A TACTICAL AIR GAME
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Summary

A discrete, linear model of a tactical air war is formulated as a multi-move game. The symmetric case in which the attrition rates are the same for both sides is solved for both finite and infinite campaigns.
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1. INTRODUCTION

A continuous model of a tactical air campaign of fixed length between two air forces engaged in either airfield strikes or ground support strikes was proposed by Arnold Mengel of The RAND Corporation in 1952.* This paper formulates a discrete analogue of Mengel's model as a multi-move game in which both sides, at each period of the campaign, simultaneously deploy their forces between the two missions. Each force suffers a fixed rate of attrition per period due to accidents, etc., and in addition loses planes proportionally to the size of the enemy's attack on his airfields. Replacements for each side are received periodwise, and these may be functions of time. The payoff is assumed to be the difference between the total number of ground support sorties flown by the two sides during the campaign, discounted for future time periods.

Section 2 describes the model in more precise terms. In section 3 we rewrite the payoff function in a convenient way for the subsequent analysis. Beginning with section 4, attention is focused on the symmetric campaign in which the attrition

*The results of this paper were obtained in 1953. Since then, various generalizations of the model studied here have been formulated and solved by M. Dresher and L. Berkovitz of The RAND Corporation.
rates are the same for both forces. This case is completely solved for both finite and infinite campaigns. Optimal strategies in the finite case consist, for both forces, of attacking airfields with maximum strength over an initial interval of the campaign, and the ground thereafter. For the infinite campaign, it is proved that either both sides should always attack airfields or both sides should always fly ground support strikes, depending on the values of the attrition and discount parameters. Section 6 concludes with a numerical example for the symmetric case and some examples for the asymmetric case which indicate that optimal strategies for the latter may be more complicated.

2. FORMULATION OF THE PROBLEM

The game may be described as follows. Let \( p_1 \geq 0 \) and \( q_1 \geq 0 \) be the initial forces possessed by Blue and Red, respectively. It is assumed that before the first move, these forces and the following additional information are known to both sides:

\[ T, \quad \text{a positive integer indicating the number of periods in the campaign;} \]

\[ r_n \geq 0, \ n = 1, \ldots, T - 1, \text{ replacements for Blue;} \]

\[ r'_n \geq 0, \ n = 1, \ldots, T - 1, \text{ replacements for Red;} \]

\[ a, \ 0 \leq a \leq 1, \quad \text{fraction of Blue force not destroyed by anti-aircraft fire, accidents, etc., during a period;} \]
\[ c, \ 0 \leq c \leq 1, \quad \text{fraction of Red force not destroyed by} \]
\[ \text{anti-aircraft fire, accidents, etc., during a period;} \]

\[ b > 0, \quad \text{Red kill potential per plane sent against} \]
\[ \text{Blue airfields;} \]

\[ d > 0, \quad \text{Blue kill potential per plane sent against} \]
\[ \text{Red airfields;} \]

\[ \rho, \ 0 < \rho \leq 1, \quad \text{a factor discounting future periods in} \]
\[ \text{the payoff.} \]

In the \( n \)-th period, \( n = 1, \ldots, T \), Blue chooses \( x_n, \ 0 \leq x_n \leq p_n \),
and Red simultaneously chooses \( y_n, \ 0 \leq y_n \leq q_n \), these choices
being the number of planes sent against opposing airfields,
where

\[
\begin{align*}
p_n &= \max \left( 0, \ ap_{n-1} - by_{n-1} + r_{n-1} \right) \\
q_n &= \max \left( 0, \ cq_{n-1} - dx_{n-1} + r'_{n-1} \right)
\end{align*}
\]

(1)

The rest of the forces \( p_n - x_n \) and \( q_n - y_n \) are automatically
dispatched on ground support strikes. Blue wishes to maximize,
and Red to minimize, the payoff

\[
P(x, y) = \sum_{n=1}^{T} \rho^n \left[ (p_n - x_n) - (q_n - y_n) \right].
\]

(2)

Since both sides know the parameters \( a, b, c, d \) and \( p_n, q_n, \]
\( r_n, r'_n \) at the beginning of each period, clearly neither side
will ever choose to send more than enough planes to wipe out
the opposing force (including replacements for the period).

