ON CONCAVE BODIES IN FREE MOLECULE FLOW

R. Schamberg

August 1967
ON CONCAVE BODIES IN FREE MOLECULE FLOW

R. Schamberg

The RAND Corporation, Santa Monica, California

I. INTRODUCTION

Until about 1960, calculation of aerodynamic forces and heat transfer for free molecule flow had been limited to bodies having convex surfaces facing the flow. For such bodies (like spheres, cones, cylinders, etc.) molecules intercepted and then reemitted by its surface cannot collide with the same body a second time, and by the definition of "free molecule flow" subsequent intermolecular collisions occur sufficiently far from the body as not to affect it at all. This simple situation does not apply to concave surfaces, such as the "inside" of the spherical or cylindrical sectors shown in Fig. 1, where a free stream molecule may collide several times with the surface before finally escaping out of the cavity. This more complicated problem was first solved by M. T. Chahine, (2) using integral equations to represent the number flux, and momentum and energy transfer to the surface of the gas molecules which make many collisions with the concave surface. A somewhat more elaborate analysis, confined however to cylindrical concave surfaces, was subsequently made by E. M. Sparrow, et al. (1-3) Sparrow also corrected an error in Chahine's equation for the tangential momentum exchange as did M. J. Pratt. (4,5) (Chahine's error was, unfortunately, perpetuated in Ref. 1; however, the corrected results are given herein.) Pratt also carried out the numerical computation for the force coefficients of spherical concave surfaces which had not been done by Chahine.

*This survey paper adds to and supersedes Ref. 1. The material contained in Sections III to VI hereof were presented at the Seminar on Environment Induced Orbital Dynamics, Marshall Space Flight Center, Huntsville, Alabama, June 6, 1967.

**Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the view of The RAND Corporation or the official opinion or policy of any of its governmental or private research sponsors.
(A) INFINITE CYLINDRICAL SEGMENT
\(0 < \omega \leq 90^\circ\)

(B) SPHERICAL SEGMENT
\(0 < \omega \leq 90^\circ\)

Fig. 1—Notation for concave and convex surfaces
This paper first summarizes some of the above cited work in order to then present numerical results in an alternate form which is more conducive to (1) a physical understanding of the flow, and (2) generalization to other geometries or surface interactions which have not yet been analyzed in detail. Specifically, data on drag of concave surfaces will be represented in the formulation of Ref. 6. In Sections III, IV, and VI only "unshielded" concave surfaces are considered, for which the entire surface is exposed to the incident free molecule flow. The effect on free molecule drag of shielding a portion of the concave surface from the incident flow is illustrated in Section V.

From a practical point of view, free molecule flow over concave bodies is of interest for possible applications including parachute-like drogue devices, high altitude air collectors or induction systems for propulsion systems, pressure probes, etc.

II. BASIC ASSUMPTIONS AND SCOPE OF ANALYSIS OF REFERENCES 2-5

Both Chahine and Pratt analyzed two basic types of concave geometries: infinite (two-dimensional) circularly cylindrical segments, and spherical segments. As shown in Fig. 1, the half angle subtended by either type of concave surface is denoted by $\psi (0 < \psi \leq 90^\circ)$. They analyzed these surfaces for the range of angles of attack $\gamma$, between the axis of symmetry and the flight direction, that results in an unshielded situation (i.e., $\gamma \leq 90^\circ - \psi$). For simplicity's sake, we consider here only the effects of concavity for zero angle of attack.*

All of the cited analyses(2-5) are based on the following simplifying assumptions:

1. The flow is hyperthermal with the uniform incident velocity of molecules $V_1$ (which is equal to the flight speed $U$).
2. The surface interaction is perfectly diffuse ($\nu \to \infty$)** with thermal accommodations defined by the accommodation coefficient $\alpha$:

---

* Except for Section V, where the results of Sparrow et al.(3) for cylindrical surfaces at several angles of attack will be used to illustrate the effects of shielding.

** Parenthetical notation defines the surface interaction parameters for the interaction model of Ref. 6.
a. Reemitted particles are distributed angularly according to the cosine law \( \varphi_0 = \frac{\pi}{2} \).

b. Reemitted particles have a Maxwellian velocity distribution.

III. NUMBER FLUX ON CONCAVE SURFACES

The number flux, \( N(S) \), incident per unit area per unit time at a particular point \( S \) of the convex surface is represented by an integral equation of the Fredholm type:

\[
N(S) = N_1(S) + \int_{\Sigma} N(S_1) K(S,S_1) \, dS_1;
\]  

(1)

here \( N_1(S) \) represents the flux due to free stream molecules, i.e.,

\[
N_1(S) = n_{\perp} V_{\perp} \cos \theta,
\]  

(2)

where \( n_{\perp} \) is the number density of molecules in the undisturbed atmosphere and \( \theta \) is defined by Fig. 1. The integral term in Equation (1) represents the flux of molecules impinging on the surface element \( S \) after having been emitted from all other surface elements \( (S_1) \) of the surface \( \Sigma \). The kernel \( K(S,S_1) \) accounts for both the angular distribution of reemitted molecules and the geometrical relationships between surface elements \( S \) and \( S_1 \). The assumed cosine distribution for the reemitted molecules (assumption 2a above) results in an algebraic form \( K(S,S_1) \) which is sufficiently symmetrical as to permit solution of Equation (1) in closed form.

