minimization of a definite integral defined over a specified interval of integration. A simple example of such a problem is: Choose the function $y(x)$ so as to minimize

$$J(y) = \int_{x_0}^{x_1} F\Big(x, y(x), y'(x)\Big)\, dx \ . \qquad (1.1.1)$$

The function F and the interval x_0 to x_1 are always specified explicitly. In addition, other constraints are sometimes placed upon admissible solution functions, $y(x)$. For example, the function y is usually required to be continuous with piecewise continuous first derivative. Also, its endpoints may be completely specified by $y(x_0) = y_0$ and $y(x_1) = y_1$. Or, some more complicated property of y, such as its arc length, may be stipulated. In what follows, we shall at first assume continuity and completely fixed endpoints. Later, the endpoint restriction will be weakened and other conditions will be added.

Note that $J(y)$ is a function of the function $y(x)$; such a relationship is called a functional.

As we shall see later, there are other equivalent statements of the problem which are better suited for

certain applications. For the time being, however, we shall think of a problem as being defined by the integrand F, the initial point (x_0,y_0), and the final point (x_1,y_1).

1.2 THE CHARACTERIZATION OF THE OPTIMAL RETURN FUNCTION

Let us consider a more general problem than the above. We shall assume that the final point (x_1,y_1) is specified, but that we seek the solution of the problem for a whole range of initial points. The solution of the problem for any particular initial point is then a special case of our general problem, and any conclusions that we draw for the general problem will hold for any particular case.

We shall regard y' as the decision variable under our control for the above generalized problem. For each different initial point (x_0,y_0), there will generally be a different optimal decision $y'(x_0,y_0)$ so that y' should be regarded as a function of both the x-coordinate and the y-coordinate.*

This observation leads us to the definition of two concepts that are essential to what is to follow.

1) The state variables of a problem are those quantities that must be known in order to render an optimal decision.

*This is a departure from the conventional variational approach where the unknown curve y(x) is regarded as the solution. For a given initial point, however, the decision function y'(x,y) determines the curve y(x) uniquely; in what follows, we shall always regard y'(x,y), rather than y(x), as the solution.

2) The <u>decision variables</u> are those quantities that are considered to be under our control and which are to be chosen optimally.

In the above example, the x- and y-coordinates are state variables and $y'(x,y)$ is the decision variable.

Note that the choice of a decision variable in a given state determines both the value of the integrand F and the infinitesimal transformation of state variables.

$$\begin{aligned} x &\to x + dx \\ y &\to y + y'dx \end{aligned} \tag{1.2.1}$$

Hence, associated with any initial state (x_0,y_0), we have an optimal decision $y'(x_0,y_0)$ which in turn generates a new state and a new optimal decision. Therefore, if the optimal decision function y' were known for all states, we could use it to assign to each initial state (x_0,y_0) a number which would be the value of

$$\int_{x_0}^{x_1} F(x,y,y')\,dx \tag{1.2.2}$$

where $y(x_0) = y_0$ and y' is chosen optimally. We shall call this value $S(x_0,y_0)$, and define

$S(x_0,y_0)$ = the value of the integral of F from initial state (x_0,y_0) to a specified final state

(x_1,y_1), where y' is chosen optimally.*

This definition establishes the value of $S(x_0,y_0)$ for any particular initial point (x_0,y_0) where $x_0 \leq x_1$.

When we speak of partial differentiation with respect to x_0 or y_0, we mean a change in the initial state and the associated initial optimal decision $y'(x_0,y_0)$.

The function $S(x_0,y_0)$, defined above, is the optimal return function for the problem. It is always a function of the state variables for a problem and not, explicitly, of the decision variables, since they are assumed in the definition of S to be determined optimally.

What can we say about the optimal return function $S(x_0,y_0)$ for the problem**

$$\text{minimize} \int_{x_0}^{x_1} F(x,y,y')dx, \quad y(x_0) = y_0 \ ? \tag{1.2.3}$$

To answer the question, we first of all consider a discrete version of the problem and then pass to the limiting continuous case. We make the problem discrete by

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*We choose the name S to conform with the terminology of the Hamilton-Jacobi theory of classical mechanics. Hamilton's principal function S, though derived quite differently, can be identified as our S. See Ref. 2 and Appendix A for elucidation; however, we recommend that the reader defer examination of the Appendix until he has finished this chapter.

**In the classical approach the analog of this question is: What can we say about the optimal solution y(x)? The intuitive answer is: Any nearby function $y(x) + \epsilon\, \eta(x)$ is inferior.

dividing the line segment (x_0,x_1) into parts of small finite length, dx. We choose a direction $\tilde{y}_0'$ at the point (x_0,y_0) and move in that constant direction until x_0 equals $x_0 + dx$. Then we again choose a direction to be used during the next interval of length dx and continue in this way. We replace the value of the integral over the interval x_0 to $x_0 + dx$ by its first order approximation $F(x_0,y_0,\tilde{y}_0')dx$, and similarly over each subsequent interval.

We now reason as follows: If we start at the point (x_0,y_0) and choose any initial decision $\tilde{y}_0'$ at that point, we find ourselves, after moving in the direction determined by $\tilde{y}_0'$ until x_0 equals $x_0 + dx$, at the point $(x_0 + dx, y_0 + \tilde{y}_0'dx)$. Certainly, if our path is to be optimal, we must proceed optimally from the point $(x_0 + dx, y_0 + \tilde{y}_0'dx)$ to the endpoint. Furthermore, our initial decision to move in direction $\tilde{y}_0'$ at the point (x_0,y_0) contributes a value $F(x_0,y_0,\tilde{y}_0')dx$ to the summation we are seeking to minimize. Hence, for any initial decision we have

$$S(x_0,y_0) \leq F(x_0,y_0,\tilde{y}_0')dx + S(x_0 + dx,\ y_0 + \tilde{y}_0'dx) \tag{1.2.4}$$

where the function on the left is defined as the minimal value of the discrete approximation of the definite integral of F, and the function on the right is an attainable value associated with an arbitrary initial decision $\tilde{y}_0'$ used for an x interval of length dx and an optimal decision thereafter. If we choose the initial $\tilde{y}_0'$ at the point (x_0,y_0)

so as to minimize the quantity on the right in the above equation, we have made the optimal decision and then have equality in (1.2.4). Therefore, we are able to write

$$S(x_0,y_0) = \min_{y_0'}\left[F(x_0,y_0,y_0')\,dx + S(x_0 + dx,\ y_0 + y_0'dx)\right]. \tag{1.2.5}$$

This is the local characterization of the optimal return function at the point (x_0,y_0) that is central to our method. The reasoning used to justify (1.2.5) is called by Bellman the Principle of Optimality.

In the above argument, we conceived of choosing a y_0' at (x_0,y_0) and using that decision until x_0 reached $x_0 + dx$. The limiting equation resulting from letting $dx \to 0$ in (1.2.5) is the local characterization of optimality in the continuous case. We now make two assumptions. We assume that the limiting and minimizing operations can be interchanged in equation (1.2.5).* Further we assume that the second partial derivatives of $S(x_0,y_0)$ are finite.** Using these assumptions, we can now expand $S(x_0 + dx,\ y_0 + y_0'dx)$ in Taylor series about (x_0,y_0), obtaining

---

*The determination of the precise conditions on F that allow this interchange poses a difficult and interesting question.

**An examination of this assumption shows it to be equivalent to Jacobi's necessary condition of the classical calculus of variations (see Ref. 3).

$$S(x_0,y_0) = \min_{y_0'}\left[\lim_{dx \to 0}\left(F(x_0,y_0,y_0')dx + S(x_0,y_0) + \frac{\partial S}{\partial x_0}\,dx + \frac{\partial S}{\partial y_0}\,y_0'\,dx + o\,(dx)\right)\right] \qquad (1.2.6)$$

where $z = o\,(dx)$ means the limit as $dx \to 0$ of $\frac{z}{dx}$ is zero. Cancelling $S(x_0,y_0)$, dividing by dx, and taking the limit as $dx \to 0$, we obtain

$$0 = \min_{y_0'}\left[F(x_0,y_0,y_0') + \frac{\partial S}{\partial x_0} + y_0'\,\frac{\partial S}{\partial y_0}\right]. \qquad (1.2.7)$$

If y_0' is unconstrained, the partial derivative with respect to y_0' of the right-hand side of (1.2.7) must equal 0 at the minimum, yielding the two equations

$$\frac{\partial F_0}{\partial y_0'} + \frac{\partial S}{\partial y_0} = 0 \qquad (1.2.8)$$

$$F_0 + \frac{\partial S}{\partial x_0} + y_0'\,\frac{\partial S}{\partial y_0} = 0\ . \qquad (1.2.9)$$

These equations have verbal interpretations. Equation (1.2.9) states that as we move along an optimal curve from a fixed initial point (so that y' is determined and y is dependent on x), the minimal value of the integral of F from the moving initial point to the endpoint must decrease at precisely the rate that the integral of F from the fixed initial point to the moving point increases. We shall call

this the proper or correct descent rate equation. Equation (1.2.8) says that our direction of descent must be chosen so as to render stationary the remaining contribution of the definite integral of F and will be called the optimal descent rate equation.

Equations (1.2.8) and (1.2.9) are necessary but not sufficient conditions for an optimal solution, since (1.2.8) requires only stationarity with respect to y_0', rather than minimality. This point is pursued further in Sections 4 and 5 of this chapter.

To summarize, equation (1.2.7) with proper boundary conditions is both necessary and sufficient for optimality so long as the two additional conditions (the admissibility of the interchange of limit and minimum, and the satisfaction of Jacobi's necessary condition) mentioned above are satisfied. Furthermore, the two equations (1.2.8) and (1.2.9) deduced from (1.2.7) and called the conditions of correct and optimal descent rate, are necessary conditions for optimality. In subsequent sections, we shall show how this characterization yields many of the classical results of the calculus of variations.*

*Our point of view in this section follows closely that of Bellman (Ref. 1) and of Caratheodory (Ref. 4). The dissimilarity between the above and the conventional characterization should be noted. The simplicity of the subsequent derivations derives from the fact that the classical necessary conditions of the calculus of variations are all local and our characterization of optimality is also local. The conventional comparison function characterization is global.

1.3 THE EULER-LAGRANGE EQUATION

For the Problem of Lagrange, the best known result, the Euler-Lagrange Equation, does not involve the optimal return function $S(x_0,y_0)$.* Consequently, we seek to eliminate the return function from our results. This is accomplished by first taking the total derivative of equation (1.2.8) with respect to x_0. To obtain the total derivative at the point (x_0,y_0), we regard y_0 as dependent upon x_0 through $y'(x_0,y_0)$. This differentiation yields

$$\frac{\partial^2 S}{\partial x_0 \partial y_0} + \frac{\partial^2 S}{\partial y_0^2} y_0' + \frac{d}{dx_0} \frac{\partial F_0}{\partial y_0'} = 0 \ . \qquad (1.3.1)$$

Then we take the partial derivative of (1.2.9) with respect to y_0, regarding x_0 and y_0 as independent but $y'(x_0,y_0)$ as dependent upon both x_0 and y_0, obtaining

$$\frac{\partial F_0}{\partial y_0} + \frac{\partial F_0}{\partial y_0'} \frac{\partial y_0'}{\partial y_0} + \frac{\partial^2 S}{\partial x_0 \partial y_0} + \frac{\partial^2 S}{\partial y_0^2} y_0' + \frac{\partial S}{\partial y_0} \frac{\partial y_0'}{\partial y_0} = 0 \ . \qquad (1.3.2)$$

These last two equations, combined with (1.2.8) which eliminates the $\frac{\partial y_0'}{\partial y_0}$ terms, yield the classical Euler-Lagrange

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*In Chapter 2 we shall find that the fundamental result for the statement of the problem there considered, called the multiplier rule, does involve $S(x_0,y_0)$. In the classical approach, of course, the multipliers are not written in terms of $S(x_0,y_0)$, nor are they so interpreted in the derivation. There is, however, an equivalence between Lagrange multipliers and the partial derivatives of our $S(x_0,y_0)$.

result*

$$\frac{d}{dx_0}\ \frac{\partial F_0}{\partial y_0'} = \frac{\partial F_0}{\partial y_0} \quad . \tag{1.3.3}$$

1.4 THE LEGENDRE CONDITION

The optimal descent equation (1.2.8) really assures only stationarity of descent rate, not optimality, since we derived it by setting a derivative with respect to y_0' equal to 0. Re-examination of equation (1.2.7) yields a further result for optimality, for, if y_0' is chosen such that

$$\frac{\partial F_0}{\partial y_0'} + \frac{\partial S}{\partial y_0} = 0 \ , \tag{1.4.1}$$

we have the further condition for minimality that the second partial derivative of the right-hand side of (1.2.7) with respect to y_0' must be ≥ 0. Therefore, we conclude

$$\frac{\partial^2 F_0}{\partial y_0'^2} \geq 0 \ . \tag{1.4.2}$$

---

*Since any point along a solution curve can be considered the initial point of a new variational problem, our results, though always expressed in terms of an initial point, can be interpreted as conditions holding all along an optimal solution curve.

This is the Legendre condition and guarantees, if strict inequality holds, that y_0' yields a relative minimum for the variational problem rather than a relative maximum or some other stationary point.

1.5 THE WEIERSTRASS CONDITION

Even the Legendre condition is not strong enough. In order to have an absolute minimum in (1.2.7) at y_0', for all other Y_0' we must have the inequality

$$F(x_0,y_0,y_0') + \frac{\partial S}{\partial x_0} + \frac{\partial S}{\partial y_0} y_0' \leq F(x_0,y_0,Y_0') + \frac{\partial S}{\partial x_0} + \frac{\partial S}{\partial y_0} Y_0' \qquad (1.5.1)$$

or

$$F(x_0,y_0,Y_0') - F(x_0,y_0,y_0') + (Y_0' - y_0') \frac{\partial S}{\partial y_0} \geq 0 \qquad (1.5.2)$$

which, using equation (1.2.8), yields

$$F(x_0,y_0,Y_0') - F(x_0,y_0,y_0') - (Y_0' - y_0') \frac{\partial F_0}{\partial y_0'} \geq 0 \ , \qquad (1.5.3)$$

the Weierstrass necessary condition.

1.6 GENERALIZATION

The above arguments easily generalize to include the case where several functions $y_i(x)$ are sought. Here, F is a scalar function of the independent variable x, the N functions $y_i(x)$, and the derivatives of these functions with respect to x. The function S is a function of the lower limit of integration x_0, and the values y_{i_0} at the point x_0.

Then, following the reasoning of Section 2, we obtain the equations

$$F_0 + \frac{\partial S}{\partial x_0} + \sum_{j=1}^{N} y'_{j_0} \frac{\partial S}{\partial y_{j_0}} = 0 \qquad (1.6.1)$$

$$\frac{\partial F_0}{\partial y'_{i_0}} + \frac{\partial S}{\partial y_{i_0}} = 0 \qquad i = 1,\ldots,N \; . \qquad (1.6.2)$$

Differentiation of (1.6.2) and partial differentiation of (1.6.1) yields

$$\frac{d}{dx_0} F_{0_{y'_{i_0}}} - F_{0_{y_{i_0}}} = 0 \qquad i = 1,\ldots,N \; . \qquad (1.6.3)$$

We thus obtain the set of Euler-Lagrange equations.

