THE MATHEMATICS OF MILITARY PAY

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Military pay has become one of the major issues of defense policy during the past year. It is an issue which is susceptible to rational quantitative analysis, although it is complicated by the fact that a man's pay is both his income and the price of his services. The income aspect raises questions of equity. Consequently, most popular discussions of military pay have been excessively if not exclusively influenced by vague notions of fairness. As a result, the price aspect has been neglected in the determination of military pay scales and the efficiency of the services has suffered.

All of the services are now facing a serious reenlistment problem. In 1956, the Defense Department appointed a special committee to study the matter and to recommend measures for its amelioration. The proximate cause of the problem is that military salaries have not kept pace since the war with the salaries offered by civilian industry, the services' principal competitor for manpower. However, the seriousness of the problem and the appointment of a special committee to investigate it are symptomatic of a deeper malaise. Supply and demand conditions for various kinds of military manpower are constantly changing, but the services are not allowed to adjust by changing (upwards or downwards) the salaries they offer. This situation ought to be remedied. A considerable improvement in the effectiveness of the military establishment could be obtained at no extra cost if the services were allowed more freedom in the allocation of their budgets for personnel. Perhaps that freedom would be forthcoming if this prospect of greater effectiveness were more generally understood. The Congress
might instruct the services to follow a policy of maximizing military
effectiveness, within the limitations of the personnel budget, by varying
salaries and bonuses within certain limits.

The purpose of this paper is to explain the mathematical theory of
maximization of military effectiveness within the limitations of a fixed
budget and to show that the theory has an appropriate application in the
pay question. In principle, the problem is not difficult. In order to
determine the optimum allocation of funds, a service could begin with its
current program and compare its actual over-all effectiveness with that
which would result from slightly different allocations. For example,
the service might consider retaining a few more radar men with funds
obtained by retaining fewer vehicle drivers. Those directions which show
improvement could be followed until a point was reached at which no
further improvement seemed possible. Such an exercise would require the
ability to compare force compositions and to select the better ones.

A further development of the logic of maximization will be valuable
for two reasons. First, it can provide criteria for the identification of
a maximum. Secondly, it can indicate the most fruitful directions of change
to the investigators seeking an optimum. Since experimentation, even of a
conceptual sort, may be expensive, this is likely to be quite important.
The significant feature of the theory is its great generality. Proceeding
from a bare minimum of general relationships between military effectiveness
and the numbers of men in the services in various categories, some definite
practical prescriptions can be developed for the optimum use of the
personnel budget.
I. The Foundations of the Analysis

We must have some notion of what military effectiveness is and how it might be measured before we can maximize it. There is no very precise way of defining military effectiveness as a function of the factors which comprise it. However, as the analysis will show, a precise measure of military effectiveness is not necessary. We need only to be able to compare and rank the effectiveness resulting from different combinations of men and equipment in order to apply the logic of maximization. In practice this ranking may not always be easy, but it is clear that it would be vastly more difficult to determine the general functional relationship between effectiveness and manpower.

Specification of a complete military effectiveness function may not be possible, but it is possible to say some things which are useful in the comparison of different force compositions. Total effectiveness can be broken down into independent components and lower level criteria can be specified as measures of the effectiveness of parts of the force. In doing this, one should be careful to see to it that the criteria used at the lower levels are the relevant ones for the effectiveness of the whole organization. For example, it is important to distinguish between combat-ready units and total numbers. Thus, the number of bombers which are in commission or which are combat ready is a much better measure of the military capability of the Strategic Air Command than is the total number of bombers in the force. It is important to avoid ratios in measuring military effectiveness. It is the absolute number of combat-ready bombers that is relevant and not the ratio of combat ready bombers to total bombers. In general, measures of any capability should be related to the
cost of attaining it. The correct form for the objective of the whole or a part of a military organization where the total budget is fixed should be to maximize total effectiveness subject to that budget or, alternatively, to obtain a stated capability at minimum cost. The two are logically equivalent.