For application to the spherical and cylindrical surfaces it is convenient to rewrite Equation (1) in normalized form

\[
\frac{N(\theta)}{N_o} = \cos \theta + \frac{\Delta N}{N_o},
\]  

(3)

where \( N_0 = n_{\perp} V_{\perp} \) is the direct flux incident on an area perpendicular to the free stream and \( \Delta N \) is the "effect of convexity," i.e., the contribution of molecules experiencing their second or higher collision with

*Parenthetical notation defines the surface interaction parameters for the interaction model of Ref. 6.
the surface. Specifically, the expressions obtained by Chahine for \( \frac{\Delta N}{N_0} \) for the convex cylindrical and spherical surfaces (at zero angle of attack) are:

\[
\frac{\Delta N}{N_0} = \frac{1}{4} (1 - \cos \omega) + \frac{1}{4} (1 - \cos \theta), \quad \theta \leq \omega \quad (4)
\]

\[
\frac{\Delta N}{N_0} = \frac{1}{2} (1 - \cos \omega) \quad (5)
\]

Equations (4) and (5) are graphed in Fig. 2. Note that for the spherical surface the number flux, \( \Delta N \), of multiple-colliding molecules is the same for all surface elements (all values of \( \theta \))--which is not the case for the cylindrical surface. Furthermore, \( \Delta N \) for the sphere turns out to be just twice \( \Delta N \) for an element of a cylindrical surface normal to the flow (\( \theta = 0 \)). Thus, as might be expected, a doubly curved concave surface is more effective in producing multiple collisions than a singly curved surface such as the infinite cylinder.

Chahine also indicates that if Equation (1) is solved by a method of successive approximations--which accounts separately for molecules which have had one, two, three, etc., prior collisions with the surface--Equation (5) takes the form of this equivalent infinite geometric series:

\[
\frac{\Delta N}{N_0} = \frac{1}{4} \sin^2 \omega \sum_{j=1}^{\infty} \left( \frac{1 - \cos \omega}{2} \right)^{j-1} \quad \text{(Sphere).} \quad (6)
\]

Each term of this series,

\[
\frac{\Delta N_j}{N_0} = \frac{1}{4} \sin^2 \omega \left( \frac{1 - \cos \omega}{2} \right)^{j-1} \quad (7)
\]

represents the number of molecules which are still "trapped" by the surface after \( j \) collisions with the surface. Hence the (constant) fraction of molecules which "escape" from the concave surface after each collision is
Fig. 2—Reflected number-flux for concave cylindrical and spherical segments (diffuse reemission)
\[ \mu_e = 1 - \frac{1}{2} (1 - \cos \omega) = \frac{1}{2} (1 + \cos \omega). \]  \hspace{1cm} (8)

Thus for a concave hemisphere (\( \omega = 90^\circ \)) half of the molecules escape after any one collision, whereas for a spherical cavity which subtends a half angle of 60° this fraction is 3/4.

According to Equation (3), the total incident flux is obtained by adding that of the multiple-colliding molecules, \( \Delta N \), to that of the incident free stream molecules, \( N_\parallel \). The latter flux is, of course, the same (for a given value of \( \theta \)) for concave and convex surfaces. Figure 3 shows the variation with position angle \( \theta \) of \( N_\parallel \), \( \Delta N \), and \( N \) for a concave hemisphere and semicylinder (\( \omega = 90^\circ \)). Evidently, the process of multiple reflections reduces the variation with \( \theta \) of the number flux. For both convex and concave shapes the maximum total number flux occurs at the "zenith" (\( \theta = 0 \)).

Perhaps the most significant result contained in Figs. 2 and 3 is that in cavities of the type analyzed (\( \omega \leq 90^\circ \)) the total number flux is at most 50 percent greater than that experienced by a flat plate normal to the hyperthermal free molecule flow.

**IV. DRAG OF UNSHIELDED CONCAVE BODIES**

In Ref. 2 it is shown that, for a rather general form of surface interaction model, the drag coefficient of convex bodies in hyperthermal free molecule flow can be expressed in the general form

\[ C_D = 2 \left[ 1 + \frac{\beta}{(\phi_0)} \frac{V_r}{V_\parallel} \cdot f(\phi) \right]. \]  \hspace{1cm} (9)

Here \( \beta / (\phi_0) \) is a pure number which depends on the shape of the beam of molecules reemitted from the surface. For the diffuse cosine distribution assumed by Chahine (1) (assumption 2a, p. 5), \( \beta \) is equal to 2/3. The ratio \( V_r/V_\parallel \) of reemitted to incident free stream molecular speeds can be expressed in terms of the flight speed U, the wall temperature \( T_w \) and the thermal accommodation coefficient \( \alpha \) by the equation
Fig. 3—Number flux distribution over concave semicylinder and hemisphere 
($\omega = 90^\circ$)
\[ \frac{V_r}{V_i} = \sqrt{1 + \alpha \left( \frac{3 \frac{k}{m} \frac{T_w}{U^2} - 1}{m U^2} \right)} \]

where \( k \) and \( m \) are the Boltzmann constant and the molecular mass, respectively. In Equation (9) the factor \( f(\psi) \) represents the influence on the "reemission drag" of the geometry of the body and of the degree of specularity (or diffuseness) of the interaction.