1.7 AN ISOPERIMETRIC PROBLEM

Let us add the restriction that

$$\int_{x_0}^{x_1} G(x,y,y')\,dx = z_0 \tag{1.7.1}$$

where z_0 is a given number. Our basic function, the value of the minimum of (1.2.3), is now a function of three variables, x_0, y_0, and z_0. Hence, we define

$$S(x_0,y_0,z_0) = \min_{\{y\}} \int_{(x_0,y_0)}^{(x_1,y_1)} F(x,y,y')\,dx \tag{1.7.2}$$

subject to the restriction (1.7.1).

Since

$$\frac{dz_0}{dx_0} = -G_0 \tag{1.7.3}$$

the analog of equation (1.2.9) is

$$F_0 + \frac{\partial S}{\partial x_0} + \frac{\partial S}{\partial y_0}\, y_0' - \frac{\partial S}{\partial z_0}\, G_0 = 0 \tag{1.7.4}$$

and of (1.2.8) is

$$\frac{\partial F_0}{\partial y_0'} + \frac{\partial S}{\partial y_0} - \frac{\partial S}{\partial z_0}\,\frac{\partial G_0}{\partial y_0'} = 0\ . \tag{1.7.5}$$

Differentiation with respect to x_0 of (1.7.5) yields

$$\frac{d}{dx_0}\ \frac{\partial F_0}{\partial y_0'} + \frac{\partial^2 S}{\partial x_0 \partial y_0} + \frac{\partial^2 S}{\partial y_0^2}\ y_0' - \frac{\partial^2 S}{\partial y_0 \partial z_0}\ G_0$$

$$- \frac{d}{dx_0}\left(\frac{\partial S}{\partial z_0}\ \frac{\partial G_0}{\partial y_0'}\right) = 0\ . \qquad (1.7.6)$$

Partial differentiation with respect to y_0 of (1.7.4) gives

$$\frac{\partial F_0}{\partial y_0} + \frac{\partial F_0}{\partial y_0'}\ \frac{\partial y_0'}{\partial y_0} + \frac{\partial^2 S}{\partial x_0 \partial y_0} + \frac{\partial S}{\partial y_0}\ \frac{\partial y_0'}{\partial y_0} + \frac{\partial^2 S}{\partial y_0^2}\ y_0'$$

$$- \left(\frac{\partial G_0}{\partial y_0} + \frac{\partial G_0}{\partial y_0'}\ \frac{\partial y_0'}{\partial y_0}\right)\ \frac{\partial S}{\partial z_0} - G_0\ \frac{\partial^2 S}{\partial z_0 \partial y_0} = 0\ . \qquad (1.7.7)$$

Partial differentiation of (1.7.4) with respect to z_0 yields

$$\frac{\partial F_0}{\partial y_0'}\ \frac{\partial y_0'}{\partial z_0} + \frac{\partial^2 S}{\partial x_0 \partial z_0} + \frac{\partial^2 S}{\partial y_0 \partial z_0}\ y_0' + \frac{\partial S}{\partial y_0}\ \frac{\partial y_0'}{\partial z_0}$$

$$- \frac{\partial^2 S}{\partial z_0^2}\ G_0 - \frac{\partial S}{\partial z_0}\ \frac{\partial G_0}{\partial y_0'}\ \frac{\partial y_0'}{\partial z_0} = 0\ . \qquad (1.7.8)$$

Equation (1.7.5) eliminates three terms of (1.7.8) and (1.7.3) replaces $-G_0$ by z_0', and we see that

$$\frac{\partial^2 S}{\partial x_0 \partial z_0} + \frac{\partial^2 S}{\partial y_0 \partial z_0}\ y_0' + \frac{\partial^2 S}{\partial z_0^2}\ z_0' = 0 \qquad (1.7.9)$$

or,

$$\frac{d}{dx_0} \frac{\partial S}{\partial z_0} = 0 \tag{1.7.10}$$

or,

$$\frac{\partial S}{\partial z_0} = \text{constant.} \tag{1.7.11}$$

Now, using (1.7.5) to eliminate the $\frac{\partial y_0'}{\partial y_0}$ terms in (1.7.7), and combining (1.7.7) and (1.7.6), we get

$$\frac{d}{dx_0} \frac{\partial F_0}{\partial y_0'} - \frac{d}{dx_0}\left(\frac{\partial S}{\partial z_0} \frac{\partial G_0}{\partial y_0'} \right) = \frac{\partial F_0}{\partial y_0} - \frac{\partial G_0}{\partial y_0} \frac{\partial S}{\partial z_0} . \tag{1.7.12}$$

If we denote $\frac{\partial S}{\partial z_0}$ by $-\lambda$ and agree to treat λ as a constant independent of y_0, we can write (1.7.12) as

$$\frac{d}{dx_0} \frac{\partial}{\partial y_0'} (F_0 + \lambda G_0) - \frac{\partial}{\partial y_0} (F_0 + \lambda G_0) = 0 . \tag{1.7.13}$$

1.8 THE LAGRANGE MULTIPLIER

In view of equation (1.7.13), we see that we may identify $\frac{\partial S}{\partial z_0}$ with the Lagrange multiplier introduced to include a constraint such as (1.7.1) in the conventional derivation. This identification shows that the Lagrange multiplier introduced in the conventional approach has a simple physical interpretation. If, for a given initial point, the value of the isoperimetric constraint z_0 is

allowed to change, the value of the multiplier is equal to the resulting rate of change of extremal value of the functional being minimized subject to the constraint. This interpretation, however, requires one additional proviso. We know that the rate of change of the optimal solution S with respect to a change in the constraint level z_0 is dependent upon y_0 and changes if the initial y_0 point changes for fixed x_0. Yet in writing (1.7.12) in the form (1.7.13), we have to agree to regard λ as independent of y_0, or (1.7.13) would disagree with the correct result, (1.7.12).

Equation (1.7.11) tells us that the multiplier λ is constant along an optimal trajectory.

1.9 NATURAL BOUNDARY CONDITIONS

Suppose that the initial y-value, y_0, is not specified in a given problem. Then the optimal initial value has the property that the change in the minimum value of the integral to the final point (x_1,y_1) caused by a change in the initial y point is zero. Otherwise, there would be a better starting point. Therefore, at the initial point

$$\frac{\partial S}{\partial y_0} = 0 \tag{1.9.1}$$

or, by (1.2.8),

$$\frac{\partial F_0}{\partial y_0'} = 0 \ . \tag{1.9.2}$$

This is the natural boundary condition associated with an unspecified initial value of y_0.

If the final value of y, y_1, is not specified, one would define $S(x_0,y_0)$ as the minimum value of the integral under consideration from the initial point (x_0,y_0) to any point on the line $x = x_1$. Then S would clearly have the value zero along the stopping line because of its definition. Consequently

$$\frac{\partial S}{\partial y_1} = 0 \tag{1.9.3}$$

and, therefore,

$$\frac{\partial F_1}{\partial y_1'} = 0 \ . \tag{1.9.4}$$

Of course, one could draw this same conclusion merely from the symmetry of initial and final point.

1.10 A TRANSVERSALITY CONDITION

Let us suppose that the y-curve sought must start somewhere on a given curve $y = h(x)$. Now the initial value is neither specified nor is it constrained to the vertical line $x = x_0$, as in the above section.

We reason as follows: At the optimal initial point, the differential change in S as the initial point varies along the specified initial curve $y = h(x)$, must be zero.

For, as in the above section, there would otherwise be a better starting point. This is equivalent to saying

$$\frac{\partial S}{\partial x_0} + h_0' \frac{\partial S}{\partial y_0} = 0 \ . \tag{1.10.1}$$

By (1.2.9) we may substitute for $\frac{\partial S}{\partial x_0}$, obtaining

$$F_0 + y_0' \frac{\partial S}{\partial y_0} - h_0' \frac{\partial S}{\partial y_0} = 0 \tag{1.10.2}$$

and, by (1.2.8),

$$F_0 + (h_0' - y_0') F_{y_0'} = 0 \ . \tag{1.10.3}$$

This condition on the initial derivatives y_0' in terms of the initial point (x_0, y_0) and the slope of $h(x)$ is a classical transversality condition.

By symmetry, the same condition would hold at a final point constrained to lie on a given curve.

1.11 ERDMANN CORNER CONDITION

Let us assume that $\frac{\partial S}{\partial x_0}$ and $\frac{\partial S}{\partial y_0}$ are continuous along an optimal curve.* This is clearly true from their definition for a well-behaved problem. A discussion of the

*There are examples such as Bushaw's problem (see p. 23 of Ref. 5, Example 1) where there exists a curve in (x_0, y_0) space across which $\frac{\partial S}{\partial x_0}$ and $\frac{\partial S}{\partial y_0}$ are discontinuous. However, no optimal curves cross this curve of discontinuity.

precise properties of the definition of the problem that could result in discontinuous partial derivatives is beyond the scope of this Paper.

We now ask under what conditions may the optimal y' be discontinuous. Examination of (1.2.8) shows that

$$F_{y_0'} \in C \tag{1.11.1}$$

across the discontinuity, and (1.2.9) now tells us that

$$F_0 - y_0' F_{y_0'} \in C \tag{1.11.2}$$

across the discontinuity. These are the Erdmann corner conditions.

1.12 SUMMARY

We have reproduced, by differentiations alone, many of the better known results of the calculus of variations for the Problem of Lagrange. We did so because these results and their conventional derivations are fairly well known and therefore furnish a good comparison of the conventional approach and our viewpoint.*

*The reader familiar with the Hamilton-Jacobi theory is now referred to Appendix A for a further example of a dynamic programming derivation of a well known result.

Most of the manipulation involved in our derivations was required to eliminate the partial derivatives of the optimal return function from the results. We will, in the next chapter, consider a formulation of the original problem that is better suited both to most physical applications and to numerical computation, and one in which the partial derivatives themselves play a fundamental role.

Chapter 2

THE PROBLEM OF MAYER

2.1 INTRODUCTION

In the previous chapter we stated a variational problem and deduced several properties of the solution. We chose a formulation called the Problem of Lagrange because that has been historically the standard variational problem statement, and the associated results are quite well known. We felt that it was advisable to introduce the reader to our rather new viewpoint by treating a rather old problem.

In what is to follow, we state a variational problem in a different format, called The Problem of Mayer. This problem, involving a terminal criterion and motion determined by a set of differential equations, is the natural one for analyzing optimal trajectory problems. Furthermore, since the results explicitly involve quantities which we recognize as partial derivatives of the optimal return function, the derivations are even simpler than in the previous case where some labor was expended eliminating the partial derivatives of S from the results.*

2.2 THE PROBLEM

We shall consider a variational problem stated in the following form: Find that function

*That is, we avoid the labor of getting from equations (1.2.8) and (1.2.9), which completely characterize a relative extremal, to equation (1.3.3), the familiar Euler-Lagrange equation.

$$z(t) \tag{2.2.1}$$

such that the set of functions

$$x_i(t) \qquad i = 1,\ldots,n \tag{2.2.2}$$

given by the differential equations

$$\dot{x}_i = \frac{dx_i}{dt} = f_i(x_1,\ldots,x_n,t,z) \quad i = 1,\ldots,n \tag{2.2.3}$$

and initial conditions

$$x_i(t_0) = x_{i_0} \quad i = 1,\ldots,n \tag{2.2.4}$$

minimize a given function

$$\Phi(x_1,\ldots,x_n,t) \tag{2.2.5}$$

evaluated at some future time T determined by the satisfaction of one or more terminal conditions,

$$\Psi_j(x_1,\ldots,x_n,t) = 0 \quad j = 1,\ldots,m < n \ . \tag{2.2.6}$$

We call the variables x_i and t in the above problem state variables; the variable z is called the decision variable.

Minimum-time problems can be considered to be a special case of the above problem where $\Phi = t$. Also, if the final time t is specified to be T_f, then $\Psi_1 = t - T_f = 0$ is the stopping condition.

We have made certain inessential specializations in the above statement of the problem. For example, a set of decision functions z_i may be sought. Such a condition would complicate the algebra, but not affect the reasoning in what is to follow.

Let us now introduce a notational simplification. We denote t by x_{n+1} and add the differential equation

$$\dot{x}_{n+1} = f_{n+1} = 1 \ . \tag{2.2.7}$$

Then t does not explicitly enter the above equations and the subscript i runs from 1 to n+1.

Any problem of the above general form may be recast as a Problem of Lagrange, and conversely. Because of the suitability of the above formulation for trajectory and related problems, and because of the physical interpretation of the multipliers that appear in the results, we choose to investigate the problem as stated above.

2.3 THE CHARACTERIZATION OF OPTIMALITY

We define

$$S(x_{1_0},\ldots,x_{n+1_0}) = \text{the value of } \Phi(x_{1_0},\ldots,x_{n+1_0})$$

at terminal conditions $\Psi_j = 0$ where we start the problem in state $(x_{1_0},\ldots,x_{n+1_0})$ and use an optimal policy.

The function S is called the optimal return function. We observe that $S(x_{1_0},\ldots,x_{n+1_0})$ satisfies the recurrence relation

$$S\left(x_{1_0},\ldots,x_{n+1_0}\right) = \min_{z_0}\left[S\left(x_{1_0} + f_{1_0}\,dt,\ldots,x_{n+1_0} + f_{n+1_0}\,dt\right)\right] \qquad (2.3.1)$$

where by f_{i_0} we mean the function f_i evaluated in terms of x_{i_0} and z_0. The above equation is justified as follows: Suppose we make any decision $\tilde{z}_0$ at the initial point P_1 at time t_1 and let it determine the trajectory until time increases by amount dt. This leads us to a new initial point P_2 and this new initial point has associated with it an optimal final value of Φ. The optimal value of Φ associated with starting at point P_1 must be less than or equal to the value of starting at the attainable point P_2. Equality occurs if we choose that $\tilde{z}_0$ at time t_1 that yields that point P_2 at time $t_1 + dt$ having the smallest attainable final value of Φ associated with it.

By expanding the right-hand side of (2.3.1) about $(x_{1_0},\ldots,x_{n+1_0})$, dividing by dt and letting dt approach

zero, we get*

$$0 = \min_{z_0} \left[\sum_{i=1}^{n+1} \frac{\partial S}{\partial x_{i_0}} \, f_{i_0} \right] . \tag{2.3.2}$$

This yields the two conditions for optimality

$$\sum_{i=1}^{n+1} \frac{\partial S}{\partial x_{i_0}} \, f_{i_0} = 0 \tag{2.3.3}$$

$$\sum_{i=1}^{n+1} \frac{\partial S}{\partial x_{i_0}} \frac{\partial f_{i_0}}{\partial z_0} = 0 . \tag{2.3.4}$$

Equation (2.3.3) states that the optimal final value of Φ should not change along the optimal path; we shall call this the proper descent rate equation. Equation (2.3.4) says that the decision variable should be chosen so as to minimize the final value of Φ; we call this the optimal descent rate condition. It should be noted that equation (2.3.3) will hold at all points of an optimal solution, whereas equation (2.3.4) applies only for points where free choice of z_0 is allowed. This observation is critical when the optimal solution of inequality constrained problems is considered.