Hence, (1) may be replaced by
\[
\begin{align*}
\begin{cases}
 p_{n+1} &= ap_n - by_n + r_n \\
 q_{n+1} &= cq_n - dx_n + r_n'
\end{cases}
\tag{1'}
\end{align*}
\]

where now \( x_n \) and \( y_n \) are subject to the inequalities
\[
\begin{align*}
\begin{cases}
 0 \leq x_n \leq \min \left( p_n, \frac{cq_n + r_n'}{d} \right) \\
 0 \leq y_n \leq \min \left( q_n, \frac{ap_n + r_n}{b} \right)
\end{cases}
\tag{3}
\end{align*}
\]

A word might be said at this point about the information pattern on replacements. It turns out in the symmetric case \( a = c, b = d \), that it is immaterial whether the replacements are known completely in advance, or are merely announced period-wise. However, it seems unlikely that pure strategy solutions will exist in the general case unless the replacement schedules are announced before the first move, and we have thus formulated the problem in this way.

3. A DIFFERENT FORM FOR THE PAYOFF

It will be convenient to rewrite the payoff in the following way. Iterating (1') yields
\[
\begin{align*}
\begin{cases}
 p_n &= a^{n-1} p_1 - b \sum_{i=1}^{n-1} a^{n-1-i} y_1 + \sum_{i=1}^{n-1} a^{n-1-i} r_i \\
 q_n &= c^{n-1} q_1 - d \sum_{i=1}^{n-1} c^{n-1-i} x_1 + \sum_{i=1}^{n-1} c^{n-1-i} r_i'
\end{cases}
\tag{4}
\end{align*}
\]
whence (2) becomes

\[ P(x, y) = \sum_{n=1}^{T} \rho^n \left( a^{n-1} p_1 - b \sum_{i=1}^{n-1} a^{n-1-i} y_i + \sum_{i=1}^{n-1} a^{n-1-i} r_i x_i \right) 
- \sum_{n=1}^{T} \rho^n \left( c^{n-1} q_1 - d \sum_{i=1}^{n-1} c^{n-1-i} x_i + \sum_{i=1}^{n-1} c^{n-1-i} r_i y_i \right), \]

or, after collecting coefficients of \( x_n \) and \( y_n \),

\[ P(x, y) = p_1 \sum_{n=1}^{T} \rho^n a^{n-1} - q_1 \sum_{n=1}^{T} \rho^n c^{n-1} + \sum_{n=1}^{T-1} r_n \rho^{n+1} \sum_{i=0}^{T-n-1} (\rho a)^i \]
\[ - \sum_{n=1}^{T-1} r_n' \rho^{n+1} \sum_{i=0}^{T-n-1} (\rho c)^i + \sum_{n=1}^{T} x_n \rho^n \left[ -1 + \rho d \sum_{i=0}^{T-n-1} (\rho c)^i \right] \]
\[ - \sum_{n=1}^{T} y_n \rho^n \left[ -1 + \rho b \sum_{i=0}^{T-n-1} (\rho a)^i \right]. \]

For simplicity, we write this as

\[ P(x, y) = K + \sum_{n=1}^{T} (x_n f_n - y_n g_n), \]

where

\[
\begin{align*}
  f_n &= \rho^n \left[ -1 + \rho d \sum_{i=0}^{T-n-1} (\rho c)^i \right] \\
  g_n &= \rho^n \left[ -1 + \rho b \sum_{i=0}^{T-n-1} (\rho a)^i \right],
\end{align*}
\]
and $K$ stands for the rest of the right side of (5). Since the factors in brackets in (7) are monotone decreasing in $n$, we see that $f_n$ and $g_n$ are monotone decreasing as long as they are positive, and upon becoming negative, remain so thereafter.

We can now dispose of a trivial case.