Using the method of Section IV of Ref. 6 one can readily show that, for diffuse reemission \((\psi \to \infty)\), \( f(\psi) \) of convex cylindrical and spherical segments is given by

\[ f = \frac{\omega + \frac{1}{2} \sin 2\omega}{2 \sin \omega}, \text{ (convex cylinder)} \]

\[ f = \frac{2}{3} \cdot \frac{1 - \cos^2 \omega}{1 - \cos^2 \omega}, \text{ (convex sphere)} \]

where \( \omega \) is the half angle subtended by the segment (Fig. 1).

Expressions for the drag coefficients of concave cylindrical and spherical segments are given by Chahine (2) and Pratt (4,5) and for cylindrical segments (with or without shielding) also by Sparrow et al. (3). These expressions are more complicated functions of the half-angle \( \omega \) than (11) and (12) above. (In fact the expression for the concave spherical segment contains a quadruple definite integral which has been evaluated numerically by Pratt.) Nevertheless, for the case of perfect accommodation, \( \alpha = 1 \), these expressions for \( C_D \) are linear in the parameter \( \sqrt{\frac{kT_w}{m U^2} - \frac{V_r}{V_i}} \). Hence the form of Equation (9) is also valid for concave cylindrical and spherical segments.

Figure 4 compares \( f(\psi) \) for concave cylindrical segments as calculated by Pratt with the corresponding values for the convex cylindrical

*See Figs. 3 and 5 of Ref. 6 for values of \( f(\psi) \), which lie between +1 and -1.

---

\( \sqrt{\frac{kT_w}{m U^2} - \frac{V_r}{V_i}} \)
Fig. 4—The function $f(\nu)$ for cylindrical and spherical segments (diffuse reemission $\nu \rightarrow \infty$, $\alpha = 1$)
and spherical segments from Equations (11) and (12). Figure 4 indicates that the reemission drag (and hence also the total drag) of convex bodies is less than that for a flat plate normal to the flow \((\nu = 0)\), for which \(f = 1.0\). The lower part of Fig. 5 shows schematically this "curvature relief" effect on the axial momentum of the reemitted molecules. As might be expected, this effect is more pronounced for the doubly curved spherical surfaces than for the singly curved cylinders, and for the larger values of the body half angle \(\nu\).

For concave bodies, however, the reemission drag exceeds that of the normal flat plate, and increases with the body half angle \(\nu\). The effect is slightly greater for concave spherical surfaces than for cylindrical ones. Evidently the effect on reemission drag of second and higher collisions of molecules with the concave surface more than compensates for the curvature relief effect. As shown schematically in the upper sketch of Fig. 5, second and higher collisions increase the upstream-component of momentum of reemitted molecules and, thus the drag.

Table 1 shows a comparison of estimated values of hyperthermal drag coefficients of concave and convex cylinders and spheres. For the cases shown the accommodation coefficient is assumed to be 0.5, which may be appropriate for bodies, at earth-satellite speeds, having clear, smooth metal surfaces. As for the calculation of the drag coefficients given in Table 3 of Ref. 6, Equations (9) and (10) are used, and the latter is simplified to

\[
\frac{V_r}{V_i} = \sqrt{1 - \alpha}.
\]

Specifically, Table 1 corresponds to line II of Table 3 of Ref. 6, and extends that data to concave bodies with diffuse reemission.*

---

*This calculation incorporates an approximation, since in calculating \(C_D\) from Equation (9), the values of \(f(\nu)\) used were those given in Fig. 4 for \(\alpha = 1.0\). Thus it is assumed implicitly that the effect of interreflections on reemission drag varies in the same manner with accommodation coefficient as does the reemission drag of convex surfaces. For cylindrical surfaces more precise numerical data is available in Ref. 3, as a function of the accommodation coefficient, but for a specific value of \((3k T_w/mU^2) = 0.075\).
LEGEND:
A₁, B₁ ~ FIRST COLLISION
A₂, B₂ ~ SECOND COLLISION

COSINE NUMBER
FLUX DISTRIBUTION

AVERAGE MOMENTUM OF
MOLECULES WHICH HAVE
ANOTHER COLLISION

AVERAGE MOMENTUM OF
MOLECULES WHICH ESCAPE

INCIDENT STREAM
V₁

CONCAVE SURFACE

CONVEX SURFACE

Fig. 5—Schematic representation of molecular reflections
Table 1

ESTIMATED HYPERTHERMAL DRAG COEFFICIENTS FOR CONCAVE AND CONVEX BODIES

Diffuse reemission ($\psi = \infty$); $\alpha = 0.50$

<table>
<thead>
<tr>
<th>Shape</th>
<th>$f(\psi)$</th>
<th>$C_D^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat plate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal to flow ($\psi = 0$)</td>
<td>1.00</td>
<td>2.94</td>
</tr>
<tr>
<td>Convex cylinder ($\psi = 90^\circ$)</td>
<td>0.785</td>
<td>2.75</td>
</tr>
<tr>
<td>Concave semicylinder ($\psi = 90^\circ$)</td>
<td>1.033</td>
<td>2.98</td>
</tr>
<tr>
<td>Convex sphere ($\psi = 90^\circ$)</td>
<td>0.667</td>
<td>2.63</td>
</tr>
<tr>
<td>Concave hemisphere ($\psi = 90^\circ$)</td>
<td>1.06</td>
<td>3.00</td>
</tr>
</tbody>
</table>