*As in Chapter 1, we assume the existence of the second partial derivatives of S at this point. The condition for their existence is Jacobi's condition for the Mayer problem. We also again assume that the limiting and minimizing operations can be interchanged.

Since equations (2.3.3) and (2.3.4) must hold for any initial point $(x_{1_0},\ldots,x_{n+1_0})$, they must hold for all points along any optimal curve. Consequently, we suppress the subscripted "nought" that we used to refer to a particular initial point.

2.4 THE MULTIPLIER RULE

Equations (2.3.3) and (2.3.4) furnish the means of deriving a result analogous to the Euler-Lagrange equation of the Problem of Lagrange. This result is called by Bliss[6] the "multiplier rule." The rule is stated by Breakwell[7] in a form quite similar to that which we shall derive.

Examination of equation (2.3.4) indicates that knowledge of the initial values of the state variables x_i and of the partial derivatives of S would allow us to compute the optimal decision z by solving an equation.

Let us suppose that we have the above information and have computed the initial optimal decision z. The state variables then change with time by the rules (2.2.3). How do the partial derivatives vary with time along the optimal path?

That is, we would like to be able to compute $\frac{d}{dt}\frac{\partial S}{\partial x_i}$ where z is determined by (2.3.4). By the rules of differentiation, we have

$$\frac{d}{dt}\frac{\partial S}{\partial x_j} = \sum_{i=1}^{n+1} \frac{\partial^2 S}{\partial x_i \partial x_j}\frac{dx_i}{dt} = \sum_{i=1}^{n+1} \frac{\partial}{\partial x_j}\left(\frac{\partial S}{\partial x_i}\right) f_i$$

$$j = 1,\ldots,n+1 \ . \qquad (2.4.1)$$

Partial differentiation of (2.3.3) with respect to x_j yields

$$\sum_{i=1}^{n+1} \frac{\partial}{\partial x_j}\left(\frac{\partial S}{\partial x_i}\right) f_i + \sum_{i=1}^{n+1} \frac{\partial S}{\partial x_i}\left(\frac{\partial f_i}{\partial x_j} + \frac{\partial f_i}{\partial z}\frac{\partial z}{\partial x_j}\right) = 0 \ .$$

(2.4.2)

Combining these two results, with the aid of (2.3.4) which eliminates the $\frac{\partial z}{\partial x_j}$ term, we obtain the equations for the time derivatives of the partial derivatives of S along an optimal curve

$$\frac{d}{dt}\frac{\partial S}{\partial x_j} = - \sum_{i=1}^{n+1} \frac{\partial S}{\partial x_i}\frac{\partial f_i}{\partial x_j} \qquad j = 1,\ldots,n+1 \ . \qquad (2.4.3)$$

This is the multiplier rule. The $\frac{\partial S}{\partial x_j}$ are written as λ_j in the classical notation and are Lagrange multipliers introduced to incorporate the constraining differential equations (2.2.3). Now that we know how the partial derivatives of the optimal return function S, as well as the state variables, vary along an optimal curve, equation

(2.3.4) determines the optimal decision variable z(t) at each time t.

2.5 TRANSVERSALITY CONDITIONS AND OTHER RESULTS

Examination of the above results shows that we have derived n+1 simultaneous linear first order differential equations for the partial derivatives of S along the optimal path. We have also been given n+1 simultaneous nonlinear first order equations (2.2.3) for the time derivatives of the state variables. Therefore, one could expect the problem to have 2(n+1) boundary conditions.

These boundary conditions are furnished by a combination of specified conditions and transversality conditions. Recall that we are minimizing a function at a final time determined by the satisfaction of a set of stopping conditions. A change, at the endpoint, in one of the state variables has two effects: it changes the value of the functional to be minimized; and it changes the terminal time by changing the value of Ψ_i, the stopping conditions. If there is only one terminal condition $\Psi = 0$, the above reasoning finds mathematical expression in the following equations which hold at the endpoint:*

*This and the subsequent result are derived in Appendix B, to which the derivation is relegated since the argument has nothing to do with our dynamic programming viewpoint.

$$\frac{\partial S}{\partial x_j} = \frac{\partial \Phi}{\partial x_j} - \frac{\dot{\Phi}}{\dot{\Psi}} \frac{\partial \Psi}{\partial x_j} \qquad j = 1,\ldots,n+1 \ . \tag{2.5.1}$$

If m simultaneous terminal conditions $\Psi_i = 0$ are specified, the equations of correct and optimal descent are not affected, but the terminal condition (2.5.1) becomes more complicated and can be written as

$$\frac{\partial S}{\partial x_j} = \frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{m} \nu_i \frac{\partial \Psi_i}{\partial x_j} \qquad j = 1,\ldots,n+1 \ . \tag{2.5.2}$$

Note that if x_j appears in neither the objective nor the constraints, $\frac{\partial S}{\partial x_j} = 0$ at the terminal point. This corresponds to the fact that changing such a state variable at the endpoint affects neither the objective function nor the terminal time and hence does not affect the optimal return function S.

These equations are n+1 conditions on the differential equations. The remaining n+1 conditions are the initial values of the state variables, if these are given, or, at initial time t_0

$$\frac{\partial S}{\partial x_j} = 0 \tag{2.5.3}$$

if state variable x_j is not specified initially. The latter expression follows from the definition of S, as it did in Chapter 1, Section 9.

Other interesting observations follow from results (2.3.3) and (2.3.4) and the definition of S:

a) If Φ and Ψ do not depend explicitly upon time t, then $\frac{\partial S}{\partial t}$ $(= \frac{\partial S}{\partial x_{n+1}})$ equals zero at the endpoint;

b) If the final time is to be minimized and Ψ does not depend explicitly upon t, then $\frac{\partial S}{\partial t} = 1$ at the endpoint;

c) If, in either of the above cases, the governing equations $\dot{x}_i = f_i$ are not time dependent, then $\frac{\partial S}{\partial t}$ is constant along the entire solution curve. Then, by equation (2.3.3),

$$\sum_{i=1}^{n} \frac{\partial S}{\partial x_i} f_i = - \frac{\partial S}{\partial t} = \text{constant} \qquad (2.5.4)$$

constitutes a first integral of the solution.

Chapter 3

DECISION VARIABLE INEQUALITY CONSTRAINTS

3.1 INTRODUCTION

In the previous chapter we deduced a set of differential equations and boundary conditions that determines at least a locally extremal decision function $z(t)$ for the Problem of Mayer. In this brief chapter we place a further restriction on admissible decision variables z and investigate the effect upon the differential equations that determine the solution. We shall find that when the additional constraint involves the decision variable explicitly, no conceptual modification of our method of derivation is necessary. The result will be a modified multiplier rule. Equivalent results were first deduced by means of a classical derivation using a Lagrange multiplier by Valentine.[8]

3.2 THE MODIFIED MULTIPLIER RULE

Let us now assume that the decision variable $z(t)$ is to be chosen subject to the inequality constraint

$$h\left(x_1,\ldots,x_{n+1},z\right) \leq 0 \ . \tag{3.2.1}$$

Such a constraint might express an angle of attack limitation, perhaps as a function of speed and altitude, on an airplane; or, perhaps, a structurally dictated acceleration constraint on a missile.

When the optimal choice of z violates the constraint (3.2.1), we determine z by the equation given by equality in (3.2.1). Therefore, the optimal descent equation (2.3.4) does not hold. As a result, when we compute $\frac{d}{dt}\frac{\partial S}{\partial x_j}$ by the method of Chapter 2, Section 4, we cannot use equation (2.3.4) to eliminate the coefficient of $\frac{\partial z}{\partial x_j}$ in (2.4.2), and the resulting equation is

$$\frac{d}{dt}\frac{\partial S}{\partial x_j} = -\sum_{i=1}^{n+1}\frac{\partial S}{\partial x_i}\frac{\partial f_i}{\partial x_j} - \left(\sum_{i=1}^{n+1}\frac{\partial S}{\partial x_i}\frac{\partial f_i}{\partial z}\right)\frac{\partial z}{\partial x_j} . \quad (3.2.2)$$

When (3.2.1) is an equality, we can take the partial derivative of (3.2.1) with respect to x_j, obtaining

$$\frac{\partial h}{\partial x_j} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial x_j} = 0 . \quad (3.2.3)$$

Using this equation to evaluate $\frac{\partial z}{\partial x_j}$, equation (3.2.2) becomes

$$\frac{d}{dt}\frac{\partial S}{\partial x_j} = -\sum_{i=1}^{n+1}\frac{\partial S}{\partial x_i}\frac{\partial f_i}{\partial x_j} + \left(\sum_{i=1}^{n+1}\frac{\partial S}{\partial x_i}\frac{\partial f_i}{\partial z}\right)\frac{\frac{\partial h}{\partial x_j}}{\frac{\partial h}{\partial z}} . \quad (3.2.4)$$

This is the modified multiplier rule.

In conventional notation this result appears as

$$\dot{\lambda}_j = - \sum_{i=1}^{n+1} \lambda_i \frac{\partial f_i}{\partial x_j} + \mu \frac{\partial h}{\partial x_j} \qquad (3.2.5)$$

where the λ and μ are Lagrange multipliers. Our derivation confirms this result and tells us that we can give the multipliers a physical interpretation by means of the relationships

$$\lambda_i = \frac{\partial S}{\partial x_i} \qquad i = 1,\ldots,n+1 \qquad (3.2.6)$$

$$\mu = \frac{\sum_{i=1}^{n+1} \frac{\partial S}{\partial x_i} \frac{\partial f_i}{\partial z}}{\frac{\partial h}{\partial z}} = \frac{\sum_{i=1}^{n+1} \lambda_i \frac{\partial f_i}{\partial z}}{\frac{\partial h}{\partial z}} \qquad (3.2.7)$$

or

$$\dot{\lambda}_j = - \sum_{i=1}^{n+1} \lambda_i \left[\frac{\partial f_i}{\partial x_j} - \frac{\partial h}{\partial x_j} \frac{\partial f_i}{\partial z} \frac{1}{\frac{\partial h}{\partial z}} \right] \text{ on } h = 0 \ . \qquad (3.2.8)$$

Note that if the constraint involves only the decision variable and no state variable--e.g., $h = z - z_0 \leq 0$--then $\frac{\partial h}{\partial x_j} = 0$, and the multiplier rule is unchanged.

3.3 THE SIGN OF THE MULTIPLIER

We have deduced the existence of a multiplier function $\mu(t)$ such that $\mu = 0$ when the constraint (3.2.1) is a strict inequality and $\mu(t)$ is given by (3.2.7) when the constraint (3.2.1) is an equality.

The numerator of (3.2.7) is the partial derivative with respect to z of the time derivative of S. Since we are minimizing $\dot{S}$, if the constraint (3.2.1) prevents our choice of the optimal z, we have two possible situations: either (1) $\dot{S}$ would decrease as z were increased, but the constraint (3.2.1) prevents us from increasing z to take advantage of this decrease; or (2) $\dot{S}$ would decrease if z were decreased, but the constraint prevents us from decreasing z.

In case (1), $\frac{\partial \dot{S}}{\partial z} < 0$, but $\frac{\partial h}{\partial z} > 0$, so we cannot increase z without violating (3.2.1).

In case (2), $\frac{\partial \dot{S}}{\partial z} > 0$, but $\frac{\partial h}{\partial z} < 0$.

In either case, we conclude that

$$\frac{\frac{\partial \dot{S}}{\partial z}}{\frac{\partial h}{\partial z}} < 0 \qquad (3.3.1)$$

and hence that

$$\mu < 0 \qquad (3.3.2)$$

when $\mu \neq 0$.

If we were maximizing, or if the sense of inequality (3.2.1) were reversed, we would conclude that μ was greater than zero by the same argument.

Chapter 4

STATE VARIABLE INEQUALITY CONSTRAINTS

4.1 INTRODUCTION

We have seen how inequality constraints explicitly involving the decision variable may be incorporated into the standard format of Chapter 2. The time derivatives of the partial derivatives of S are merely defined differently along a boundary. Otherwise, the results are just as in the unconstrained case.

However, matters are quite different if the constraint equation does not contain the decision variable. This can be seen in several ways. For one, the device of the preceding chapter whereby $\frac{\partial z}{\partial x_j}$ was computed from the boundary equation

$$h(x_1,\ldots,x_{n+1},z) = 0 \tag{4.1.1}$$

fails if h is independent of z. From another viewpoint, on a boundary given by

$$h(x_1,\ldots,x_{n+1}) = 0 \tag{4.1.2}$$

the state variables x_i are not all independent, so we must regard one or more of the x_i as dependent upon the other state variables when we perform differentiation. This was not done in any of the preceding derivations. In the

following sections we shall consider a variational problem of the type defined in Chapter 2, but where the admissible region of state variable space is restricted by the inequality

$$h(x_1,\ldots,x_{n+1}) \leq 0 \ . \tag{4.1.3}$$

Problems containing more than one inequality constraint are more difficult and are not considered here.*

4.2 STATE VARIABLES ALONG A CONSTRAINT

Let us suppose that the solution curve of a problem contains a portion that lies along the boundary given by equality in (4.1.3).

We observe that the specification that the solution curve satisfies the equation

$$h(x_1,\ldots,x_{n+1}) = 0 \tag{4.2.1}$$

also implies that during any time interval when the constraint holds

$$\frac{d}{dt} h = 0 \tag{4.2.2}$$

*Since the writing of this Paper, the results of this chapter and of Chapter 8 have been deduced from another viewpoint and generalized (see Ref. 9).

and, furthermore,

$$\frac{d^{\ell}}{dt^{\ell}} h = 0 \qquad \ell = 2,3,\ldots . \tag{4.2.3}$$

Each of the above equations has the effect of making one additional x_i dependent upon the others until that derivative is reached where z enters through some $\dot{x}_i$. For that and all higher-order derivatives, z can be chosen so as to satisfy the equations and no new dependency among the x's is introduced. We shall assume that z enters first in the k'th derivative, where k is an integer greater than, or equal to, one.

In the next section we shall illustrate this phenomenon by two examples. Then we shall discuss the characterization of the optimal return function along a boundary specified by state variables alone.

4.3 TWO EXAMPLES

Suppose that we are considering an airplane trajectory problem in which we are programming the angle of attack, α. Our kinematic and dynamic equations are

$$\begin{aligned} \dot{x} &= v \cos \gamma \\ \dot{h} &= v \sin \gamma \\ m\dot{v} &= T(v,h) - D(v,h,\alpha) - mg \sin \gamma \\ mv\dot{\gamma} &= L(v,h,\alpha) - mg \cos \gamma \end{aligned} \tag{4.3.1}$$

where x = horizontal range

h = altitude

v = velocity

γ = inclination of the plane to the horizontal

T = thrust

D = drag

L = lift.

For flight free of any boundary constraint, we have four state variables (x, h, v, γ) in this model. The decision variable is α.

Suppose now that the constraint

$$\gamma \leq \frac{\pi}{4} \tag{4.3.2}$$

were added to the problem. Then, when the airplane had the inclination

$$\gamma = \frac{\pi}{4} \tag{4.3.3}$$

only three state variables would be independent. These state variables are independent since

$$\frac{d}{dt}\left(\gamma - \frac{\pi}{4}\right) = \dot{\gamma} = \dot{\gamma}(h, v, \alpha) = 0 \tag{4.3.4}$$

implies a value for α, but no further relationship among x, h, and v.