**Theorem 1.** If $f_1 \leq 0$, $g_1 \leq 0$, then an optimal strategy for Blue (Red) is to take $x_n = 0$ ($y_n = 0$) for all $n$, i.e., both sides should fly only ground support strikes.

For denoting these strategies by $x^O$ and $y^O$ respectively, we have

$$P(x^O, y) = K - \sum_{n=1}^{T} y_n g_n$$

$$P(x^O, y^O) = K$$

$$P(x, y^O) = K + \sum_{n=1}^{T} x_n f_n .$$

By assumption $f_1 \leq 0$, hence $f_n \leq 0$ all $n$. Similarly $g_n \leq 0$. Thus,

$$P(x^O, y) \geq P(x^O, y^O) \geq P(x, y^O) .$$

Note that the hypothesis of Theorem 1 is automatically satisfied, irrespective of $T$, if $\rho(a + b) \leq 1$ and $\rho(c + d) \leq 1$, since, for example,

$$-1 + \rho b \sum_{i=0}^{\infty} (\rho a)^i = -1 + \rho b \left( \frac{1}{1 - \rho a} \right) \leq 0 \iff \rho(a + b) \leq 1.$$
4. THE SYMMETRIC CASE

In this section we suppose \( a = c, b = d \), and in view of Theorem 1, that \( f_1(= g_1) > 0 \). Thus, there is a positive integer \( N < T \) such that \( f_1 > f_2 > \ldots > f_N > 0, f_{N+1}, \ldots, f_T \leq 0 \). We shall prove

Theorem 2. In the symmetric case an optimal strategy for
\underline{Blue (Red)} is to take \( x_n = \max (y_n = \max) \) for \( n = 1, \ldots, N, \)
\( x_n = 0(y_n = 0) \) for \( n > N, \) i.e., both sides should attack airfields with maximum strength over the initial interval \( 1, \ldots, N, \)
and the ground thereafter.

First of all, note that it is obvious from Theorem 1 that regardless of what policy is employed in the first \( N \) periods, one should concentrate on ground strikes in the remainder of the campaign. Now consider the case \( N = 1 \). Let \( x^0, y^0 \) denote the strategies described in the theorem, so that

\[
P(x^0, y) = K + (\max x_1 - y_1)f_1 - \sum_{n=2}^{T} y_n f_n
\]

\[
P(x^0, y^0) = K + (\max x_1 - \max y_1)f_1
\]

\[
P(x, y^0) = K + (x_1 - \max y_1)f_1 + \sum_{n=2}^{T} x_n f_n.
\]

Since

\[
(\max x_1 - y_1)f_1 \geq (\max x_1 - \max y_1)f_1 \geq (x_1 - \max y_1)f_1,
\]

\[
\sum_{n=2}^{T} y_n f_n, \sum_{n=2}^{T} x_n f_n \leq 0,
\]
we have the saddlepoint inequality

\[ P(x^0, y) \geq P(x^0, y^0) \geq P(x, y^0) , \]

and hence Theorem 2 is verified for \( N = 1 \).

It may not be amiss to point out that the kind of naive inequalities we have been writing down so far are not evident for \( N > 1 \). Consequently we shall proceed by induction on \( N \), but first we state two lemmas which will be useful in the induction step.

**Lemma 1.** If \( \sum_{i=1}^{n} k_i \geq 0 \) for \( n = 1, \ldots, N \) with strict inequality for some \( n \), and if \( f_1 > f_2 > \ldots > f_N > 0 \), then

\[ \sum_{i=1}^{N} k_i f_i > 0. \]

For, denoting the partial sums \( \sum_{i=1}^{n} k_i \) by \( K_n \), where \( K_0 = 0 \), we have

\[ \sum_{i=1}^{N} k_i f_i = \sum_{i=1}^{N} (K_i - K_{i-1}) f_i = \sum_{i=1}^{N-1} K_i (f_i - f_{i+1}) + K_N f_N > 0. \]