Let us define, then, a general index of military effectiveness, a continuous differentiable function which depends upon all of the manpower and equipment inputs used by the services. We may write this in the form

\[ E(x_1, \ldots, x_n) \]

What can we say about the shape of this function? First, it should increase with increments in each of the equipment and manpower inputs, \( x_1, \ldots, x_n \). Letting the subscript \( i \) denote partial differentiation with respect to \( x_i \), we may express this property by the inequalities

\[ E_i > 0. \]

Second, the analysis requires that \( E \) obey what economists call "the generalized law of diminishing marginal returns" at least over the range of \( x_i \) with which we are concerned. The mathematical term for this property is concavity. A function is said to be strictly concave if any plane profile of its graph has the property that the chord joining any two points of the profile lies entirely below the graph.\(^4\) For a twice differentiable function, this property may be described by the inequality

\[ \frac{d^2 E}{d x_i d x_j} < 0 \]

for all \( d x_i \), not all equal to zero.\(^5\)

The economic name for "\( E_i \)" is "the marginal product of \( x_i \)." The law of diminishing marginal returns implies, first, that the marginal product
of each \( x_i \) diminishes as the corresponding input is increased, that is \( E_{ii} < 0 \). For example, the gain in the military effectiveness of a force obtained by increasing the number of radar men from one thousand and one to one thousand and two is less than the gain obtained by increasing the number from one hundred and one to one hundred and two. Secondly, it implies that the marginal product of any group of inputs diminishes as the size of the group is increased. Alternatively, suppose that we were to fix the value of \( E \) and of all but two of the inputs, say, \( x_i \) and \( x_j \).

Starting with any pair of values of \( x_i \) and \( x_j \) which are consistent with the value of \( E \), we might inquire how the increment of \( x_j \) required to keep \( E \) constant varies as \( x_i \) is reduced a unit at a time. If \( E \) is concave, then ever increasing increments in \( x_j \) will be required to offset the successive reductions in \( x_i \). It is easy to see why such a law would hold for a consumer if \( E \) were to represent a fixed level of "utility" and if \( x_i \) and \( x_j \) were to represent goods which he consumes. Can we assume that the military effectiveness function satisfies this law? As a general rule, we cannot.

There are military situations in which groups of inputs will display increasing marginal returns. For example, if a flight of bombers is attempting to penetrate enemy defenses, the number that penetrate successfully will rise more rapidly than the number which attempt to penetrate, as the latter is increased, at least over a range. The reason for this is that the bombers interact favorably with each other in confusing and saturating the defenses. Similarly, against a bomber force of fixed size the defenses may have increasing returns over a range of expenditure. However, there are many other situations in which the inputs
have diminishing marginal returns. It will be argued later in the paper that personnel is such an input. A valuable feature of the maximization analysis is that it may be applied to a subset of the variables in $E$, while the others are held constant, if $E$ is concave in those variables. Thus, the analysis can be applied to the maximization of $E$ subject to the limitations of the personnel budget if $E$ is concave in the personnel inputs when all other variables are held constant.

II. Maximization of Military Effectiveness Within the Limitations of a Fixed Budget

The problem which we now proceed to analyze is the maximization of $E$ subject to the limitation of a fixed budget. Suppose initially that the service is free to select any non-negative set of inputs that is consistent with its budget limitations. Symbolically, its constraints may be written

(2.1) $G(x_1, \ldots, x_n) = 0$ and

(2.2) $x_i \geq 0$ \hspace{1cm} (i = 1, \ldots, n) .

(2.1) is a more general form of the budget constraint which appears in (2.9) in the traditional form. The method of Lagrange multipliers provides a convenient mathematical tool. We form the expression

(2.3) $L(x, u) = E(x_1, \ldots, x_n) + \sum_k G^k(x_1, \ldots, x_n)$

in which the $u_k$ are undetermined multipliers and in which the $G^k$ are $m$ constraints ($m < n$) of the form (2.1). Then, we may draw upon the following general theorem to reach conclusions of practical interest in particular cases.6/ Let $L(x, u)$ be a differentiable function of $x_i \geq 0$ and $u_k$ and let $L_{x_i}^0$ and $L_{u_k}^0$ represent the partial derivatives of $L(x, u)$ evaluated at the point $x_i^0$ and $u_k^0$. Then, if $L(x, u^0)$ is a concave function of $x$ and if $L(x^0, u)$ is a convex function of $u$, the following are
necessary and sufficient conditions for \( L \) to have a saddle point at \( x^0_1, x^0_k \), i.e.,

\[
L(x, u^0) \leq L(x^0, u^0) \leq L(x^0, u) \quad : \\
L^0_{x_1}(x, u) = 0 \\
x^0_1 L^0_{x_1}(x, u) = 0 \\
x_i \geq 0 \quad \quad (i = 1, ..., n)
\]

and

\[
L^0_{u_k}(x, u) = 0 \quad \quad (k = 1, ..., m).
\]

Therefore, if \( L(x, u) \) has the appropriate concavity, \( E \) has a constrained maximum at \( x^0_i \) if and only if either

\[
E_1 + \sum_{k=1}^{m} \frac{\partial E}{\partial \lambda_i} \bigg|_{u} = 0 \\
x_i = 0 \quad \quad (i = 1, ..., n).
\]

If there is only one constraint, \( G \), the equations (2.7) can be written

\[
\frac{\partial E}{\partial r} = \ldots = \frac{\partial E}{\partial s}
\]

for those \( x_i \) whose maximizing value is not zero. This set of equations expresses the basic necessary condition for efficient allocation of the budget.