$C_D^* = 2\left[1 + \frac{2}{3}\sqrt{1 - \alpha \cdot f(\psi)}\right]$
Table 1 shows that drag coefficients for the concave satellites differ very little for that of the normal flat plate, i.e., by 2 percent or less. This difference is apt to be masked by uncertainties in the accommodation coefficient ($\alpha$).

For "rough" or porous satellite surfaces, thermal accommodation is expected to be complete ($\alpha \rightarrow 1$), which results in $C_D = 2$ for all shapes, irrespective of curvature or concavity.

V. DRAG OF A SHIELDED CONCAVE BODY

Sparrow et al$^{(3)}$ also considered "shielded" cylindrical surfaces of the type illustrated in Fig. 6. A portion of such an interior surface, i.e., the arc $\overline{FB}$, is not impinged on by the incident hyperthermal flow.* The incident flow is at an angle of attack $\gamma$ such that $\gamma > 90^\circ - \omega$. As Sparrow gives their numerical results for axial and side forces on the cylinder in a form which obscures the physical effects, the present discussion is intended to clarify what contribution to the total drag is made by the shielded portion of the surface.

Table 2 shows the results of the drag analysis of the illustrative surface of Fig. 6, which is a concave semicylinder ($\omega = 90^\circ$) at an angle of attack of $45^\circ$. The reemission drag contributed by the presence of the shielding arc $\overline{BF}$ can be calculated as the difference between (1) the reemission drag of the entire arc $\overline{AEF}$ (for which $\omega = 90^\circ$, $\gamma = 45^\circ$) and (2) the reemission drag of the "exposed arc" $\overline{AEF}$ (for which $\omega' = 45^\circ$, $\gamma' = 0$). It is convenient to base all drag coefficients on the area $\overline{AEF}$ of the chord plane of the exposed arc $\overline{AEF}$.

As shown by the upper part of Table 2, interreflections due to the shielded surface can contribute as much to reemission drag as interreflections from the exposed part of the concave surface. However, the dominant contribution to the reemission drag is that associated with the first collision of the molecules with the exposed surface. The lower part of Table 2 shows that the total drag coefficient of the shielded surface $\overline{AEF}$ differs from that of the unshielded one $\overline{AEF}$, or from that

*The additional drag due to the impingement of the flow on the upstream portion of the exterior surface of the arc $\overline{BF}$ is not considered here.
\[ \gamma \geq (90^\circ - \omega) \]

**Fig. 6—Shielded concave cylinder**
Table 2

DRAG OF SHIELDED CONCAVE CYLINDER
\((\omega = 90^\circ, \gamma = 45^\circ)\)

Distribution of reemission drag:

<table>
<thead>
<tr>
<th></th>
<th>(\Delta F)</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>First reemission</td>
<td>.910</td>
<td>83.1</td>
</tr>
</tbody>
</table>

Interreflections from:

<table>
<thead>
<tr>
<th></th>
<th>(\Delta F)</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exposed surface</td>
<td>.093</td>
<td>8.5</td>
</tr>
<tr>
<td>Shielded surface</td>
<td>.091</td>
<td>8.4</td>
</tr>
<tr>
<td>(f = 1.094)</td>
<td></td>
<td>100.0</td>
</tr>
</tbody>
</table>

(For flat plate \(\overline{AF}\): \(f = 1.000\))

\(C_D\) Based on area \(\overline{AF}\):

\((\gamma = 45^\circ, \alpha = 0.50)\)

- 3.03 Shielded, \(\overline{AEFB}\)
- 2.95 No shield, \(\overline{AE}\)
- 2.94 Flat plate, \(\overline{AF}\)
of the normalizing flat plate $\overline{AF}$ by less than 3 percent. This is as expected in view of the result of Table 1.

VI. DRAG OF CONCAVE BODIES IN NON-HYPERTHERMAL FLOWS

Throughout the preceding discussion it was assumed that incident flow was hyperthermal, i.e., that its thermal motion could be neglected. As will be briefly indicated here, it appears possible to extend at least some of the results of the present paper (or of Ref. 3) to free molecule flows of lower free stream speeds, for which the effects of the thermal motion is not negligible. Fortunately, the algebraic complexity inherent in precise integration over all molecular speeds (having the continuous Maxwell distribution) can be avoided by the following device, first used by Joule in 1851 for his kinetic derivation of the perfect gas law. The ambient (Maxwellian) gas is approximated by a "Joule gas" which has only 6 discrete values for the velocity vector of its molecules, i.e., all molecules are constrained to move in either direction parallel to three cartesian axes, with a fixed (mean) thermal speed $c$. If the flight speed $U$ of the body is added vectorially to each of the six molecular velocities, one obtains six different "equivalent hyperthermal flows," having different incident speeds $V_i = U \pm c, \sqrt{U^2 + c^2}$, and different directions relative to the true flight speed vector $\overline{U}$, as shown schematically in Fig. 7. By definition, in free molecule flow the various molecular species cannot interact, so that the momentum transferred by the six separate hyperthermal flows can be added to represent the forces due to the finite-speed-ratio free molecule flow.