**Lemma 2.** Let \( s_n, t_n, n = 2, \ldots, N \), be two arbitrary sequences of 0's and 1's, and suppose

\[ \alpha_n = - s_n b \sum_{i=1}^{n-1} a^{n-i-1} \beta_i - (1 - s_n) \sum_{i=1}^{n-1} a^{n-i} \alpha_i \]

\[ \beta_n = - t_n b \sum_{i=1}^{n-1} a^{n-i-1} \alpha_i - (1 - t_n) \sum_{i=1}^{n-1} a^{n-i} \beta_i , \quad n = 2, \ldots, N, \]

where \( \alpha_1 = 1, \beta_1 = 0, b > 0, 0 \leq a \leq 1 \). If \( f_1 > f_2 > \ldots > f_N > 0 \),
then \(\sum_{i=1}^{N} (\alpha_i - \beta_i) r_i > 0\).

By Lemma 1, since \(\alpha_1 - \beta_1 = 1 > 0\), it suffices to prove that all the partial sums \(\sum_{i=1}^{n} (\alpha_i - \beta_i)\), \(n = 2, \ldots, N\), are non-negative.

We note first of all that
\[
\sum_{i=1}^{n} a^{n-i} \alpha_i \geq 0, \quad \sum_{i=1}^{n} a^{n-i} \beta_i \leq 0
\]
for all \(n\). This is obvious for \(n = 1\), and assuming it true for \(n - 1\), we have that either
\[
\sum_{i=1}^{n} a^{n-i} \alpha_i = \sum_{i=1}^{n-1} a^{n-1} \alpha_i - b \sum_{i=1}^{n-1} a^{n-1-i} \beta_i \geq 0
\]
or
\[
\sum_{i=1}^{n} a^{n-i} \alpha_i = \sum_{i=1}^{n-1} a^{n-1} \alpha_i - \sum_{i=1}^{n-1} a^{n-1} \alpha_i = 0
\]
according as \(s_n = 1\) or \(s_n = 0\). Similarly for \(\sum_{i=1}^{n} a^{n-i} \beta_i\).

For given sequences \(s_i, t_i\) of length \(n - 1\), these inequalities show that the minimum \(\alpha_n - \beta_n\) is given by taking \(s_n = t_n = 0\). To see this, note that
\[ \alpha_n - \beta_n = \begin{cases} 
- b \sum_{i=1}^{n-1} a^{n-1-i} \rho_i + b \sum_{i=1}^{n-1} a^{n-1-i} \alpha_i & \text{if } s_n = 1, t_n = 1 \\
- b \sum_{i=1}^{n-1} a^{n-1-i} \rho_i + \sum_{i=1}^{n-1} a^{n-1-i} \beta_i & \text{if } s_n = 1, t_n = 0 \\
- \sum_{i=1}^{n-1} a^{n-1-i} \alpha_i + b \sum_{i=1}^{n-1} a^{n-1-i} \alpha_i & \text{if } s_n = 0, t_n = 1 \\
- \sum_{i=1}^{n-1} a^{n-1-i} \alpha_i + \sum_{i=1}^{n-1} a^{n-1-i} \beta_i & \text{if } s_n = 0, t_n = 0. 
\end{cases} \]

If \( s_n = 1, t_n = 1 \), then \( \alpha_n - \beta_n \geq 0 \), whereas if \( s_n = 0, t_n = 0 \), then \( \alpha_n - \beta_n \leq 0 \). Comparing the last expression for \( \alpha_n - \beta_n \) with each of the other remaining two amounts to the inequalities

\[ \sum_{i=1}^{n-1} a^{n-1-i} \alpha_i \geq b \sum_{i=1}^{n-1} a^{n-1-i} \rho_i, \quad b \sum_{i=1}^{n-1} a^{n-1-i} \alpha_i \geq \sum_{i=1}^{n-1} a^{n-1} \beta_i, \]

which certainly hold since the sums on the left are non-negative, those on the right non-positive.