In this example, k (as defined in Section 2) equals one. However, were the constraint

$$h \geq 0 \tag{4.3.5}$$

introduced, then when the altitude h was at its bound, not only would h cease to be an independent state variable, but since

$$\frac{d}{dt} h = \dot{h} = \dot{h}(v,\gamma) = v \sin \gamma = 0 \tag{4.3.6}$$

is a further relationship involving only state variables, a second state variable is not independent. That is, if we know we are on the boundary

$$h = 0 \tag{4.3.7}$$

we also know that

$$\gamma = 0 \, . \tag{4.3.8}$$

Hence, at most x and v there are state variables along the boundary (4.3.7). Since

$$\begin{aligned} \frac{d}{dt} v \sin \gamma &= v \cos \gamma \, \dot{\gamma} + \dot{v} \sin \gamma \\ &= f(v,\gamma,\alpha) = 0 \end{aligned} \tag{4.3.9}$$

no further dependency is implied. Here k = 2.

In general, along a state variable boundary there are n + 1 - k independent state variables where $1 \leq k \leq n$.

4.4 CHARACTERIZATION OF THE OPTIMAL RETURN FUNCTION

As might be expected, the optimal return function for the state variable constrained problem is defined at each admissible interior point of state variable space by

$S(x_1,\ldots,x_{n+1})$ = the value of Φ at terminal conditions $\Psi_i = 0$ where we start in initial state $(x_1,\ldots,x_{n+1})$ and use an optimal policy.

Along a boundary of state variable space we define a different function $\overline{S}(x_1,\ldots,x_{n+1-k})$ of the independent state variables by

$\overline{S}(x_1,\ldots,x_{n+1-k})$ = the value of Φ at terminal conditions $\Psi_i = 0$ where we start in state $(x_1,\ldots,x_{n+1-k})$ on the boundary and use an optimal policy.

We have already investigated the properties of S. Let us now characterize $\overline{S}$.

As the solution curve moves along the boundary, we still have the correct descent rate requirement

$$\frac{d\overline{S}}{dt} = \sum_{i=1}^{n+1-k} \frac{\partial \overline{S}}{\partial x_i} f_i = 0 \ . \qquad (4.4.1)$$

Since we are on a constraint, we have no optimal descent rate equation.*

We now proceed as in Chapter 2, Section 4, where the multiplier rule for the time derivatives of the partial derivatives of S was derived.

By the rules of differentiation,

$$\frac{d}{dt}\frac{\partial \overline{S}}{\partial x_j} = \sum_{i=1}^{n+1-k} \frac{\partial^2 \overline{S}}{\partial x_i \partial x_j} f_i \qquad j = 1,\ldots,n+1-k \ . \qquad (4.4.2)$$

Partial differentiation of (4.4.1) with respect to one of the independent variables x_j gives

$$\sum_{i=1}^{n+1-k} \frac{\partial^2 \overline{S}}{\partial x_i \partial x_j} f_i + \sum_{i=1}^{n+1-k} \left\{ \frac{\partial \overline{S}}{\partial x_i} \left(\frac{\partial f_i}{\partial x_j} + \sum_{\ell=n+1-k+1}^{n+1} \frac{\partial f_i}{\partial x_\ell} \frac{\partial x_\ell}{\partial x_j} + \frac{\partial f_i}{\partial z} \frac{\partial z}{\partial x_j} \right) \right\} = 0 \qquad (4.4.3)$$

where, we repeat, $x_1,\ldots,x_{n+1-k}$ are considered independent and $x_{n+1-k+1},\ldots,x_{n+1}$ are considered dependent variables. Combination of the above two equations produces the result

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*We have assumed, as is the case for trajectory problems, that z is uniquely determined if we are to maintain a constraint. If this is not the case, z would be chosen optimally subject to the requirement that the constraining equality be maintained. This complicates the algebra but not the theory.

$$\frac{d}{dt}\frac{\partial \overline{S}}{\partial x_j} = -\sum_{i=1}^{n+1-k}\left\{\frac{\partial \overline{S}}{\partial x_i}\left(\frac{\partial f_i}{\partial x_j} + \sum_{\ell=n+1-k+1}^{n+1}\frac{\partial f_i}{\partial x_\ell}\frac{\partial x_\ell}{\partial x_j} + \frac{\partial f_i}{\partial z}\frac{\partial z}{\partial x_j}\right)\right\} \qquad j = 1,\ldots,n+1-k. \qquad (4.4.4)$$

We can evaluate the $\frac{\partial x_\ell}{\partial x_j}$ by means of the boundary equation and its first k-1 derivatives (recalling that z first enters in the k'th differentiation), and we can evaluate $\frac{\partial z}{\partial x_j}$ by using the k'th derivative of the constraint equation.

Hence, we have a rule for evaluating the partial derivatives of $\overline{S}$ along a curve lying on a boundary.

We have a further result relating $\frac{\partial \overline{S}}{\partial x_j}$ and $\frac{\partial S}{\partial x_j}$ at any point where a free solution curve first touches a boundary:

$$\frac{\partial \overline{S}}{\partial x_j} = \frac{\partial S}{\partial x_j} + \sum_{\ell=n+1-k+1}^{n+1}\frac{\partial S}{\partial x_\ell}\frac{\partial x_\ell}{\partial x_j} \qquad j = 1,\ldots,n+1-k. \qquad (4.4.5)$$

This equation simply evaluates the change in optimal Φ with respect to a change in a state variable in two equivalent ways, once with certain other state variables defined to be dependent, and once where they are merely treated that way. We would not expect $\frac{\partial \overline{S}}{\partial x_j}$ to equal $\frac{\partial S}{\partial x_j}$, since our definition of "a change in x_j holding all other variables

constant" is different for S and $\overline{S}$. It is this obvious corner condition relating partial derivatives along the free curve and along the boundary, that seems to have been overlooked previously. Of course, this relationship is not at all obvious when the partial derivatives appear merely as artificially introduced Lagrange multipliers, as they do in the classical theory.

Two additional observations are in order here. Equation (4.4.1) furnishes a corner condition, much as the correct descent equation did in Chapter 1, Section 11. Although it is possible, and quite likely, that the decision variable z will experience a finite discontinuity at the point where the optimal curve reaches a constraint, equation (4.4.1) requires that a particular combination of well defined partial derivatives and f_i's must equal zero on both sides of the discontinuity.

Also, since z enters only into the derivatives of the state variable, the state variables will be continuous, even across a finite jump in z. This presents no difficulties if k = 1, such as in the first example of the previous section. However, if k is two or greater, this represents a severe requirement on admissible curves. For example, in the second example of the previous section, where altitude greater than or equal to zero was specified, no trajectory reaching the ground with inclination γ other than zero is admissible, since we have deduced that γ equals zero along the boundary and γ cannot be discontinuous. The

upshot of this argument is the result that k-1 continuity conditions are implied at any corner where a free curve intersects a boundary.

Let us now consider a general problem with given initial and final points not on a boundary, but with an intermediate segment of the trajectory lying on a state variable boundary. We are in a position to count degrees of freedom and requirements upon them. Since the initial point is not on a constraint boundary, and the state variables are all specified, there are n unknown multiplier values. The remaining multiplier and the initial decision are determined by equations (2.3.3) and (2.3.4). At the point of intersection with the boundary there are k-1 continuity conditions, as discussed above. Also, equation (4.4.1) represents a corner condition since the rate of descent, where the multipliers are determined by (4.4.5), must be correct for the new z on the boundary after the corner. At the time at which the solution curve leaves the boundary, we have kept track of only n+1-k multipliers by equations (4.4.4) and we have no information about the missing k multipliers. Therefore, k-1 multipliers are unspecified, with equations (2.3.3) and (2.3.4) determining the remaining multiplier and z. At the stopping condition $\Psi_i = 0$, n conditions on the state variables and multipliers must be satisfied. The (n+1)st condition is automatically satisfied since it is the stopping condition.

Summarizing, corresponding to an extremal curve with specified initial point, there must exist n+k numbers (n initial multipliers, the time off the boundary, and k-1 multipliers at the time off the boundary) which yield a solution curve that satisfies n+k conditions (a corner condition where the curve first intersects the boundary, k-1 continuity conditions on state variables at the corner, and n final conditions on multipliers and state variables). Any curve satisfying these conditions and the Euler-Lagrange equations (2.4.3) off the boundary and (4.4.4) on the boundary, will be a relative extremal for the variational problem.

Chapter 5

NUMERICAL SOLUTION: SUCCESSIVE APPROXIMATION TO THE PROBLEM

5.1 INTRODUCTION

Having developed a set of necessary conditions for the solution of either unconstrained or constrained problems of the Mayer type, we turn to matters of computational solution.

In this chapter we present and dismiss as impractical one approach to numerical solution. Then in Chapters 6, 7, and 8 we develop and implement an alternative method of solution.

The methods of numerical solution discussed here are independent of the method of derivation of necessary conditions. Both the classical and dynamic programming viewpoints lead to the same numerical situation.

5.2 THE CONVENTIONAL APPROACH

Analysis of variational problems generally leads to sets of simultaneous nonlinear differential equations (the Euler-Lagrange or multiplier equations, and the kinematic equations of the problem). Almost invariably one is faced with the problem of finding a solution satisfying mixed boundary conditions--some variables (usually state variables) being known initially and others (usually a mixture of state variables and Lagrange multipliers) being specified at the final point.

The usual approach[7] involves a process of guessing the unknown initial values of the Lagrange multipliers and numerically integrating the set of nonlinear differential equations. When the final values of the variables are not as specified, the guessed initial values are adjusted with the aim of correcting the wrong final values. If all goes well, each incorrect guess contributes information about the effects of initial multipliers on the final values, and the process converges to the solution.

This problem can be thought of as one involving a successive approximation to the problem, since at each step one determines a stationary solution to an unwanted problem--a solution with the wrong final values for states and multipliers. The solutions do represent locally extremal trajectories to the final state variables that they yield. Optimistically, we suggest the name "successive approximations to the problem" for this technique, in hope that its "conventional" title is transient and will die out in time. This name also has the merit of accentuating the contrast with the approach we shall advocate below and shall call "successive approximation to the solution."

Experience, rather than mathematical analysis, seems to condemn the problem-approximation approach. Many attempts in a variety of problem areas have uncovered unsuspected numerical instabilities which lead to great sensitivity to initial value changes and have stymied convergence. Minute

changes in the initial value of guessed multipliers lead to wildly fluctuating solution curves, and successive improvement of the solution rapidly gives way to successive, but not often successful, modification of the procedure. At best, the technique must be classified as art rather than science, and one readily turns to a computational algorithm that avoids the instabilities that stem from the simultaneous integration of the physical equations and the Euler equations.

5.3 THE CONSTRAINED PROBLEM

If one considers constrained state variable problems of the type discussed in Chapter 4, one is confronted by conditions necessary for optimality specified not only at the beginning and end of the solution curve, but also at intermediate points where the solution joins or leaves a boundary.

Consequently, one is faced with the problem of guessing many multipliers that yield a curve satisfying many conditions specified at various points of the trajectory. But the manner in which these numbers must be guessed makes matters even worse. One would have to discover, experimentally first of all, the subset of initial multipliers that leads to curves that satisfy the corner conditions at the point of intersection of optimal curve and boundary. Then one would have to search this subset for that set of initial multipliers that, in conjunction with more unknowns guessed

at the point of departure from the boundary, satisfied the final conditions of the problem.

This procedure would then yield a relative extremal, if such exists, consisting of a free interior curve, a portion of the boundary, and then a free curve. If the optimal curve contained several different segments of the boundary, the process would be even more complex. This hierarchical guessing game is clearly unplayable when we are stymied by the numerical solution of even unconstrained problems.

Consequently, we discard the concept of successive approximation to the problem and turn, in the next chapter, to a technique of successive approximation to the solution.

Chapter 6

SUCCESSIVE APPROXIMATION TO THE SOLUTION

6.1 INTRODUCTION

In the previous chapter we discussed a technique which we called "successive approximation to the problem." The unknown initial conditions for a set of differential equations that determine a stationary solution to a problem are guessed and the equations integrated. The resulting curve is locally extremal to whatever points in state variable space it reaches. If it does not reach the required final point, with proper final values for the multipliers, we adjust the initial guesses and solve again. We stated that due to inherent instabilities resulting from the coupling of Euler-Lagrange equations and equations of motion, the above straightforward method of solution usually fails in practice. For complicated problems with state variable inequality constraints it is totally infeasible.

In this chapter we develop an alternative approach to numerical solution which we call the method of "successive approximation to the solution." While this method is known to a few practitioners of the numerical art,[10,11] it is not in general use at present.* Our method of

*Use of the computational method discussed in this chapter (often called a gradient technique or a method of steepest ascent), is, at the time of the 1963 revision of this Paper, rather frequent.

derivation, the embedding technique used throughout this Paper, is new, and our inclusion of state variable inequality constraints also represents an advance.

Our general procedure involves guessing a non-optimal nominal starting decision function which results in a curve which satisfies the state variable endpoint conditions of the problem.* We shall then perform an analysis upon this presumably non-optimal decision function that will lead us to a better solution. Successive application of this procedure will produce successive curves that approach at least a relative extremal for the problem.

6.2 CHARACTERIZATION OF THE NON-OPTIMAL RETURN FUNCTION

We consider first, as a simple example, the problem of minimizing a specified function

$$\Phi\ (x_1,\ldots,x_n,t) \tag{6.2.1}$$

of the state variables at the first time t_1 that a terminal condition

$$\Psi\ (x_1,\ldots,x_n,t) = 0 \tag{6.2.2}$$

*One can actually start with a nominal curve not satisfying all final conditions, as we shall see in Section 3.

is satisfied. We consider only one terminal or stopping condition; we generalize this problem in Section 3. Let $z(t)$ be the decision variable. Let the system be defined by the difference equations

$$x_i(t + \Delta) = x_i(t) + f_i(x_1,\ldots,x_n,z)\Delta$$

$$i = 1,\ldots,n \qquad (6.2.3)$$

where the initial values of the variables are given by

$$x_i(t_0) = x_{i_0} \ . \qquad (6.2.4)$$

We seek the sequence $\{z(k\Delta)\} = \{z_k\}$ such that the given objective function $\Phi(x_1(t_1),\ldots,x_n(t_1),t_1)$ of the state variables at the unspecified final time is minimum.*

We begin by guessing a decision variable as a <u>function of time but not of the other state variables</u>. Clearly such a decision variable cannot be optimal for the problem with general starting point. However, we shall ultimately adjust z so that it is locally optimal for a particular problem with specified starting point. Because we conceive of the decision sequence being specified as a function of time, time plays a special role in what follows and we

*Note that we have stated the problem discretely out of deference to the discrete nature of digital computations.

choose to write it separately rather than call it x_{n+1} as we did previously. We also have assumed, as is generally the case, that the equations of motion (6.2.3) are not time dependent.