For the sake of a simple illustration, suppose that the service is limited in the maximization of \( E \) by a fixed over-all budget, \( B \), and that it can buy any amount which it may require of each input at a fixed price, \( p_i \), for \( x_i \). Then we may express the budget limitation by the equation

\[
B - \sum_{i=1}^{n} p_i x_i = 0.
\]
The appropriate Lagrangean expression is

\[(2.10) \quad L(x, u) = E(x_1, \ldots, x_n) + u(B - \sum_{i=1}^{n} p_i x_i)\]

and the maximizing values of \(x_i\) are either zero or are determined by the equations

\[(2.11) \quad E_i - u p_i = 0\]

and (2.9). In this example, the concavity of \(L(x, u^0)\) is assured by the concavity of \(E\). The appropriate form of (2.8) for this example is

\[(2.12) \quad \frac{E_B}{P_B} = \frac{E_B}{P_B} = \ldots = \frac{E_C}{P_C} .\]

Military effectiveness will be maximized when the marginal product of each input, whose maximizing value is not zero, is proportional to the price of the input and when the budget is just used up. /7/ 

An important feature of this analysis and of the proportionality conditions is that they are independent of the specific form chosen for \(E\). They depend only upon the rankings implied by the military effectiveness function. Thus, if the service is always able to decide that one combination of inputs is more or less effective than another, it has, implicitly, an \(E\) function within the meaning of this discussion. If we replace \(E\) by any concave monotonically increasing function of itself, we shall find that all of the conclusions demonstrated so far continue to be valid. A transformation of this kind will change the scale in which effectiveness is measured, but it will not change the relative rankings of different combinations of inputs.

Let \(E\) be replaced by \(TE\), with

\[(2.13) \quad \frac{dT}{dE} = T E^* > 0 ,\]
and maximize $T^*E_1$ subject to (2.1). The Lagrangean expression is

$$T^*E(x_1, \ldots, x_n) + uG(x_1, \ldots, x_n)$$

and the maximizing values of $x_i$ are either zero or must satisfy

$$T^*E_i + uG_i = 0.$$  

The equations (2.8) are implied directly by (2.15). Since

$$\frac{\partial}{\partial x_1} \frac{T}{x_j} = T_{ij} = T^*E_{ij} + T^*E_{i}E_j,$$

the concavity of (2.14) can be expressed by the inequality

$$T^*\Sigma E_{ij}d_{i1}d_{x_i}d_{x_j} + T^*\Sigma E_{i}E_{j}d_{x_i}d_{x_j} + u\Sigma \Sigma G_{ij}d_{x_i}d_{x_j} < 0,$$

which is the only restriction on the shape of $T$ other than (2.13).

(2.17) can be written

$$T^*\Sigma E_{ij}d_{i1}d_{x_i}d_{x_j} + T^*\{E_{i}d_{x_i}\}^2 + u\Sigma \Sigma G_{ij}d_{x_i}d_{x_j} < 0.$$  

Thus, if $E$ and $G$ are each concave, and they will be in all of the cases of interest to us, then it is sufficient, but not necessary, for (2.15) to define a maximum, that $T[E]$ be concave, i.e.,

$$T'[E] \leq 0.$$  

Thus we have shown that the maximizing values of $x_i$ are invariant under a wide class of monotonic transformations and hence that the logic of maximization is dependent upon the ranking of different states of effectiveness rather than upon the actual measurement of effectiveness.  