Now assume inductively that

\[ \sum_{i=1}^{n} (\alpha_i - \beta_i) \geq 0, \text{ } n = 1, \ldots, N - 1. \]

Then, by what was just proved, we have

\[ \sum_{i=1}^{N} (\alpha_i - \beta_i) \geq \sum_{i=1}^{N-1} (\alpha_i - \beta_i) - \sum_{i=1}^{N-1} a^{N-1}(\alpha_i - \beta_i) \]

\[ \geq \sum_{i=1}^{N-1} (1 - a^{N-1}) (\alpha_i - \beta_i). \]
Since $1 - a^{N-1}$ is monotone decreasing in $1$, Lemma 1 and the induction assumption $\sum_{i=1}^{N} (\alpha_i - \beta_i) \geq 0$, $n \leq N - 1$, insure that $\sum_{i=1}^{N} (\alpha_i - \beta_i) \geq 0$. This establishes Lemma 2.

We return to the proof of Theorem 2. Assume the theorem for all games in which the initial interval is of length $< N$, or, what is the same thing, assume that if the hypothesis of Theorem 2 holds, both sides should attack airfields with maximum strength in periods $2, \ldots, N$, and, of course, the ground thereafter. What should Blue, for example, do in the first period? The payoff $P(x, y)$ may now be considered a function of $x_1$ and $y_1$ only. By (3) and (4), $x_n$ and $y_n$ for $1 < n \leq N$ are given by

\[
\begin{align*}
x_n &= \min \left[ a^{n-1}p_1 + b \sum_{i=1}^{n-1} a^{n-1-i} y_1 + \sum_{i=1}^{n-1} a^{n-1-i} r_1, \\
&\quad \frac{1}{b}(a^{n}q_1 - b \sum_{i=1}^{n-1} a^{n-1} x_1 + \sum_{i=1}^{n-1} a^{n-1-i} r_1'), \\
&\quad \frac{1}{b}(a^n p_1 - b \sum_{i=1}^{n-1} a^{n-1-i} y_1 + \sum_{i=1}^{n-1} a^{n-1-i} r_1 + r_n') \right] \\
y_n &= \min \left[ a^{n-1}q_1 + b \sum_{i=1}^{n-1} a^{n-1-i} x_1 + \sum_{i=1}^{n-1} a^{n-1-i} r_1, \\
&\quad \frac{1}{b}(a^{n}p_1 - b \sum_{i=1}^{n-1} a^{n-1-i} y_1 + \sum_{i=1}^{n-1} a^{n-1-i} r_1 + r_n) \right],
\end{align*}
\]

and hence, since $x_n$ is a piecewise linear function of $x_1$, for all but a finite number of values of $x_1$, ...
\[ \frac{\partial x_n}{\partial x_1} = \begin{cases} 
- b \sum_{i=1}^{n-1} a^{n-1-i} \frac{\partial y_1}{\partial x_1} \\
- \sum_{i=1}^{n-1} a^{n-1} \frac{\partial x_i}{\partial x_1} 
\end{cases} \quad 1 \leq n \leq N, \]

according as Blue can send all of his force against Red airfields or not. Similarly, at all but a finite number of points,

\[ \frac{\partial y_n}{\partial x_1} = \begin{cases} 
- b \sum_{i=1}^{n-1} a^{n-1-i} \frac{\partial x_1}{\partial x_1} \\
- \sum_{i=1}^{n-1} a^{n-1} \frac{\partial y_1}{\partial x_1} 
\end{cases} \quad 1 \leq n \leq N, \]

according as Red can send all of his force against Blue airfields or not.

We are now in a position to apply Lemma 2 with

\[ \alpha_i = \frac{\partial x_1}{\partial x_1}, \beta_i = \frac{\partial y_1}{\partial x_1}, \text{ and } s_n = 1 \text{ or } 0(t_n = 1 \text{ or } 0) \]

according as Blue (Red) can or cannot attack Red (Blue) airfields with his entire force in period \( n \). Thus,

\[ \frac{\partial P(x_1,y_1)}{\partial x_1} = \sum_{n=1}^{N} \left( \frac{\partial x_n}{\partial x_1} - \frac{\partial y_n}{\partial x_1} \right) f_n > 0. \]

In other words, the payoff is monotone increasing in \( x_1 \), and accordingly Blue should choose \( x_1 = \text{max} \). In a similar way, the payoff is decreasing in \( y_1 \), and Red should choose \( y_1 = \text{max} \). This completes the proof of Theorem 2.
Observe, as was asserted earlier, that optimal strategies in the symmetric case do not depend on the assumption that the complete replacement schedules are known in advance. All that is needed is an announcement prior to each period in the initial phase of the game of the replacements for that period, so that each side will know how many planes are required to wipe out the opposing force. We will give an example later on for the asymmetric case in which more complete information about replacements is essential for one of the players.