The guessed decision sequence $\{z_k\}$ determines, by (6.2.3), a nominal trajectory from each starting point. We now define

$$T(x_{1_0},\ldots,x_{n_0},t_0) = \text{the value of } \Phi(x_1(t_1),\ldots,x_n(t_1),t_1)$$
at the time t_1 when $\Psi = 0$, where we start in state $(x_{1_0},\ldots,x_{n_0})$ at time t_0 and use a particular nominal decision sequence $\{z_k\}$.*

Note that T may depend on t_0 in three ways. A change in t_0 results in a change in: (1) the duration of the process if Ψ depends on t; (2) the final objective function if Φ depends on t; and (3) the decision z_0, since z_0 is given as a function of t. Also note that $\{z_k\}$ is a trial decision function and not, presumably, an optimal one. The function T is immediately seen to satisfy the recurrence relation

---

*We use T to distinguish the non-optimal return function from S, the optimal return function. Of course, the value of T depends on the particular sequence $\{z_k\}$, the terminal point and criterion, etc., but we suppress this in the notation.

$$T\left(x_1,\ldots,x_n,t\right) = T\left(x_1+f_1\Delta,\ldots,x_n+f_n\Delta,t+\Delta\right) \qquad (6.2.5)$$

where the f are evaluated using the guessed $\{z_k\}$ and associated trajectory.

In order to discover the first order effect of a change in the decision variable at time t, we seek to evaluate

$$\frac{\partial T}{\partial z}\Big|_t$$

where this notation means $\frac{\partial T}{\partial z}$ evaluated in terms of the state and decision variables at time t.

By partial differentiation of (6.2.5) with respect to z where the f_i are seen to depend on z, we conclude that

$$\frac{\partial T}{\partial z}\Big|_t = \left\{\sum_{i=1}^{n}\left(\frac{\partial T}{\partial x_i}\Big|_{t+\Delta}\frac{\partial f_i}{\partial z}\Big|_t\right)\right\}\Delta\ . \qquad (6.2.6)$$

To evaluate this expression we see that we need to know

$$\frac{\partial T}{\partial x_j}\Big|_{t+\Delta}\ .$$

A recurrence relation for that quantity is obtained by partial differentiation with respect to x_j of (6.2.5):

$$\frac{\partial T}{\partial x_j}\Big|_t = \left\{ \sum_{i=1}^{n} \left(\frac{\partial T}{\partial x_i}\Big|_{t+\Delta} \frac{\partial f_i}{\partial x_j}\Big|_t \right) \right\} \Delta + \frac{\partial T}{\partial x_j}\Big|_{t+\Delta} \qquad j = 1,\ldots,n \qquad (6.2.7)$$

and

$$\frac{\partial T}{\partial t}\Big|_t = \left\{ \sum_{i=1}^{n} \left(\frac{\partial T}{\partial x_i}\Big|_{t+\Delta} \frac{\partial f_i}{\partial z}\Big|_t \right) \right\} \frac{dz}{dt}\Big|_t \Delta + \frac{\partial T}{\partial t}\Big|_{t+\Delta},$$

since f_i depends on t through the fact that z is given as a function of t.

Both equations (6.2.6) and (6.2.7) have obvious verbal interpretations. Equation (6.2.6) states that the rate of change of T with respect to z at time t equals the rate at which the state of the system at time $t + \Delta$ changes as z varies, multiplied by the rate at which T changes as the state of the system changes at time $t + \Delta$. Equation (6.2.7) adds the change in T due to the change in state at time

$t + \Delta$ (due to the effect of a change in x_j on the f_i) to the direct effect of the change in $x(t)$ on $x(t + \Delta)$ to obtain the net change in T.

Equation (6.2.7) is seen to be the discrete analog of the Multiplier Rule

$$\frac{d}{dt}\frac{\partial T}{\partial x_j} = -\sum_{i=1}^{n}\frac{\partial T}{\partial x_i}\frac{\partial f_i}{\partial x_j} \qquad j = 1,\ldots,n \qquad (6.2.8)$$

derived in Chapter 2. If $\frac{\partial T}{\partial z} \equiv 0$, no improvement is possible and the nominal curve is locally optimal. This observation reduces (6.2.6) to the optimality condition derived in Chapter 2.

We now have two ordinary recurrence relations (6.2.6) and (6.2.7) that permit us to evaluate the effect of a change in z at any time upon the final objective function Φ. We determine the boundary conditions for the recurrence relations by observing that a change in a state variable at the final time t_1 has two effects, the immediate change in Φ and the change in Φ due to the change in the final time determined by $\Psi = 0$. Applying this reasoning we have*

$$\frac{\partial T}{\partial x_j}\Big|_{t_1} = \frac{\partial \Phi}{\partial x_j}\Big|_{t_1} - \frac{\dot{\Phi}}{\dot{\Psi}}\Big|_{t_1}\frac{\partial \Psi}{\partial x_j}\Big|_{t_1} \qquad j = 1,\ldots,n \qquad (6.2.9)$$

*This result has already been discussed in Chapter 2, Section 5, and derived in Appendix B.

while, for a change in time at the endpoint we have

$$\frac{\partial T}{\partial t}\Big|_{t_1} = \frac{\partial \Phi}{\partial t}\Big|_{t_1} - \frac{\dot{\Phi}}{\dot{\Psi}}\Big|_{t_1} \frac{\partial \Psi}{\partial t}\Big|_{t_1} . \tag{6.2.10}$$

We have derived, by two simple differentiations, expressions evaluating the first order effect of a decision change at any time upon the final value of Φ. These results can be thought of as "influence functions," or adjoint equations, and are usually derived from theorems concerning the representation of solutions of linear differential equations.

The manner in which these results can be used most efficiently for the successive improvement of a non-optimal solution is largely an experimental matter in the realm of numerical analysis. In the following section we shall present one seemingly very efficient way of using these results due to Bryson.[11]

6.3 THE MEANS OF IMPROVEMENT

We postulate the rule

$$z_{new}(t) = z_{old}(t) + \delta z(t) \tag{6.3.1}$$

for adjusting z and seek an expression for δz. We start by adopting the reasonable policy (there are alternatives) of changing z at each time t proportionally to the rate

at which the final Φ changes with z, that is $\frac{\partial T}{\partial z}$, which equals the quantity

$$\sum_{i=1}^{n} \frac{\partial T}{\partial x_i}\Big|_{t+\Delta} \frac{\partial f_i}{\partial z}\Big|_t \Delta$$

evaluated at time t. That is, where the potential payoff rate $\frac{\partial T}{\partial z}$ is greater, we will act more decisively. Writing

$$\delta z = K \left\{ \sum_{i=1}^{n} \frac{\partial T}{\partial x_i}\Big|_{t+\Delta} \frac{\partial f_i}{\partial z}\Big|_t \right\} \Delta , \qquad (6.3.2)$$

where K is an as yet undetermined constant of proportionality, and recognizing that (since we are considering only first order effects) the total change in Φ is the sum of the changes during each time interval,

$$\Delta\Phi = \sum_{t=0}^{t_1} \frac{\partial T}{\partial z}\Big|_t \delta z = K \sum_{t=0}^{t_1} \left\{ \sum_{i=1}^{n} \frac{\partial T}{\partial x_i}\Big|_{t+\Delta} \frac{\partial f_i}{\partial z}\Big|_t \right\}^2 \Delta^2 \qquad (6.3.3)$$

where $\Delta\Phi$ is the change in the final value of Φ due to the changes of $\delta z(t)$ in $z(t)$ at all time $0 \le t \le t_1$. The summand in the expression for $\Delta\Phi$ is easily computable along a given trajectory by means of the recurrence relations (6.2.6) and (6.2.7).

If we desire an improvement of $\overline{\Delta\Phi}$ in the value of Φ,

we choose

$$K = \frac{\overline{\Delta\Phi}}{\sum_{t=0}^{t_1} \left\{ \sum_{i=1}^{n} \frac{\partial T}{\partial x_i}\Big|_{t+\Delta} \frac{\partial f_i}{\partial z}\Big|_t \right\}^2 \Delta^2} \tag{6.3.4}$$

and use, for the next iteration, the new decision function given by

$$z_{new}(t) = z_{old}(t) + \frac{\left\{ \sum_{i=1}^{n} \frac{\partial T}{\partial x_i}\Big|_{t+\Delta} \frac{\partial f_i}{\partial z}\Big|_t \right\} \overline{\Delta\Phi}}{\sum_{t=0}^{t_1} \left\{ \sum_{i=1}^{n} \frac{\partial T}{\partial x_i}\Big|_{t+\Delta} \frac{\partial f_i}{\partial z}\Big|_t \right\}^2 \Delta} . \tag{6.3.5}$$

We would be well advised to seek only a modest improvement $\overline{\Delta\Phi}$ at each successive iteration since our analysis is first order and only accurate for small changes.

Let us now introduce some notation and recapitulate results before deriving successive approximation techniques for more complicated problems. We shall write here and throughout $\lambda_{x_i}(\Phi)$ for $\frac{\partial T}{\partial x_i}$, remembering that $\lambda_{x_i}(\Phi)$ can be interpreted as the effect of a change in x_i on the value of Φ at the terminal point. For

$$\sum_{i=1}^{n} \frac{\partial T}{\partial x_i} \frac{\partial f_i}{\partial z} \Delta$$

we write $\lambda_z(\Phi)$. In this notation the technique of successive improvement is:

1) Guess $\{z_k\}$;
2) Integrate the equations of motion (6.2.3);
3) Evaluate $\lambda_{x_i}(\Phi)$ at the final time t_1 by means of (6.2.9);
4) Determine $\lambda_{x_i}(\Phi)$ along the nominal trajectory by backwards recursion of (6.2.7) and simultaneously compute

$$\lambda_z(\Phi) \text{ and } \sum \left(\lambda_z(\Phi)\right)^2$$

from (6.2.6);
5) Determine z_{new} for a specified small $\overline{\Delta\Phi}$ by (6.3.5);
6) Return to step (2).

Suppose now that an additional relationship

$$\theta(x_1,\ldots,x_n,t) = 0 \qquad (6.3.6)$$

must be satisfied at the terminal point. The same arguments as in the preceding paragraphs allow us to compute the influence of a change in z on the final value of θ by means of the formulas

$$\lambda_{x_j}(\theta)\Big|_t = \left\{ \sum_{i=1}^{n} \lambda_{x_i}(\theta)\Big|_{t+\Delta} \frac{\partial f_i}{\partial x_j}\Big|_t \right\} \Delta + \lambda_{x_j}(\theta)\Big|_{t+\Delta}$$

$$j = 1,\ldots,n \qquad (6.3.7)$$

$$\lambda_z(\theta)\Big|_t = \left\{ \sum_{i=1}^{n} \lambda_{x_i}(\theta)\Big|_{t+\Delta} \frac{\partial f_i}{\partial z}\Big|_t \right\} \Delta \qquad (6.3.8)$$

$$\lambda_{x_j}(\theta)\Big|_{t_1} = \frac{\partial \theta}{\partial x_j}\Big|_{t_1} - \frac{\dot{\theta}}{\dot{\Psi}}\Big|_{t_1} \frac{\partial \Psi}{\partial x_j}\Big|_{t_1}$$

$$j = 1,\ldots,n\ . \qquad (6.3.9)$$

We now let δz have the form

$$\delta z = K_1\, \lambda_z(\Phi) + K_2\, \lambda_z(\theta) \qquad (6.3.10)$$

and conclude, from linearity, that

$$\Delta\Phi = \sum_{t=0}^{t_1} \lambda_z(\Phi)\ \delta z \qquad (6.3.11)$$

$$\Delta\theta = \sum_{t=0}^{t_1} \lambda_z(\theta)\ \delta z\ . \qquad (6.3.12)$$

If the nominal trajectory, due to either numerical roundoff, nonlinearity, or difficulties in finding an initial feasible trajectory, does not satisfy the auxiliary terminal condition (6.3.6), we choose $\overline{\Delta\theta}$ as minus the deviation from the desired final condition. If the nominal trajectory is feasible, $\overline{\Delta\theta}$ is taken to be zero.

We now solve the simultaneous linear equations

$$\overline{\Delta\Phi} = \left[\sum_{t=0}^{t_1} \lambda_z(\Phi)^2 \right] K_1 + \left[\sum_{t=0}^{t_1} \lambda_z(\Phi)\, \lambda_z(\theta) \right] K_2 \tag{6.3.13}$$

$$\overline{\Delta\theta} = \left[\sum_{t=0}^{t_1} \lambda_z(\Phi)\, \lambda_z(\theta) \right] K_1 + \left[\sum_{t=0}^{t_1} \lambda_z(\theta)^2 \right] K_2 \tag{6.3.14}$$

for K_1 and K_2 to be used in equation (6.3.10) to achieve an improvement $\overline{\Delta\Phi}$ in the objective function and a correction $\overline{\Delta\theta}$ in the final value of the subsidiary condition.

The above device can be used to include any reasonable number of auxiliary final conditions.

Bryson[11] has extended the above technique so that one can ask for the maximum improvement in the objective function given a specified value for

$$\sum_{t=0}^{t_1} (\delta z)^2 \tag{6.3.15}$$

and has also developed techniques that prevent one from asking for incompatible changes in the objective and auxiliary conditions. The analysis, however, is beyond the scope of this discussion.

Chapter 7

THE NUMERICAL SOLUTION OF AN UNCONSTRAINED PROBLEM

7.1 INTRODUCTION

We have now developed a technique for the numerical solution of variational problems. We illustrate this technique by studying a rocket trajectory problem for which the analytic solution is known.

7.2 A ROCKET PROBLEM

We wish to launch a rocket in fixed time to a given altitude with a given final vertical velocity component with maximum horizontal velocity component. We simplify the problem by making the following assumptions and approximations:

1) Thrust varies with mass so as to produce constant acceleration;
2) The earth is flat;
3) Gravitational acceleration is constant.

We define the following relevant quantities:

x = horizontal range
y = altitude
u = velocity component in the horizontal direction
v = velocity component in the vertical direction
a = constant acceleration due to thrust
g = constant gravitational acceleration
γ = inclination of the thrust vector to the horizontal.

Our problem takes the following mathematical form:

Given: x_0, y_0, u_0, v_0, a, g, and t_1;

Determine: $\gamma(t)$;

Such that: at a specified time t_1, u is maximum and

$$\begin{aligned} y &= y_1 \\ v &= v_1 \end{aligned} \tag{7.2.1}$$

where the motion of the rocket is governed by the differential equations

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= a \cos \gamma \\ \dot{v} &= a \sin \gamma - g \ . \end{aligned} \tag{7.2.2}$$

7.3 THE VARIATIONAL EQUATIONS

Identifying with our previous notation and results, we have:

$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= u \\ x_4 &= v \\ x_5 &= t \\ z &= \gamma \\ \Phi &= u \\ \Psi &= t-t_1 = 0 \\ \theta_1 &= y-y_1 = 0 \\ \theta_2 &= v-v_1 = 0 \ . \end{aligned} \tag{7.3.1}$$

We have a problem involving a stopping condition and two auxiliary terminal constraints. From Chapter 6, Section 2:

$$
\begin{aligned}
\lambda_x(\Phi)\,|_t &= \lambda_x(\Phi)\,|_{t+\Delta} \\
\lambda_y(\Phi)\,|_t &= \lambda_y(\Phi)\,|_{t+\Delta} \\
\lambda_u(\Phi)\,|_t &= -\lambda_x(\Phi)\,|_t\,\Delta + \lambda_u(\Phi)\,|_{t+\Delta} \\
\lambda_v(\Phi)\,|_t &= -\lambda_y(\Phi)\,|_t\,\Delta + \lambda_v(\Phi)\,|_{t+\Delta} \\
\lambda_t(\Phi)\,|_t &= \lambda_t(\Phi)\,|_{t+\Delta} \\
\lambda_\gamma(\Phi)\,|_t &= \lambda_v(\Phi)\,|_{t+\Delta}(-a \sin\gamma)\,|_t\,\Delta + \lambda_v(\Phi)\,|_{t+\Delta} \\
&\qquad (a\cos\gamma)\,|_t\,\Delta
\end{aligned}
\tag{7.3.2}
$$

and similarly for $\lambda_{x_i}(\theta_1)$ and $\lambda_{x_i}(\theta_2)$.

