MARKETS AS COOPERATIVE GAMES

L. S. Shapley

P-629

7 March 1955

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

SUMMARY

Markets are formulated as cooperative n-person games in which the players are divided into two camps, the buyers and the sellers, with some kind of profitable exchange going on between them. The solutions of several examples are discussed and a general notion of "abstract market game" is defined.

MARKETS AS COOPERATIVE GAMES

L. S. Shapley

We consider n-person games in which the sets of players is divided into "sellers" and "buyers," with some kind of profitable exchange going on between the two camps but not within them. Such "markets" are most commonly analyzed by the method of equilibrium points — i.e., as if they were noncooperative games — but there are some attractive reasons for using instead the "cooperative" theory of von Neumann and Morgenstern. For one thing, money transfers can be handled by side payments, without explicit determination of prices. Moreover, all bargaining can be regarded as part of the coalition-forming process, so that no formal rules for bids and offers, etc., are required.

Before defining the general "abstract market game," we shall discuss several examples. For the first, let us assume (1) that each buyer wants precisely one item of the commodity involved, (2) that each seller has precisely one item for sale, and (3) that each sale increases the sum of the utilities of buyer and seller by one unit. Then the characteristic function will have the following simple form:

$$\text{(I)} \qquad v(S) = \min\left(|S \cap A|,\ |S \cap B|\right),$$

where A and B denote the sets of sellers and buyers.

This game can be generalized in two divergent ways. One way is to assume that the commodity is perfectly divisible and to let the players have various supplies and demands, represented by numbers

$c_i \geq 0$, i ranging over A and B respectively. Then if buyers are allowed to split their trade among several sellers and vice versa, the generalized characteristic function becomes:

$$\text{(II)} \qquad v(S) = \min\Big(c(S \cap A),\ c(S \cap B)\Big) ,$$

if we let $c(X)$ stand for the sum $\sum_X c_i$. A second generalization would be to assume that the commodity comes in indivisible items, with no one having or wanting more than one item, but with different values attached to them by the different players. Let $a_{ij} \geq 0$ be the total profit to $\{i, j\}$ if $i \in A$ sells his item to $j \in B$. Then we have, for all $i, i' \in A$, $j, j' \in B$:

$$\text{(III)} \qquad \begin{cases} v(\{i\}) = v(\{j\}) = v(\{i, i'\}) = v(\{j, j'\}) = 0 \\ v(\{i, j\}) = a_{ij} \\ v(S) = \max \sum\limits_{T \in \pi} v(T) , \end{cases}$$

the maximum being taken over all partitions π of S into one- and two-element sets. That is, $v(S)$ is as small as possible consistent with superadditivity, and is determined by solving the "optimal assignment" matrix $\|a_{ij}\|$.

In (I) and (III) (but not in (II)) it is easily shown that only coalitions consisting of one buyer and one seller can be "vital," in the sense of Gillies[1], and hence that the other coalitions can be disregarded in the search for solutions. It is therefore a good idea to take as a starting point the characteristic function

$$\text{(IV)} \qquad v(S) = \min\Big(|S \cap A|,\ |S \cap B),\ 1\Big)$$

which assumes only the values 0 and 1 and is not superadditive, but nevertheless agrees with (I) on all vital sets. It represents not a game but a "pseudogame"[2], which happens to decompose into a product of two sums of elementary one-person games. As a result the complete set of solutions is easily obtained; they turn out to be all monotonic curves running from the "A" face ($x_j = 0$, all $j \varepsilon B$) to the "B" face ($x_i = 0$, all $i \varepsilon A$) of the simplex of imputations.

The solutions of (I) are the same, except that the inequalities $x_i + x_j \leq 1$, all $i \varepsilon A$, $j \varepsilon B$, must hold as well as monotonicity. This further restriction fixes at least one endpoint of the curves, namely, the one in the face corresponding to the scarcer type of player. If $|A| = |B|$ the solution is unique — it is the line segment of points $x(p)$ where $x_i(p) = p$ and $x_j(p) = 1 - p$. Thus a uniform market price can be inferred, although its level, as measured by the sellers' profit p, is not determined.

Uniqueness persists in (II) if the total supply and total demand are equal: $c(A) = c(B)$. The solution in this case is in general not just the line segment $\{y(p) | 0 \leq p \leq 1\}$ where $y_i(p) = pc_i$, $y_j(p) = (1-p)c_j$, but a closed convex neighborhood of that segment, comprising the full set of undominated imputations, or "core" of the game. In other words, some price variation may occur in the market because of the inefficient matching of supply and demand within small trading coalitions. If total supply and total demand are unequal, then there will be many solutions, and effects which can be interpreted as favoritism, discrimination, etc., begin to appear.

All the solutions so far discovered to games of type (III), as well as (II), have been connected sets touching both the "A" face and the "B" face in single points. Particular games are easily solved once their cores have been found. The existence of a nonempty core is trivial in cases (I) and (II); in (III) the core is the solution to the dual of the optimal assignment problem, since the minimum of $\sum_{A \cup B} x_k$ subject to $x_i + x_j \geq a_{ij}$ is equal to $v(A \cup B)$.[3]

An abstract market game is defined by imposing certain convexity–concavity conditions on the characteristic function. For brevity, write S_i for $S - \{i\}$, etc. Then we require

$$\begin{cases} v(S) - v(S_i) - v(S_{i'}) + v(S_{ii'}) \leq 0 \\ v(S) - v(S_j) - v(S_{j'}) + v(S_{jj'}) \leq 0 \\ v(S) - v(S_i) - v(S_j) + v(S_{ij}) \geq 0 \end{cases}$$

for all distinct $i, i' \in S \cap A$, $j, j' \in S \cap B$, and for all S. These inequalities hold for the pseudogame (IV) as well as for the games (I)–(III) we have been considering. They state, e.g., that the value of a seller to a coalition is increased if an additional buyer is brought in, or if some other seller is thrown out. We conjecture (1) that all abstract market games possess cores, and (2) that their solutions are always connected sets meeting the "A" and "B" faces of the imputation simplex in single points.

FOOTNOTES

1. D. B. Gillies, "Some theorems on n-person games," thesis, Princeton University, 1953.

2. See L. S. Shapley, "Theory of n-person games," mimeographed lecture notes, Princeton, or "Simple games," RAND research memorandum RM-1384, 23 November 1954.

3. The core vectors (x_k), divided by $v(A \cup B)$, are precisely the optimal strategies of player I in J. von Neumann's "A certain 0-sum 2-person game equivalent to the optimal assignment problem," Annals of Mathematics Study No. 28, pp. 5-12.