It is apparent also that Theorem 2 is valid for a wider class of weight functions \( w_n \) in the payoff than \( w_n = \rho^n \). All that is required is that the function

\[
f_n = -w_n + b \sum_{i=0}^{T-n-1} w_{n+1+i} a_i
\]

have the property that it is monotone decreasing when \( f_n > 0 \), and upon becoming non-positive, remains so thereafter.

5. **THE SYMMETRIC CASE, INFINITE CAMPAIGN**

It seems intuitively clear from the analysis of the finite game, that if \( T = \infty \), and if, to insure convergence, we assume \( \rho < 1 \) and bounded replacements, then an optimal strategy for each side in the infinite game is to attack the ground always if \( \rho(a + b) \leq 1 \), the opposing force always if \( \rho(a + b) > 1 \). The first part of this assertion may be proved trivially as in Theorem 1, since the payoff for the infinite game can be written as

\[
P(x, y) = K + \left( \frac{\rho b}{1 - \rho c} - 1 \right) \sum_{n=1}^{\infty} \rho^n (x_n - y_n)
\]
and if $\rho(a + b) \leq 1$, then $\frac{\rho b}{1 - \rho a} - 1 \leq 0$. On the other hand, if $\rho(a + b) > 1$, note that for $T$ sufficiently large, one has

$$(13) \quad f_1 > f_2 > \cdots > f_{T-k} > 0; \quad f_{T-k+1}, \ldots, f_T \leq 0$$

where $k$ is a fixed positive integer independent of $T$. Let $x^0$ denote the strategy which assigns $x_n$ its maximum value for all $n$ in the infinite game, and let $(x^0, y)$ denote the play which results from using $x^0$ against an arbitrary $y$, $P(x^0, y)$ the resulting payoff. By (13) and the assumption $\rho < 1$, the truncation of $x^0$ to a finite game of length $T$ is $\varepsilon$-effective for $T$ sufficiently large, so that if $P_T(x^0, y)$ is the payoff truncated at $T$ moves, and $V_T$ is the value of the $T$-move game, then

$$P_T(x^0, y) \geq V_T - \varepsilon$$

for $T$ sufficiently large. But clearly $P_T(x^0, y)$ converges to $P(x^0, y)$, $V_T$ converges to some value $V$, and hence,

$$P(x^0, y) \geq V.$$

Similarly,

$$P(x, y^0) \leq V.$$ 

Thus, the infinite game has value $V = P(x^0, y^0)$ and $x^0$, $y^0$ are optimal strategies. This proves

**Theorem 3.** In the symmetric infinite game with $\rho < 1$ and bounded replacements, an optimal strategy for Blue (Red) is to take $x_n = 0 (y_n = 0)$ for all $n$ or $x_n = \max (y_n = \max)$ for all $n$ according as $\rho(a + b) \leq 1$ or $\rho(a + b) > 1$. 
6. Some Examples and Conjectures