The terminal conditions, from equation (6.2.9), are

$$\left.\begin{aligned} &\lambda_x(\Phi)\,|_{t_1} = \lambda_y(\Phi)\,|_{t_1} = \lambda_v(\Phi)\,|_{t_1} = 0 \\ &\lambda_u(\Phi)\,|_{t_1} = 1 \\ &\lambda_t(\Phi)\,|_{t_1} = -a\cos\gamma\,|_{t_1} \end{aligned}\right\}$$

$$\left.\begin{aligned} &\lambda_x(\theta_1)\,|_{t_1} = \lambda_u(\theta_1)\,|_{t_1} = \lambda_v(\theta_1)\,|_{t_1} = 0 \\ &\lambda_y(\theta_1)\,|_{t_1} = 1 \\ &\lambda_t(\theta_1)\,|_{t_1} = -v\,|_{t_1} \end{aligned}\right\} \quad (7.3.3)$$

$$\left.\begin{aligned} &\lambda_x(\theta_2)\,|_{t_1} = \lambda_y(\theta_2)\,|_{t_1} = \lambda_u(\theta_2)\,|_{t_1} = 0 \\ &\lambda_v(\theta_2)\,|_{t_1} = 1 \\ &\lambda_t(\theta_2)\,|_{t_1} = -(a\sin\gamma - g)\,|_{t_1} \end{aligned}\right\} .$$

The simplicity of the above equations allows us to write the results as functions of time immediately, without recourse to numerical integration. We see that:

$$
\left.
\begin{aligned}
\lambda_x(\Phi)\,|_t &= 0 \\
\lambda_y(\Phi)\,|_t &= 0 \\
\lambda_u(\Phi)\,|_t &= 1 \\
\lambda_v(\Phi)\,|_t &= 0
\end{aligned}
\right\}
$$

$$
\left.
\begin{aligned}
\lambda_x(\theta_1)\,|_t &= 0 \\
\lambda_y(\theta_1)\,|_t &= 1 \\
\lambda_u(\theta_1)\,|_t &= 0 \\
\lambda_v(\theta_1)\,|_t &= t-t_1
\end{aligned}
\right\} \qquad (7.3.4)
$$

$$
\left.
\begin{aligned}
\lambda_x(\theta_2)\,|_t &= 0 \\
\lambda_y(\theta_2)\,|_t &= 0 \\
\lambda_u(\theta_2)\,|_t &= 0 \\
\lambda_v(\theta_2)\,|_t &= 1
\end{aligned}
\right\}
$$

and, finally,

$$\lambda_\gamma(\Phi)\,|_t = -a \sin \gamma\,(t)\ \Delta$$

$$\lambda_\gamma(\theta_1)\,|_t = (t+\Delta-t_1)\ a \cos \gamma\,(t)\ \Delta$$

$$\lambda_\gamma(\theta_2)\,|_t = a \cos \gamma\,(t)\ \Delta\ .$$

Letting, as in equation (6.3.10),

$$\delta\gamma(t) = K_1\lambda_\gamma(\Phi) + K_2\lambda_\gamma(\theta_1) + K_3\lambda_\gamma(\theta_2) \qquad (7.3.5)$$

where K_1, K_2, and K_3 are determined, in terms of the improvement $\overline{\Delta\Phi}$ and corrections $\overline{\Delta\theta_1}$, and $\overline{\Delta\theta_2}$, by simultaneous solution of

$$\overline{\Delta\Phi} = \left[\sum_{t=0}^{t_1} \lambda_\gamma(\Phi)^2\right] K_1 + \left[\sum_{t=0}^{t_1} \lambda_\gamma(\Phi)\ \lambda_\gamma(\theta_1)\right] K_2 + \left[\sum_{t=0}^{t_1} \lambda_\gamma(\Phi)\ \lambda_\gamma(\theta_2)\right] K_3$$

$$\overline{\Delta\theta_1} = \left[\sum_{t=0}^{t_1} \lambda_\gamma(\Phi)\ \lambda_\gamma(\theta_1)\right] K_1 + \left[\sum_{t=0}^{t_1} \lambda_\gamma(\theta_1)^2\right] K_2 + \left[\sum_{t=0}^{t_1} \lambda_\gamma(\theta_1)\ \lambda_\gamma(\theta_2)\right] K_3$$

$$\overline{\Delta\theta_2} = \left[\left[\sum_{t=0}^{t_1} \lambda_\gamma(\Phi)\ \lambda_\gamma(\theta_2)\right] K_1 + \left[\sum_{t=0}^{t_1} \lambda_\gamma(\theta_1)\lambda_\gamma(\theta_2)\right] K_2 + \left[\sum_{t=0}^{t_1} \lambda_\gamma(\theta_2)^2\right] K_3 \right. \quad (7.3.6)$$

we have a rule for the correction of a nominal $\gamma(t)$.

7.4 NUMERICAL RESULTS

Letting

$$\begin{aligned} x_0 &= 0 \text{ ft} \\ y_0 &= 0 \text{ ft} \\ u_0 &= 0 \text{ ft/sec} \\ v_0 &= 0 \text{ ft/sec} \\ t_0 &= 0 \text{ sec} \\ t_1 &= 100 \text{ sec} \\ y_1 &= 100{,}000 \text{ ft} \\ v_1 &= 0 \text{ ft/sec} \\ g &= 32 \text{ ft/sec}^2 \\ a &= 64 \text{ ft/sec}^2 \\ \Delta &= .5 \text{ sec} \end{aligned} \quad (7.4.1)$$

the above problem was solved on an IBM 7090 computer at NASA's Langley Field location.*

*We wish to thank Jerry Elliott and Bob Boland of NASA for mathematical and programming assistance on this computation.

The initially guessed thrust program led to a final altitude of 111,031 ft and vertical velocity of 1304.9 ft/sec.

The errors in terminal conditions were automatically corrected without recourse to objective function variations. Then the horizontal velocity was increased and the terminal constraints maintained.

After 29 iterations and three minutes of computing, the terminal altitude was within 1 in. of 100,000 ft, the vertical velocity within 1 in./sec of 0, and the horizontal velocity within 1 in./sec of 3507.9 ft/sec, the analytically determined optimum.

The approach of the control variable to optimal is shown in Fig. 7.1.

The remarkably accurate results were obtained by use of the following empirical rule: When the trajectory was nearly optimal, first an improvement in objective function for the specified terminal values was requested and, on alternate iterations, the terminal values, which had drifted, were corrected without reference to the objective function. The gain in objective during the first iteration always exceeded the loss during the terminal value correction phase.

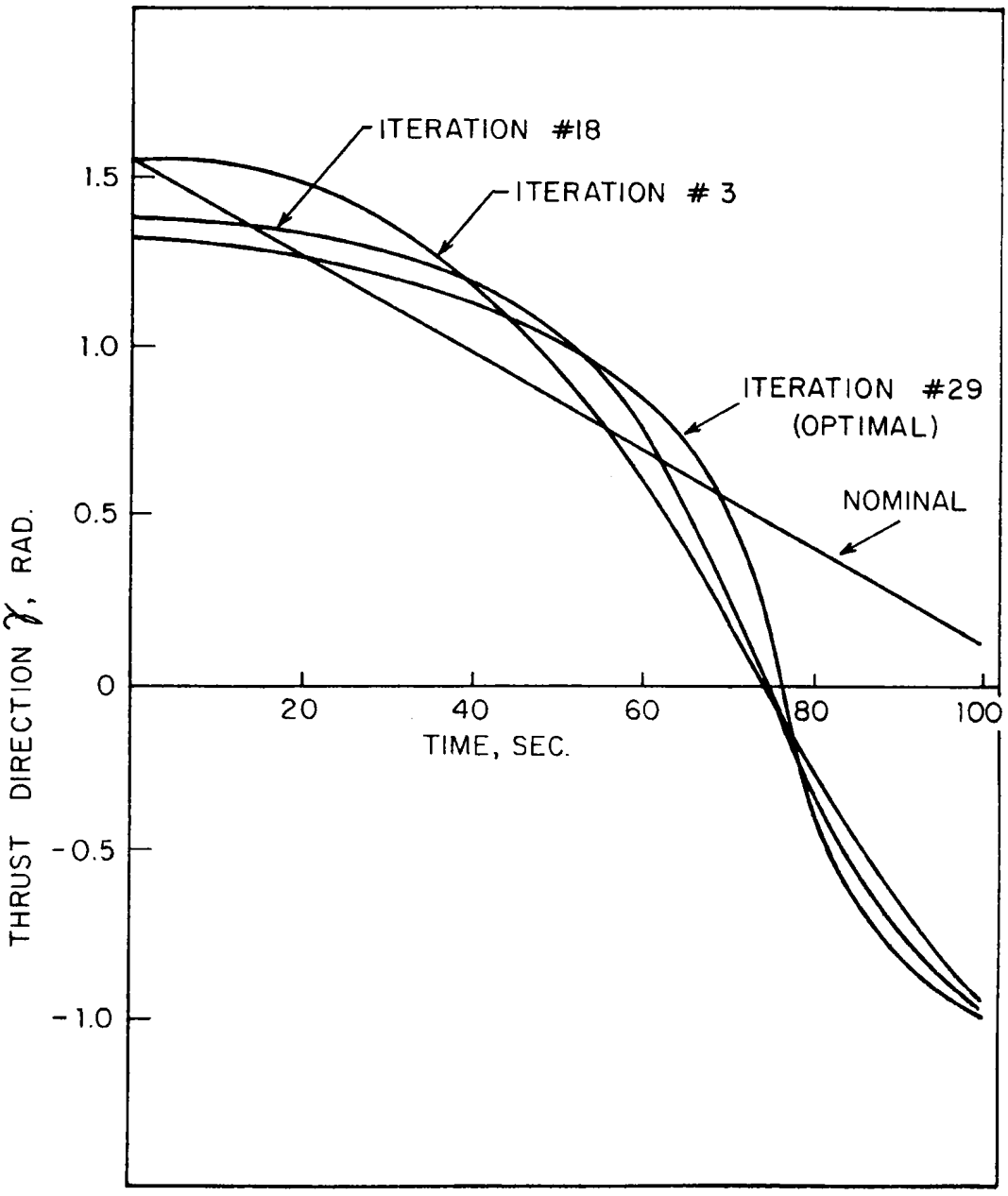

Fig. 7.1--The Rocket Thrust Direction Function

Chapter 8

THE BOUNDED BRACHISTOCHRONE PROBLEM

8.1 INTRODUCTION

The classical brachistochrone problem, the study of which gave considerable impetus to the development of the calculus of variations in the eighteenth century, is stated:

> Find that path down which a particle, under the influence of gravity alone, would slide in order to reach a final destination in minimum time.

We shall augment the traditional problem by specifying that the solution path should remain in a particular region of space.

The analytic solution to the unconstrained problem is a cycloid. It can be shown that for a problem constrained by a sloping straight line, the solution consists of: (1) the cycloid through the initial point and tangent to the line; (2) a segment of the boundary line; and (3) a cycloid through the final point and tangent to the boundary. If the final value of x, x_f is specified but the final y value is not, the second cycloid will be tangent to the boundary and perpendicular to the line $x = x_f$. It is this problem we shall consider numerically, using the analytic solution only for the evaluation of the accuracy of our numerical solution.*

*Since the original writing of this Paper, more up-to-date and more complete information on the numerical solution of state variable constrained problems has become available. The reader is referred to Ref. 9.

8.2 THE PROBLEM

Mathematically, we wish to determine the decision function

$$\gamma(t) \tag{8.2.1}$$

such that, given,

$$\begin{aligned} x(0) &= x_0 \\ y(0) &= y_0 \\ v(0) &= v_0 \\ x(t_1) &= x_f \end{aligned} \tag{8.2.2}$$

and the constraint that

$$y(t) \geq a\, x(t) + b \tag{8.2.3}$$

for all t, and the kinematic equations

$$\begin{aligned} \dot{x} &= v \cos \gamma \\ \dot{y} &= v \sin \gamma \\ \dot{v} &= -g \sin \gamma \end{aligned} \tag{8.2.4}$$

the final time, t_1, is minimum.

In our previous notation

$$
\begin{aligned}
x_1 &= x \\
x_2 &= y \\
x_3 &= v \\
x_4 &= t \\
z &= \gamma \\
f_1 &= v \cos \gamma \\
f_2 &= v \sin \gamma \\
f_3 &= -g \sin \gamma \\
\Phi &= t \\
\Psi &= x - x_f = 0 \ .
\end{aligned}
\tag{8.2.5}
$$

Graphically, we have the situation shown in Fig. 8.1,

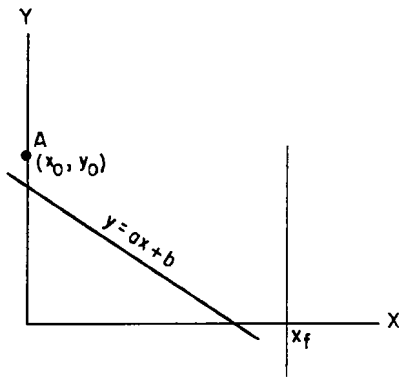

Fig. 8.1--Bounded Brachistochrone Problem

and we wish to connect A and the line $x = x_f$ with the curve of minimum descent time.

8.3 THE MULTIPLIER EQUATIONS

We guess a curve of the form ABCD, as shown in Fig. 8.2, where we specify the curve in the following way:

1) We guess a sequence $\{\gamma_k\} = \{\gamma(k\ \Delta)\}$ $k = 0,1,\ldots,$ to be used until we reach the boundary;
2) We guess a time t_c at which we will leave the boundary;
3) We guess a sequence $\{\gamma_k\}$ $k = 0,1,\ldots,$ of decisions to be used from C to the stopping line.

Note that changing the time t_c of the corner does not change our decision sequence <u>after</u> the corner and that the sequence in rule (1) above is not the same sequence as in rule (3).