Figure 8 shows, for illustration, the rather remarkable agreement of such a "Joule-gas" calculation of the drag of a convex sphere with the "exact" solution for the Maxwell gas, even for flight to thermal speed ratios as low as one. Similarly, it would be possible to carry out a Joule-gas computation for concave surfaces by superposing 6 appropriate solutions (for different $\gamma = \Theta_i$) of Sparrow for cylindrical surfaces, or of Pratt for spherical concave surfaces.
\[
\tan \delta = \frac{c}{U} \quad \sin \theta'_i = \frac{\sin \theta_i}{\sqrt{1 + (c/U)^2}}
\]

<table>
<thead>
<tr>
<th>Thermal component of joule gas ((c_j))</th>
<th>Effective value of:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Incident beam speed ((V_i)_j)</td>
</tr>
<tr>
<td>(c_1)</td>
<td>(U + c)</td>
</tr>
<tr>
<td>(c_2)</td>
<td>(U - c)</td>
</tr>
<tr>
<td>(c_3)</td>
<td>(\sqrt{U^2 + c^2})</td>
</tr>
<tr>
<td>(c_4)</td>
<td>(\sqrt{U^2 + c^2})</td>
</tr>
<tr>
<td>(c_5)</td>
<td>(\sqrt{U^2 + c^2})</td>
</tr>
<tr>
<td>(c_6)</td>
<td>(\sqrt{U^2 + c^2})</td>
</tr>
</tbody>
</table>

Fig. 7—Geometry for flat plate in joule gas
Fig. 8—Drag coefficients for convex sphere
VII. HEAT TRANSFER

The convective heat transfer $Q$ to a body is given by

$$Q = \int_S \left( q_i - q_r \right) \, dS,$$  \hspace{1cm} \text{(13)}

where $S$ is the portion of the surface exposed to the free molecule flow, and $q_i$ and $q_r$ are the local values of the incident and reemitted energy flux, respectively.

For a convex surface in a hyperthermal flow the energy incident upon a surface element inclined at an angle $\theta_i$ to the flow is

$$q_i = \left( \rho_i V_i \sin \theta_i \right) \cdot \left( \frac{1}{2} V_i^2 \right)$$  \hspace{1cm} \text{(14)}

If molecules are reemitted from the surface with an average speed $V_r$, which depends on the surface temperature $T_w$ and accommodation coefficient $\alpha$ (see Equation (10)), the reemitted energy flux is

$$q_r = \left( \rho_i V_i \sin \theta_i \right) \cdot \left( \frac{1}{2} V_r^2 \right)$$  \hspace{1cm} \text{(15)}

Substitution of Equations (14) and (15) into (13), together with the assumption that $V_r$ is constant over the surface, yields

$$Q = \frac{1}{2} \rho_i V_i^3 \left( 1 - \frac{V_r^2}{V_i^2} \right) \int_S \sin \theta_i \, dS$$  \hspace{1cm} \text{(16)}

The integral in this expression is simply the cross-sectional area $A$ of the body normal to the incident stream, and $V_r/V_i$ is given by Equation (10), so that (16) is equivalent to

$$Q = \frac{1}{2} \rho_i V_i^3 A \cdot \alpha \left( 1 - \frac{T_w}{T_i} \right),$$  \hspace{1cm} \text{convex body} \hspace{1cm} \text{(17)}$$
where, for convenience, the "kinetic temperature" $T_i$ of the incident flow is used to denote

$$T_i = \frac{2}{3} \cdot \frac{1}{k} \left( \frac{1}{2} m V_i^2 \right) = \frac{m}{3k} V_i^2$$  \hspace{1cm} (18)

By analogy with the drag coefficient, it is convenient to define a heat transfer coefficient $C_Q$ by

$$C_Q = {\frac{Q}{\frac{1}{2} \rho_i V_i^3 A}}$$  \hspace{1cm} (19)

Since the denominator represents the total kinetic energy in that portion of the hyperthermal stream which is intercepted by the body, $C_Q$ represents the fraction of this energy which is transferred to the body. From Equations (17) and (19) we obtain for all convex bodies in hyperthermal free molecule flow

$$C_{Q_{\text{convex}}} = \alpha \left( 1 - \frac{T_w}{T_i} \right)$$  \hspace{1cm} (20)