We give the results of some computations with \( a = c = .9, b = d = .2, T = 20, P = 1, r_n = r'_n = 100 \) all \( n \). Since
\[ f_{n-1} = af_n + a + b - 1, \]
one gets
\[ f_{20} = -1 \]
\[ f_{19} = .1 - .9 = - .8 \]
\[ f_{18} = .9(- .8) + .1 = -.62 \]
\[ \ldots \]
\[ f_{14} = -.062382 \]
\[ f_{13} > 0, \]
and thus both sides should hit airfields in the first 13 periods, the ground in the last 7 periods. For \( p_1 = 5000, q_1 = 5000 \), the value of the game is of course 0; for \( p_1 = 4000, q_1 = 5000 \), the value is approximately \(-15,650\); for \( p_1 = 3000, q_1 = 5000 \), the value is approximately \(-24,650\). In the last case, for example, Blue is wiped out in the fourth period, and has no planes until the fifteenth period when Red has ceased attacking Blue airfields. At the end of the game, Blue has 470 planes, Red has 1345. A poor strategy for Red cuts his take considerably. If it is assumed that Red sends half of his planes against Blue airfields in each period, the payoff for \( p_1 = 3000 \) becomes \(-16700\), Blue is never wiped out and ends the game with 100 planes, Red with 990 planes. Thus, this poor strategy for Red is roughly equivalent to giving Blue an extra 1000 planes to begin with. The examples we have computed seem to support the statement that the payoff is usually fairly sensitive to the strategies used.
The form of the solution for the symmetric case might lead one to suspect that in the asymmetric case, optimal strategies would consist of concentrating on airfields in some initial phase, on the ground thereafter, where, of course, the change-over points for the two players would, in general, be distinct. We will analyze an example with \( T = 5, \rho = 1 \), which shows that this is false. Let

\[
p_{n+1} = p_n - y_n + r_n
\]

\[
q_{n+1} = -0.9x_n + 100
\]

where \( p_1 = 100, q_1 = 90, r_1 = r_2 = 0, r_3 = 100, r_4 = 0 \). Then (5) becomes

\[
P(x, y) = 210 - 0.1(x_1 + x_2 + x_3 + x_4) - x_5 - (3y_1 + 2y_2 + y_3) + y_5.
\]

Let \( y^o \) be the strategy \( y^o_n = \max, n = 1, 2, 3, y^o_5 = 0 \). Thus,

\[
y_1^o = \min (90, 100) = 90
\]

\[
y_2^o = \min (q_2, p_2) = \min (q_2, 10) = 10
\]

since \( q_2 = -0.9x_1 + 100 \geq 10 \), and

\[
y_3^o = \min (q_3, p_3 + 100) = q_3 = -0.9x_2 + 100.
\]

Hence,

\[
P(x, y^o) = -180 - 0.1x_1 + 0.8x_2 - 0.1x_3 - 0.1x_4 - x_5.
\]

What is the best countering strategy for Blue? It is easy to see that \( x_5 = x_4 = x_3 = 0 \) (in fact, \( p_3 = 0 \) regardless of what Blue does in the first two periods, so of course, \( x_3 = 0 \)).
It follows that

\[ x_2 = \max = \min \left( p_2, \frac{100}{q} \right) = p_2 = p_1 - y_1^0 = 10, \]

and since the upper limit on \( x_2 \) is independent of \( x_1 \), Blue should choose \( x_1 = 0 \). In other words, if Red hits airfields the first three periods, then the ground, the best Blue can do is to hit the ground the first period, attack airfields the second, and the ground the last three. Now, denote this strategy for Blue by \( x^0 \), and look for the best countering strategy for Red. We have

\[ P(x^0, y) = 210 - 0.1x_2^0 - 3y_1 - 2y_2 - y_3 + y_5, \]

or, since \( x_2^0 = 100 - y_1 \),

\[ P(x^0, y) = 200 - 2.9y_1 - 2y_2 - y_3 + y_5. \]

Hence, \( y_5 = 0 \) in a minimizing solution, and \( y_3 = \max \). Since \( r_3 = 100 \), this means \( y_3 = q_3 \). The choice of \( y_2 \) doesn't alter \( q_3 \), and choosing \( y_1 \) as large as possible certainly doesn't decrease \( q_3 \). Hence, so far as \( y_3 \) is concerned we should take \( y_1 \) and \( y_2 \) at their maximum values. Thus,

\[ y_2 = \min (q_2, p_2) = \min (100, 100 - y_1) = 100 - y_1, \]

and since

\[ -2.9y_1 - 2(100 - y_1) = -0.9y_1 - 200, \]

we also have

\[ y_1 = \max = 90. \]
Consequently, $x^o, y^o$ is a saddlepoint of $P(x, y)$.