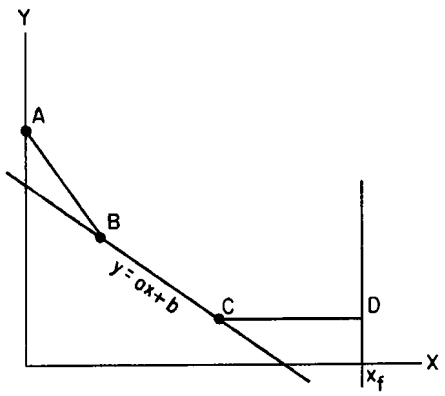

Fig. 8.2--Typical Nominal Solution

Along the segment CD we have the multiplier equations

$$
\begin{aligned}
\lambda_x(\Phi)\,|_t &= \lambda_x(\Phi)\,|_{t+\Delta} \\
\lambda_y(\Phi)\,|_t &= \lambda_y(\Phi)\,|_{t+\Delta} \\
\lambda_v(\Phi)\,|_t &= \Big(\lambda_x(\Phi)\,|_{t+\Delta}\cos\gamma\,|_t \\
&\qquad + \lambda_y(\Phi)\,|_{t+\Delta}\sin\gamma\,|_t\Big)\Delta + \lambda_v(\Phi)\,|_{t+\Delta} \\
\lambda_\gamma(\Phi)\,|_t &= \Big(-\lambda_x(\Phi)\,|_{t+\Delta} v\sin\gamma\,|_t \\
&\qquad + \lambda_y(\Phi)\,|_{t+\Delta} v\cos\gamma\,|_t - \lambda_v(\Phi)\,|_{t+\Delta} g\cos\gamma\,|_t\Big)\Delta
\end{aligned}
\tag{8.3.1}
$$

where the $\lambda(\Phi)$ are the influence functions on the time at the endpoint.

At the endpoint, we have, by (6.2.9),

$$
\begin{aligned}
\lambda_x(\Phi)\,|_{t_1} &= -\frac{1}{v\cos\gamma}\,|_{t_1} \\
\lambda_y(\Phi)\,|_{t_1} &= 0 \\
\lambda_v(\Phi)\,|_{t_1} &= 0\ .
\end{aligned}
\tag{8.3.2}
$$

These are the initial conditions for the backwards iteration of (6.2.6) and (6.2.7).

At the corner C, we have

$$\bar{\lambda}_x(\Phi) = \lambda_x(\Phi) + a\,\lambda_y(\Phi)$$

$$\bar{\lambda}_v(\Phi) = \lambda_v(\Phi) \qquad (8.3.3)$$

$$\bar{\lambda}_t(\Phi) = -\bar{\lambda}_x(\Phi)\,\dot{x}\,|_{bdy} - \bar{\lambda}_v(\Phi)\,\dot{v}\,|_{bdy}$$

where x, v, and t are taken as independent state variables along the boundary. We use the barred multiplier to represent $\frac{\partial T}{\partial x_i}$ (the partial derivative of the non-optimal return function starting on the boundary) defined and characterized in Chapter 4. The equation for $\bar{\lambda}_t(\Phi)$ requires some explanation since it is new. Along the boundary a change in time t implies a change in the location of the corner C, since it is specified in terms of time, and no other change. If t is increased by amount Δt, all other state variables and t_c being held fixed, the corner C will occur sooner. The result of such a change is a smaller x and v at the corner. The effect of this change is evaluated to obtain $\bar{\lambda}_t(\Phi)$ in (8.3.3). The symbol $\dot{x}\,|_{bdy}$ means $\dot{x}$ evaluated in terms of the γ on the boundary, not the γ on the free curve just past the boundary. Later we shall use the expression $\dot{x}\,|_{free}$ with the obvious interpretation. The multiplier $\lambda_t(\Phi)$ is discontinuous across C because a

"change in t" implies a different effect on either side of C. On CD it implies a change in decision choice, while on BC it results in a different corner followed by an unchanged decision sequence along segment CD. We do not compute $\lambda_t(\Phi)$ on CD since it is of no use.

Along BC, by equation (4.4.4),

$$\begin{aligned}
\bar{\lambda}_x(\Phi)\,|_t &= \bar{\lambda}_x(\Phi)\,|_{t+\Delta} \\
\bar{\lambda}_v(\Phi)\,|_t &= \left(\bar{\lambda}_x(\Phi)\,|_{t+\Delta} \cos \gamma_t\right)\Delta \\
&\quad + \bar{\lambda}_v(\Phi)\,|_{t+\Delta} \qquad (8.3.4) \\
\bar{\lambda}_t(\Phi)\,|_t &= \bar{\lambda}_t(\Phi)\,|_{t+\Delta} .
\end{aligned}$$

At B, by (4.4.5),

$$\begin{aligned}
\bar{\lambda}_x(\Phi) &= \lambda_x(\Phi) + a\,\lambda_y(\Phi) \\
\bar{\lambda}_v(\Phi) &= \lambda_v(\Phi) \\
\bar{\lambda}_t(\Phi) &= \lambda_t(\Phi) \qquad (8.3.5) \\
\lambda_x(\Phi)\,\dot{x}\,|_{free} &+ \lambda_y(\Phi)\,\dot{y}\,|_{free} \\
&+ \lambda_v(\Phi)\,\dot{v}\,|_{free} + \lambda_t(\Phi) = 0 .
\end{aligned}$$

The last equation is a statement that relation (4.4.1) must hold, and allows us to recover $\lambda_y(\Phi)$ to initiate the

backwards solution along AB.

Along AB we have the same difference equations, (8.3.1), as along CD.

This gives us all the equations necessary to compute $\lambda_\gamma(\Phi)$ along AB and CD and $\overline{\lambda}_t(\Phi)$ at C (this is the effect of changing the location of the corner that we earlier noted was needed). Now γ is varied according to the rules of Chapter 6 and C is varied according to the sign and magnitude of $\overline{\lambda}_t(\Phi)$.

When, after several iterations, $\lambda_\gamma(\Phi)$ is identically zero and $\overline{\lambda}_t(\Phi)$ is one, a relative extremal has been found. The reader can verify that when this occurs, all the necessary conditions of Chapter 4 are satisfied.

8.4 NUMERICAL SOLUTION

Before presenting numerical results, let us discuss some computational aspects of this problem.

As the optimal solution is approached, the improvement $\overline{\Delta\Phi}$ that can be sought naturally must be reduced. If too large a $\overline{\Delta\Phi}$ is specified, the resulting trajectory will overshoot the optimal one and may be worse, not better, than the "improved" old one. Consequently, it is useful to store the old decision sequence $\{z_{old}\}$ in the computer memory, even after $\{z_{new}\}$ has been computed. Then, if the resulting new trajectory is inferior, $\overline{\Delta\Phi}$ is reduced to, say, $\frac{\overline{\Delta\Phi}}{2}$ and a new $\{z_{new}\}$ is computed from $\{z_{old}\}$.

Convergence can be considered attained when the $\overline{\Delta\Phi}$ asked for, and not achieved due to overshoot, is smaller than some prespecified ϵ.

It is possible that a longer decision sequence will be required for some segment of a new trajectory than was for the old. If this occurs, for k greater than the old final k value, there is no z_k available to be modified. Some provision must be made for this case. Letting $z_{k+1} = z_k$ (i.e., fixing z at its old final value) works satisfactorily.

Note should be made of the last equation of set (8.3.5). If the free curve is tangent to the boundary, this equation is automatically satisfied and contributes nothing to the determination of the missing multiplier $\lambda_y(\Phi)$. However, $\lambda_y(\Phi)$ can be determined by considering the limit as $\gamma_{free} \rightarrow \gamma_{bdy}$ of the equation set. This yields the result

$$\lambda_y(\Phi) = \frac{(\overline{\lambda}_y(\Phi)\ g \cos\gamma + \overline{\lambda}_x(\Phi)\ v \sin\gamma)\cos\gamma}{v} \qquad (8.4.1)$$

for the segment AB, a result verified by the numerical computation. Care must be exercised to assure that (8.3.5) applies for non-tangent initial trajectories, but that (8.4.1) is used as the corrected curve approaches tangency.

All of these possible pitfalls are mentioned to forewarn the reader that while unconstrained problems can now be considered routine, constrained problems still raise

many problems and must be handled with care.

The bounded brachistochrone problem was programmed for the IBM 7090 computer using the following initial and final conditions

$$\begin{aligned} x_0 &= 0 \\ y_0 &= 6 \\ v_0 &= 1 \\ x_f &= 6 \end{aligned} \tag{8.4.2}$$

and the boundary

$$y \geq -\frac{1}{2} x + 5 \ . \tag{8.4.3}$$

The initial guessed decision sequence was

$$\gamma = -.785 \text{ rad} \tag{8.4.4}$$

until the boundary is reached,

$$t_c = .7 \text{ sec} \tag{8.4.5}$$

when the boundary is left, and

$$\gamma = 0 \text{ rad} \tag{8.4.6}$$

to the end-line x = 6.

The time off the boundary was modified only after the procedure had converged to the optimal solution for the given corner; i.e., $\lambda_\gamma(\Phi) \equiv 0$. Then the computation was allowed to converge again for the new corner which was then modified, etc., until both the curve and corner were optimal.

The new C-corner position, after two corners had been tried and corresponding $\bar{\lambda}_t(\Phi)$ evaluated, was found automatically by the computer, without human intervention, by linear extrapolation which determined the corner position such that $\bar{\lambda}_t(\Phi)$ would have its optimal value, 1. A corner value of 1 implies stationarity since it means that changing the time by Δt on the boundary changes the final time by Δt, with no further modification of final time due to the change in the corner position.

In Figure 8.3 at the end of the chapter we show the initial non-optimal solution. Descent required .7766 sec.

After 15 iterations, the curve was almost optimal for the given corner C. This curve is shown in solid line in Fig. 8.4. Descent time was .7445 sec, and $\bar{\lambda}_t(\Phi)$ equaled .963.

The corner was then modified on the basis of $\bar{\lambda}_t(\Phi)$ to occur at t = .594 sec. The dotted line in Fig. 8.4 shows the optimal curve for this corner position, attained on the 31st iteration, with descent time of .7422 sec and $\bar{\lambda}_t(\Phi) = .993$.

After a total of 50 iterations and four corner modifications, requiring 10 min of computer time, the computed optimal curve looked like the solid line in Fig. 8.5 with descent time of .7420 sec, and with $\overline{\lambda}_t(\Phi) = .999$.

Finally, Fig. 8.6 shows the $\lambda_\gamma(\Phi)$ functions corresponding to the initial curve and the computed optimal curve. Recall that $\lambda_\gamma(\Phi)$ is only defined off the boundary, and that $\lambda_\gamma(\Phi) \equiv 0$ implies optimality, for a specified corner C.

The true optimal solution obtained analytically is shown by a dotted line in Fig. 8.5. The convergence of the numerical solution was hampered by the extreme flatness of the optimal time curve in the neighborhood of its minimum, as can be verified by comparing the time for the solid-line curve shown in Fig. 8.4 to the time attained 35 iterations later, an improvement of just .3 per cent. Difficulties were also introduced by the tangency of the optimal curve. Since, within reason, the shape of the descent curve makes little difference in the time of descent, very accurate integration and a small integration step size ($\Delta t = .001$ sec) were required. While the solution curve is not quite as accurate as one might hope, the descent time is essentially minimal.