For concave surfaces the effects of multiple collisions between gas molecule and surface must be considered. This was done by Chahine\(^{(2)}\) for (unshielded) concave cylindrical and spherical segments; and by Sparrow et al.\(^{(3)}\) for shielded or unshielded cylindrical concave surfaces. Their algebraic expressions for $Q_{\text{concave}} (\omega, \alpha, \frac{T_w}{T_i})$ are quite lengthy and are not reproduced here. Their numerical examples were calculated for

$$\frac{T_w}{T_i} = \frac{3}{2} \cdot \frac{1}{20} = .075,$$

the results being presented in ratio from $Q_{\text{concave}}/Q_{\text{convex}}$, as shown in Fig. 9. Since for $\alpha \to 0$, the denominator of this ratio also goes to zero, it is more illuminating to replot the data of Fig. 9 in the form shown in Figs. 10 and 11.
Fig. 9—Comparison of heat transfer for concave and convex cylindrical and spherical segments

\[ \left(\frac{u^2}{2 \frac{k}{m} T_w}\right) = 20 \]

\( \text{OR } \frac{T_w}{T_i} = 0.075 \)
Fig. 10—Effect of accommodation coefficient on heat transfer
Fig. 11—Effect of concavity on heat transfer
Figure 10 compares the variation of \( C_Q \) with the accommodation coefficient \( \alpha \) for concave hemisphere and semi-cylinder (\( \omega = 90^\circ \)) with the linear dependence for convex bodies (including also flat plates) given by Equation (20).\(^*\) In the limit when \( \alpha = 0 \), \( C_Q \) is zero for concave as well as convex bodies, since no matter how many times a molecule collides with the surface, no energy is transferred in any one collision. On the other hand, for complete accommodation (\( \alpha = 1 \)) all molecules are reemitted after their first collision with the wall at the wall temperature, so that no more energy can be transferred in subsequent collisions; hence \( C_Q(\alpha = 1) \) is the same for concave and convex bodies.

For any value of \( \alpha \) lying between 0 and 1 concave surfaces experience greater heat transfer than convex surfaces (with the same cross-sectional area \( A \)) because during each successive collision with the surface a molecule transfers a fraction \( \alpha \) of its remaining energy excess to the surface. Since concave spherical surfaces experience a greater number of interreflections than do cylindrical surfaces (see Figs. 2 and 3), the heat transfer to a spherical cavity exceeds that to a cylindrical cavity.

Figure 11 shows that \( C_Q \) increases gradually with increasing concavity, \( \omega \), and that for half angles of less than \( 45^\circ \) the effect of interreflections on heat transfer is essentially negligible.

The close connection between the effects of concavity on number flux and heat transfer can be used to derive an approximate but simple formula for the heat transfer coefficient of concave spherical and cylindrical surfaces. The derivation is given in the Appendix, the result being

\[
C_{Q_{\text{concave}}} = \frac{\alpha (1 - \frac{T_w}{T_1})}{1 - \frac{1}{a} (1 - \alpha) (1 - \cos \omega)}
\]

(21)

where \( a = \begin{cases} 3 \text{ for cylindrical surfaces} \\ 2 \text{ for spherical surfaces} \end{cases} \)

\(^*\)Figures 10 and 11 are drawn for the case of \( T_w/T_1 \to 0 \) consistent with the drag calculation of Table 1. As shown later, the ratio \( C_{Q_{\text{concave}}}/C_{Q_{\text{convex}}} \) is approximately independent of \( T_w/T_1 \), so that the data of Fig. 9 can be used directly.
Table 3 compares values of $C_Q$ calculated from the approximation of Equation (28) with the values computed in greater detail by Chahine (2) (Figs. 10 and 11). Evidently the difference between any pair of values is always less than .06, or six percent of the incident energy, and in most cases the agreement is considerably better. Since Chahine's more detailed calculation also involved some approximations, Equation (21) appears to be a useful approximation. Furthermore, as observed by the writer in Ref. 9, values of $C_Q$ calculated from the simple Equation (21) for cylinders differ by only about one percent from those calculated from the elaborate equations of Ref. 3.

In concluding this discussion of heat transfer in free molecule flow, it should be noted that the Joule-gas superposition procedure discussed earlier (for drag determination) should work equally well for extending the preceding hyperthermal heat transfer results to convex or concave surfaces, provided that their temperature $T_w$ is specified a-priori. To do this, one would use directly the hyperthermal results given above by assigning to the free stream velocity $U$ and angle of attack $\gamma$ the six pairs of "effective values" tabulated in Fig. 7.

VIII. SUMMARY OF RESULTS

The principal results concerning the effects of concavity on the number flux, drag, and convective heat transfer experienced by bodies in hyperthermal free molecule flow are summarized below:

1. In all cases, the effect of concavity (as measured by the subtended half-angle $\omega$) is more pronounced for doubly curved spherical segments than for infinitely long cylindrical segments with single curvature.