If the replacements in this example are changed to $r_n = r'_n = 100$, then Blue's optimal strategy shifts to taking $x_1 = \max, x_2 = \max, x_3 = x_4 = x_5 = 0$, and Red's optimal strategy remains the same. To see this, note that now

\[
\begin{align*}
y_1^o &= \min (90, 100) = 90 \\
y_2^o &= \min (q_2, p_2 + 100) = q_2 = -0.9x_1 + 100 \\
y_3^o &= \min (q_3, p_3 + 100) = q_3 = -0.9x_2 + 100,
\end{align*}
\]

so that, except for a constant,

\[
P(x, y^o) = 1.7x_1 + 0.8x_2 - 0.1x_3 - 0.1x_4 - 0.1x_5.
\]

It is easily seen that this is maximized by taking $x_1 = \max, x_2 = \max, x_3 = x_4 = x_5 = 0$. On the other hand, if Blue plays this strategy, Red can do no better than the $y^o$ strategy.

Thus it seems improbable that pure strategy solutions will exist in the asymmetric case if replacements are not known in advance.

Our final example shows that in the asymmetric case, a player must sometimes deliberately split his force between the two missions. Let $T = 5$, $\rho = 1$, and

\[
\begin{align*}
p_{n+1} &= p_n - y_n + r_n \\
q_{n+1} &= 0.4q_n - 0.5p_n + r'_n
\end{align*}
\]

where $p_1 = 100$, $q_1 = 50$, $r_1 = 0$, $r_2 = 10$, $r_3 = 100$, $r_4 = 0$, $r'_1 = 90$, $r'_2 = 0$, $r'_3 = 100$, $r'_4 = 0$. Then, except for a constant,
$P(x, y) = -.188x_1 -.22x_2 -.3x_3 -.5x_4 - x_5 - 3y_1 - 2y_2 - y_3 + y_5$.

Let $y^o$ denote the same strategy for Red as in the previous example. Then

$y_1^o = \min (50, 100) = 50$

$y_2^o = \min (q_2, p_2 + 10) = \min (q_2, 60) = 60$

$y_3^o = \min (q_3, p_3 + 100) = q_3$.

Ignoring constants, we have

$P(x, y^o) = .012x_1 + .28x_2 - .3x_3 - .5x_4 - x_5$.

Hence, Blue should take $x_3^o = x_4^o = x_5^o = 0$, and

$x_2^o = \min (p_2, .8q_2) = \min (50, 88 - .4x_1)$.

Putting these in the payoff, we are left with the following function of $x_1$ to maximize:

$P(x_1, x_2^o, x_3^o, x_4^o, x_5^o, y^o) = \begin{cases} 
.012x_1 + 14 & \text{if } x_1 \leq 95 \\
-.1x_1 + 24.64 & \text{if } x_1 > 95
\end{cases}$

Consequently the best countering strategy to $y^o$ means a choice of $x_1^o = 95 < \max x_1$. Since one can also show that $P(x^o, y)$ is minimized by $y^o$, we have a case where one player must deliberately split his force.

It should be pointed out that $\rho(c + d) < 1$, $\rho(a + b) > 1$ in both of these examples. We conjecture that in case $\rho(a + b) > 1$, $\rho(c + d) > 1$, both sides have optimal strategies
of the same form as in the symmetric case with possibly different points at which activity is shifted from the air to the ground. If \( g_n \) changes sign later than \( f_n \), say \( g_N > 0 \) with \( g_{N+1}, \ldots, g_T \leq 0 \), we conjecture that an optimal strategy for Red is to take \( y_n = \max, n = 1, \ldots, N \), \( y_n = 0 \) for \( n > N \), and that Blue shifts earlier in the campaign.\(^2\)

In conclusion, we would like to express our thanks to I. Glicksberg, O. Gross, and G. Dantzig for much helpful discussion.

\(^2\)These conjectures have since been verified by M. Dresher.