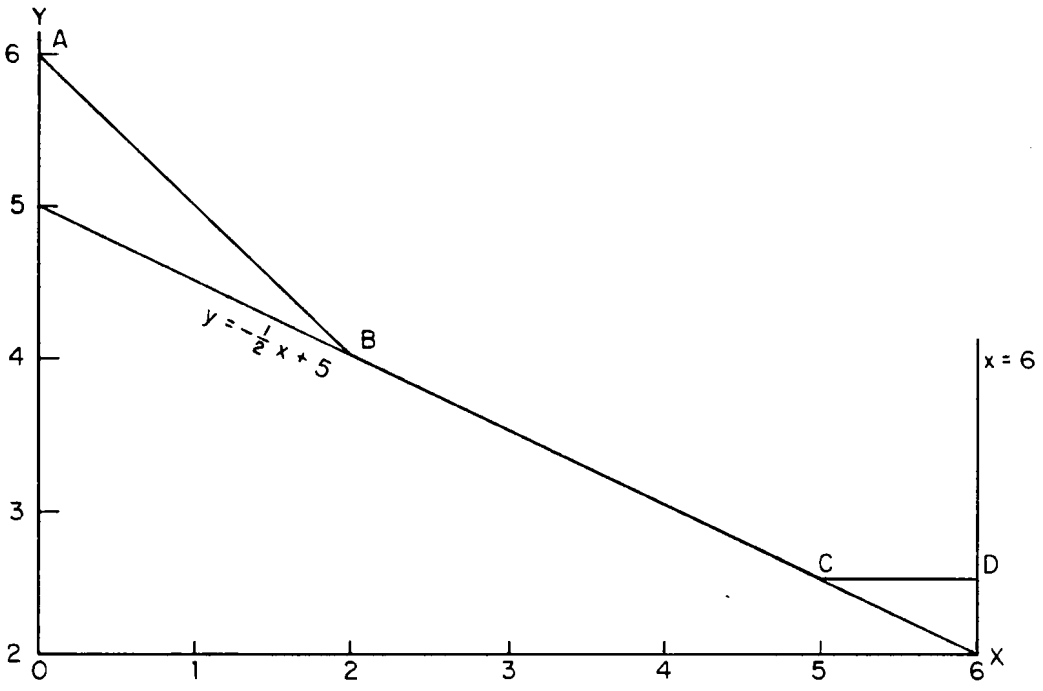

Fig. 8.3--Actual Nominal Solution

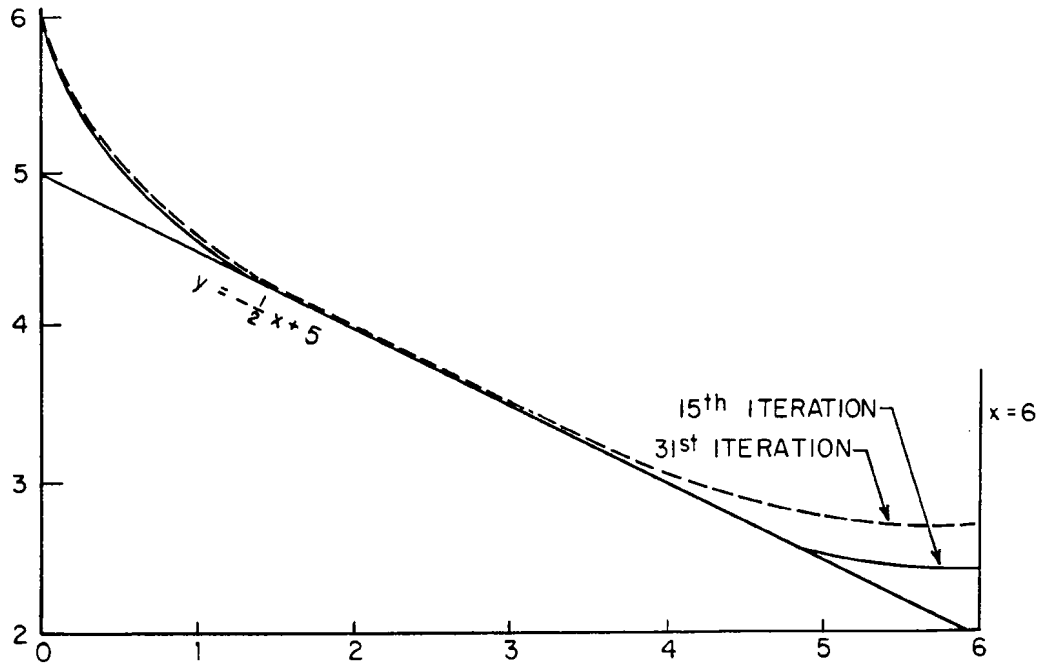

Fig. 8.4--Some Non-Optimal Paths

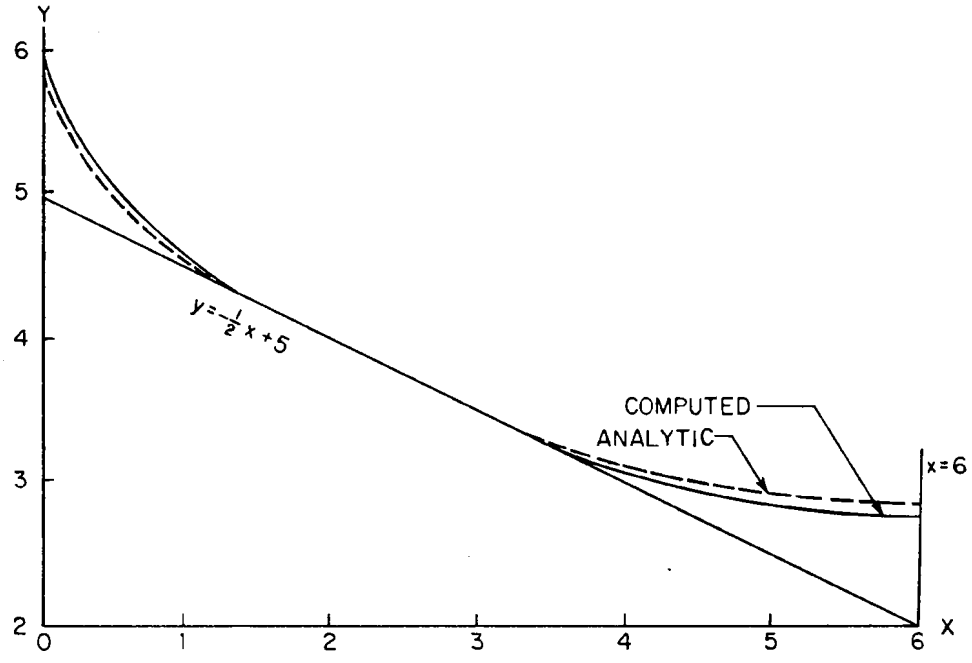

Fig. 8.5--The Optimal Solution

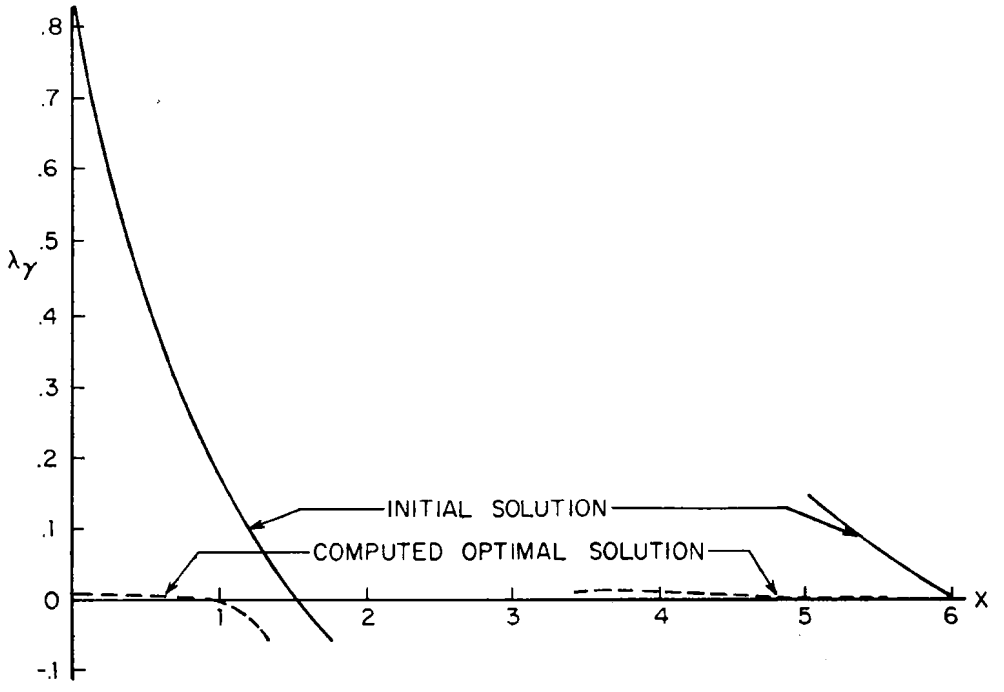

Fig. 8.6--Impulse Response Function, λ_Y

Appendix A

THE HAMILTON-JACOBI EQUATION

We shall show in this appendix that the ideas developed in Chapter 1 lead immediately to the Hamilton-Jacobi partial differential equation of mathematical physics. The usual method of derivation deduces this equation from the Euler equation quite laboriously by means of a sequence of transformations.

It is well known to physicists using classical mechanics that all information about the motion of a particle is embodied in the Hamilton-Jacobi equation. It is not known, apparently, that this equation follows readily from first principles if the embedding technique of Chapter 1 is used.

Let q, an N-component vector, describe the state of a system (i.e., q is a point in configuration space, in the terminology of classical mechanics). We make use of Hamilton's principle that a particle moves so as to minimize the integral of the Lagrangian function $L(q, \dot{q}, \tau)$ between any two points of its trajectory. The derivatives $\dot{q} = \frac{dq}{d\tau}$ will be viewed as the decision variables to be chosen optimally.

Given an initial state Q at time t_0, we wish to choose that trajectory determined by $\dot{q}(\tau)$ that transforms the state Q into a new state q at the time t and minimizes

$$\int_{t_0}^{t} L(q, \dot{q}, \tau)\, d\tau \ . \qquad (A.1)$$

There will be a different trajectory associated with each different final point and time. Later we shall see that if the initial decision $\dot{Q}$ is given, the optimality conditions determine q thereafter as a function of time.

We define

$S(q,t;\ Q,t_0)$ = the value of $\int_{t_0}^{t} L(q,\dot{q},\tau)\, d\tau$ where $\dot{q}(t)$ has been chosen so as to transform state Q at time t_0 into state q at time t while minimizing $\int_{t_0}^{t} L\, d\tau$.

The correct and optimal descent relationships (1.6.1) and (1.6.2) imply, at time t_0, the two equations

$$L\ (Q,\dot{Q},t_0) + \frac{\partial S}{\partial t_0} + \sum_{i=1}^{N} \frac{\partial S}{\partial Q_i}\, \dot{Q}_i = 0 \qquad (A.2)$$

$$\frac{\partial L}{\partial \dot{Q}_i} + \frac{\partial S}{\partial Q_i} = 0 \qquad i = 1,\ldots,N \ . \qquad (A.3)$$

These are the conditions that the trajectory leave the initial point optimally.

The further condition that the trajectory arrive at the final point (q,t) with correct and optimal descent rate implies, by the principle of optimality,

$$S(q,t;Q,t_0) = \min_{\dot{q}} \left[S(q-\dot{q}\Delta t,\ t-\Delta t;\ Q,\ t_0) + L(q-\dot{q}\Delta t,\ \dot{q},\ t-\Delta t)\Delta t \right] \tag{A.4}$$

which gives,

$$L(q,\dot{q},t) - \frac{\partial S}{\partial t} - \sum_{i=1}^{N} \frac{\partial S}{\partial q_i}\dot{q}_i = 0 \tag{A.5}$$

$$\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial S}{\partial q_i} = 0 \qquad i = 1,\ldots,N\ . \tag{A.6}$$

Defining $\frac{\partial L}{\partial \dot{q}_i}$ as momenta p_i equation (A.6) implies

$$p_i = \frac{\partial S}{\partial q_i} \qquad i = 1,\ldots,N \tag{A.7}$$

and equation (A.3) becomes

$$P_i = -\frac{\partial S}{\partial Q_i} \qquad i = 1,\ldots,N \tag{A.8}$$

Defining the Hamiltonian function $H(q,p,t)$ as

$$\sum_{i=1}^{N} p_i\dot{q}_i - L\ ,$$

(A.5) becomes the Hamilton-Jacobi partial differential equation

$$H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0 \ . \qquad (A.9)$$

The general solution of the above equation contains N+1 constants of integration. One constant is additive and is determined from the fact that

$$S(Q, t_0; Q, t_0) = 0 \qquad (A.10)$$

by definition. The remaining N constants can be determined in terms of the initial configuration Q and momenta P by evaluating equation (A.7) at the initial time t_0 when $q = Q$ and $p = P$. Knowing the constants of integration in terms of Q and P, equation (A.8) is used to determine q as a function of time and the initial conditions. Then equation (A.7) yields p as a function of time and the initial conditions. Hence, solution of the Hamilton-Jacobi equation (A.9) together with relations (A.7) and (A.8) determines the position and momenta as functions of time and of the initial configuration.

Appendix B

TERMINAL CONDITIONS FOR MULTIPLIERS

In this appendix we derive analytic relationships that must be satisfied by the terminal values of the partial derivatives of the optimal return function.

We consider first that case of an objective function

$$\Phi(x_1,\ldots,x_{n+1}) \tag{B.1}$$

with one terminal condition

$$\Psi(x_1,\ldots,x_{n+1}) = 0 \ . \tag{B.2}$$

We denote independent variations in a quantity x by δx, and dependent changes in x resulting from these independent changes by dx. Then, varying x_j independently and considering the effect of this variation on the values of Φ, Ψ, and the final time, we have

$$d\Phi = \frac{\partial \Phi}{\partial x_j}\,\delta x_j + \dot{\Phi}\,dt \tag{B.3}$$

$$d\Psi = \frac{\partial \Psi}{\partial x_j}\,\delta x_j + \dot{\Psi}\,dt \tag{B.4}$$

where dt is the change in the terminal time. Using the fact that $d\Psi$ equals zero if $\Psi = 0$ is the stopping condition

both before and after the variation, we have

$$dt = -\frac{1}{\dot{\Psi}} \frac{\partial \Psi}{\partial x_j} \delta x_j \tag{B.5}$$

and, therefore,

$$d\Phi = \left(\frac{\partial \Phi}{\partial x_j} - \frac{\dot{\Phi}}{\dot{\Psi}} \frac{\partial \Psi}{\partial x_j} \right) \delta x_j \tag{B.6}$$

or, since $\frac{d\Phi}{\delta x_j}$ is the change in the final value of Φ due to a change in x_j and a resultant change in endpoint,

$$\frac{\partial S}{\partial x_j} = \frac{\partial \Phi}{\partial x_j} - \frac{\dot{\Phi}}{\dot{\Psi}} \frac{\partial \Psi}{\partial x_j} \tag{B.7}$$

at the endpoint.

Let us now consider the general problem with objective function

$$\Phi(x_1, \ldots, x_{n+1}) \tag{B.8}$$

and m terminal constraints

$$\begin{aligned} &\Psi_1(x_1, \ldots, x_{n+1}) = 0 \\ &\vdots \\ &\Psi_m(x_1, \ldots, x_{n+1}) = 0 \ . \end{aligned} \tag{B.9}$$

If we vary the values of $x_1,\ldots,x_{n+1}$ at the terminal point of a trajectory in such a way that the constraints (B.9) remain satisfied, we see that the change in the value of the optimal solution, since we are at the terminal point, is merely the change in Φ. That is,

$$dS = \sum_{j=1}^{n+1} \frac{\partial S}{\partial x_j} \delta x_j = \sum_{j=1}^{n+1} \frac{\partial \Phi}{\partial x_j} \delta x_j \qquad \text{(B.10)}$$

where, since (B.9) must remain satisfied, we have the equations relating the δx_j

$$\sum_{j=1}^{n+1} \frac{\partial \Psi_i}{\partial x_j} \delta x_j = 0 \qquad i = 1,\ldots,m \ . \qquad \text{(B.11)}$$

The satisfaction of (B.10) subject to constraints (B.11) is equivalent to

$$\sum_{j=1}^{n+1} \frac{\partial S}{\partial x_j} \delta x_j = \sum_{j=1}^{n+1} \frac{\partial \Phi}{\partial x_j} \delta x_j - \sum_{i=1}^{m} \nu_i \sum_{j=1}^{n+1} \frac{\partial \Psi_i}{\partial x_j} \delta x_j \qquad \text{(B.12)}$$

where the δx_j can now be varied independently and the ν_i are suitably chosen constant Lagrange multipliers. Equation (B.12) can be written

$$\sum_{j=1}^{n+1} \frac{\partial S}{\partial x_j} \delta x_j = \sum_{j=1}^{n+1} \left(\frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{m} \nu_i \frac{\partial \Psi_i}{\partial x_j} \right) \delta x_j \qquad \text{(B.13)}$$

and, since the δx_j are independent, this implies the n+1 equations

$$\frac{\partial S}{\partial x_j} = \frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{m} \nu_i \frac{\partial \Psi_i}{\partial x_j} \qquad j = 1,\ldots,n+1 \qquad \text{(B.14)}$$

asserted in the text, Chapter 2, Section 5.

REFERENCES

1. Bellman, Richard, Dynamic Programming, Princeton University Press, Princeton, New Jersey, 1957.
2. Goldstein, Herbert, Classical Mechanics, Addison-Wesley, Reading, Massachusetts, 1950, p. 276.
3. Dreyfus, Stuart E., Dynamic Programming and the Jacobi Condition of the Calculus of Variations, The RAND Corporation, RM-3714-PR, August 1963.
4. Caratheodory, C., Variationrechnung and Partielle Differential Gleichungen Erster Ordnung, Teubner, Leipzig, 1935.
5. Pontryagin, L. S., et al., The Mathematical Theory of Optimal Processes, K. N. Trirogoff, Trans., Interscience Publishers, New York, 1962.
6. Bliss, Gilbert A., Lectures on the Calculus of Variations, University of Chicago Press, Chicago, 1946.
7. Breakwell, John V., "The Optimization of Trajectories," J. Soc. Indust. Appl. Math., Vol. 7, No. 2, June 1959, pp. 215-247.
8. Valentine, F. A., "The Problem of Lagrange with Differential Inequalities as Added Side Conditions," Contributions to the Calculus of Variations, 1933-1937, University of Chicago Press, Chicago, 1937, pp. 407-448.
9. Denham, Walter F., Steepest-Ascent Solution of Optimal Programming Problems, Raytheon Company, Space and Information Systems Division, BR-2393, Bedford, Massachusetts, April 2, 1963.
10. Kelley, Henry J., "Gradient Theory of Optimal Flight Paths," ARS Journal, Vol. 30, No. 10, October 1960, pp. 947-954.
11. Bryson, A. E., and W. Denham, "A Steepest Ascent Method for Solving Optimum Programming Problems," J. Appl. Mech., Vol. 29, No. 2, June 1962, pp. 247-257.