2. Number flux
   a. The reflected number flux at any point of a concave surface is at most equal to one-half the incident flux intercepted by a flat plate normal to the flow (Fig. 2).
   b. The fraction of molecules which escape from a concave spherical surface, after a specified number of collisions, decreases with increasing concavity ($\omega$) to a minimum of 1/2 for the concave hemisphere ($\omega = 90^\circ$), (Equation (8)).
Table 3

COMPARISON OF HEAT TRANSFER COEFFICIENTS FOR CONCAVE SURFACE

<table>
<thead>
<tr>
<th>Accommodation Coefficient ( \alpha )</th>
<th>Surface Half-Angle ( \psi )</th>
<th>Heat Transfer Coefficient, ( C_q ) concave</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( 0 )</td>
<td>Chahine ( 0.100 )</td>
</tr>
<tr>
<td></td>
<td>( 30^\circ )</td>
<td>( 0.106 )</td>
</tr>
<tr>
<td></td>
<td>( 45^\circ )</td>
<td>( 0.117 )</td>
</tr>
<tr>
<td></td>
<td>( 60^\circ )</td>
<td>( 0.135 )</td>
</tr>
<tr>
<td></td>
<td>( 90^\circ )</td>
<td>( 0.198 )</td>
</tr>
<tr>
<td>0.5</td>
<td>( 0 )</td>
<td>( 0.500 )</td>
</tr>
<tr>
<td></td>
<td>( 30^\circ )</td>
<td>( 0.520 )</td>
</tr>
<tr>
<td></td>
<td>( 45^\circ )</td>
<td>( 0.547 )</td>
</tr>
<tr>
<td></td>
<td>( 60^\circ )</td>
<td>( 0.595 )</td>
</tr>
<tr>
<td></td>
<td>( 90^\circ )</td>
<td>( 0.725 )</td>
</tr>
</tbody>
</table>
3. **Drag**
   a. The reemission drag of a concave body is not only larger than that of the corresponding convex body (same \( \varphi \)), but also larger than that of the flat plate which subtends the concave surface (Fig. 4).
   b. The reemission drag of an unshielded concave surface exceeds that of the flat plate which subtends it by less than 6 percent (Table 1). For practical satellite drag estimation this difference is negligible given the uncertainties in accommodation coefficient and surface interaction.
   c. Where part of a concave surface is shielded from the incident flow, interreflections from the shielded portion can contribute as much to reemission drag as interreflections from the exposed surface (Table 2).

4. **Heat Transfer**
   a. Only in the two extreme limits of zero and complete accommodation (\( \alpha = 0, 1 \)) is there no effect of concavity on hyperthermal heat transfer.
   b. For all intermediate values of the accommodation coefficient (\( 0 < \alpha < 1 \)), convective heat transfer to concave bodies is greater than that to convex bodies which present the same cross-sectional area to the flow. This increment in heat transfer for concave bodies is less than about 1/4 of the heat transfer to a flat plate normal to the flow and having perfect accommodation (Figs. 10 and 11).

While no specific calculations have been made, it was suggested that the Joule-gas technique may be useful for extending to lower flight-to-thermal speed ratios some of the hyperthermal drag and heat transfer calculations contained in Refs. 3 and 5.
Appendix

DERIVATION OF HEAT TRANSFER COEFFICIENT FOR CONCAVE SURFACES

The derivation of expressions for heat transfer to concave surfaces is here carried out in dimensionless form by dividing all heat fluxes by \( \frac{1}{2} \rho_1 V_1^2 A \), as in Equation (19). Spherical surfaces are considered first; this is followed by a somewhat more approximate analysis for cylindrical concave surfaces.

SPHERICAL SURFACES

The energy transferred in the first collision, \( Q_o \), is the same as that for a convex surface, i.e.,

\[
Q_o = \alpha \left( 1 - \frac{T_w}{T_i} \right),
\]

and the energy retained by the reemitted molecules is \( \left[ 1 - \alpha \left( 1 - \frac{T_w}{T_i} \right) \right] \).

The fraction of those molecules which do not escape from the cavity, but experience their "first interreflection" is \( \frac{1}{2} (1 - \cos \omega) \), as follows from the discussion preceding Equation (8). Therefore, the incident heat flux associated with the molecules experiencing their first interreflection is

\[
\frac{1}{2} (1 - \cos \omega) \cdot \left[ 1 - \alpha \left( 1 - \frac{T_w}{T_i} \right) \right].
\]

of which a fraction \( \alpha \left( 1 - \frac{T_w}{T_i} \right) \) is transferred to the surface. Here \( T_{i1} \) is the temperature of the incident interreflecting molecules, which is the same as the reemitted temperature \( T_{r0} \) of the molecules after their first collision with the surface. Thus, the energy transferred to the surface during the "first interreflection" is given by
\[ C_{Q_1} = \frac{1}{2} (1 - \cos \omega) \left[ 1 - \alpha \left( 1 - \frac{T_w}{T_1} \right) \right] \cdot \alpha \left( 1 - \frac{T_w}{T_{R_o}} \right) \]

Equation (23)

\[ \alpha = \frac{T_{R_o} - T_{R_i}}{T_{i} - T_w} \]

Equation (24)

Solving Equation (24) for \( T_{R_o} \), substituting this value into Equation (23), and simplifying the resulting expression, one obtains

\[ C_{Q_1} = \frac{1}{2} (1 - \cos \omega) \cdot \alpha(1 - \omega) \left[ 1 - \frac{T_w}{T_1} \right] = C_{Q_o} \cdot \left[ \frac{1}{2} (1 - \cos \omega)(1 - \omega) \right] \]

Equation (25)

By repeating this process, the energy transferred by the second interreflections is shown to be

\[ C_{Q_2} = C_{Q_o} \cdot \left[ \frac{1}{2} - \cos \omega \right] (1 - \omega) \]

Equation (26)

The total heat transfer to the concave sphere can then be represented by

\[ C_{Q_{\text{concave}}} = C_{Q_o} + C_{Q_1} + C_{Q_2} + \cdots \]

\[ = C_{Q_o} \left[ 1 + \lambda + \lambda^2 + \lambda^3 + \cdots \right] \]

Equation (27)

\[ = \frac{C_{Q_o}}{1 - \lambda} \]

where \( \lambda = \frac{1}{2} (1 - \cos \omega) (1 - \omega) \)

Equation (28)
Substituting Equation (28) into Equation (27), one obtains the heat transfer coefficient for the concave sphere.

\[
C_Q^{\text{concave sphere}} = \frac{\alpha (1 - \frac{T_w}{T_1})}{1 - \frac{1}{2} (1 - \omega)(1 - \cos \omega)}
\]  

(29)

Table 3 shows a comparison of values of \( C_Q \) calculated from Equation (29) with the more detailed calculations of Chahine (2) (Figs. 10 and 11). Evidently the difference in \( C_Q \) between the two methods is less than .06, or six percent of the incident energy. Since Chahine's detailed calculations also involve some approximations, Equation (29) appears to be a useful approximation.

**CYLINDRICAL SURFACES**

The chief complication here is that the reflected number flux \( \Delta N/N_o \) varies over the surface, according to Equation (4), as contrasted with the spherical surface (Equation (5)) for which it is constant. We try to circumvent this difficulty, approximately, by calculating the average value of \( \Delta N/N_o \) over the surface, namely

\[
\left( \frac{\Delta N}{N} \right)_{\text{cyl.}} = \frac{1}{\omega} \int_0^\omega \left( \frac{\Delta N}{N} \right)_{\text{cyl.}} d\Theta = \frac{1}{4} \left[ (1 - \cos \omega) + (1 - \frac{\sin \omega}{\omega}) \right]
\]  

(30)

The only difference between the cylindrical case and the spherical case, for purposes of calculating \( C_Q^{\text{concave}} \), lies in the fraction of molecules which do not escape after each reemission. For the spherical surfaces this fraction was \( \frac{1}{2} (1 - \cos \omega) \). For the cylindrical surface this fraction is not known, and it probably varies from one collision to the next. We will, however, approximate the situation by assuming that this fraction differs from that for the sphere in the same way as the average number flux (Equations (30) and (5)). With this assumption, \( C_Q \) for the concave cylinder is given by Equation (27) with \( \lambda \) redefined
as
\[ \lambda_{\text{cyl.}} = \frac{1}{2} (1 - \cos \omega) (1 - \alpha) \cdot \frac{\langle \Delta N/N_0 \rangle_{\text{cyl.}}}{\langle \Delta N/N_0 \rangle_{\text{sphere}}} \]  
(31)

From Equations (3) and (5) we find that
\[ \frac{\langle \Delta N/N_0 \rangle_{\text{cyl.}}}{\langle \Delta N/N_0 \rangle_{\text{sphere}}} = \frac{1}{2} \left[ 1 + \frac{1 - \frac{\sin \omega}{\omega}}{1 - \cos \omega} \right], \]  
(32)

which, as shown below, has nearly the same value for all half-angles of the cylinder

<table>
<thead>
<tr>
<th>(\omega)</th>
<th>(\frac{\langle \Delta N/N_0 \rangle_{\text{cyl.}}}{\langle \Delta N/N_0 \rangle_{\text{sphere}}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0.667 = \frac{2}{3})</td>
</tr>
<tr>
<td>30º</td>
<td>0.668</td>
</tr>
<tr>
<td>45º</td>
<td>0.670</td>
</tr>
<tr>
<td>60º</td>
<td>0.673</td>
</tr>
<tr>
<td>90º</td>
<td>0.682 = 1 - \frac{1}{\pi}</td>
</tr>
</tbody>
</table>

For practical purposes this variation with \(\omega\) can be neglected, and we adopt, for convenience, the value
\[ \frac{\langle \Delta N/N_0 \rangle_{\text{cyl.}}}{\langle \Delta N/N_0 \rangle_{\text{sphere}}} = \frac{2}{3}. \]  
(33)

Substitution of (33) into Equation (31), and the latter into (27) results in this approximate formula for heat transfer coefficient of the convex cylindrical surface:
\[ C_{\text{concave cyl.}} = \alpha \left( 1 - \frac{T_0}{T_i} \right) \left( 1 - \frac{1}{3} (1 - \alpha) (1 - \cos \omega) \right) \]  
(34)
Table 3 compares values of $C_Q$ calculated from Equation (34) with the values calculated for cylinders by Chahine (2) (and also plotted in Figs. 10 and 11). Apparently, the maximum difference in $C_Q$ between the two methods is less than .04 which is less than for the case of the spheres. Evidently the additional approximations introduced into the derivation of the formula for $C_Q_{cylinder}$ are not of significance.

Equations (29) and (34) are of the same form and have been combined in the text into Equation (21).
REFERENCES