The simple example used for illustrative purposes above is lacking in realism, and therefore applicability, in two important respects. First, the military services face not one but many constraints. They are not permitted to allocate their entire budgets as they see fit. Second, they do not face fixed prices for the goods and services they buy. It would
not be appropriate to explore the complexities of the budgetary process in this paper. Let it suffice to say that allocation decisions must be made somewhere, and the services are able to exert an important influence upon the decisions which they do not actually make themselves. This is particularly true of allocations within particular sub-categories of the military budget. An interesting aspect of the logic of maximization is that it is applicable at any level of allocation. The same principles which guide over-all optimization can be applied to sub-optimization. For example, suppose that the service is given one fixed budget for personnel, say $B_p$, and another for equipment, say $B_e$, and assume that no reallocation between them is permitted. Assuming fixed prices, for the sake of simplicity, let the first $r$ inputs be items of personnel and the $r+1^{st}$ to $n^{th}$, equipment. Then the service may seek to maximize $E$ subject to

\begin{align}
(2.19) \quad & B_p - \sum_{i=1}^{r} p_i x_i = 0 \quad \text{and} \\
(2.20) \quad & B_e - \sum_{i=r+1}^{n} p_i x_i = 0.
\end{align}

Forming the appropriate Lagrangean expression with $u_1$ and $u_2$ as the undetermined multipliers related to (2.19) and (2.20) respectively, and differentiating as before, we obtain

\begin{align}
(2.21) \quad & E_i - u_1 p_i = 0 \quad (i = 1, \ldots, r) \\
(2.22) \quad & E_i - u_2 p_i = 0 \quad (i = r+1, \ldots, n)
\end{align}

for those goods $x_i$ whose maximizing value is not zero. $u_1$ and $u_2$ are not necessarily equal. Thus, the same proportionality condition holds within each budget, though not between them. If the service were free to allocate its personnel budget, though not its entire budget, it could be guided by
the same general logic of maximization. The same argument could be extended
to more sub-constraints (and to sub-constraints within sub-constraints).
For example, a service might have freedom to allocate within a budget for
enlisted men and a budget for officers, but not between them.

A simple if frequently overlooked point is related to this. The
imposition of any new and binding constraint will lower the maximum
attainable value of E, and the removal of any binding constraint will
increase the maximum attainable value of E. In the example used above,
if the separate constraints prevent the equalization of \( u_1 \) and \( u_2 \), then the
maximum value of E will be lower than in the absence of the separate
constraints. Stated formally, the point is trivial. Yet in practice many
constraints are needlessly imposed upon the services which prevent them
from allocating their budgets efficiently. For example, they are required
to offer the same reenlistment bonuses to men in all specialties. The
imposition of even such a seemingly innocuous constraint has a real cost
in terms of over-all effectiveness.

The fact that the logic of maximization can be applied to any subset
of the inputs limited by a common budget constraint suggests that we may
restrict our attention to the allocation of particular parts of the
over-all military budget, e.g., the personnel budget. For any given
composition of military equipment we may define a new military effectiveness
function dependent only upon the manpower inputs, i.e.,

\[
(2.23) \quad F(x_1, \ldots, x_r) = E(x_1, \ldots, x_r, \bar{x}_{r+1}, \ldots, \bar{x}_n)
\]

where \( \bar{x}_{r+1}, \ldots, \bar{x}_n \) are fixed. Of course, \( F \) will change with changing
combinations of equipment. But by singling out the personnel inputs we
can maximize the contribution of manpower to over-all effectiveness by
appropriate allocation of the personnel budget, for any set of equipment
which has been specified. This is sure to be an easier problem than the
over-all maximization. Further, F may be concave even if E is not (i.e.,
E may be concave in the personnel inputs, though not concave in all \( x_i \)).

An important aspect of the problem which has not been considered yet
is that the services do not face fixed prices for manpower: they face
rising supply prices. If they wish to hire more men in any category
or specialty, generally they must offer a higher price for their services.
Let the supply curve for men in the \( i \)th specialty be

\[
(2.24) \quad p_i = S_i(x_i)
\]

where \( p_i \) is the salary or wage rate. By assumption

\[
(2.25) \quad S'_i(x_i) \geq 0 \quad \text{and} \quad S''_i(x_i) \geq 0 \quad (i = 1, \ldots, r).
\]

Substituting (2.24 into (2.19), we now maximize \( F \) subject to

\[
(2.27) \quad B_p - \sum_{i=1}^{r} x_i S_i(x_i) = 0.
\]

The new Lagrangean expression is concave in the \( x_i \). Applying our earlier
results, at the maximum, either

\[
(2.28) \quad F_i - u_1 \left[ S_i(x_i) + x_i S'_i(x_i) \right] = 0 \quad \text{or} \quad x_i = 0.
\]

The expression in square brackets represents the increase in the total
costs of the service which would result from hiring an additional man in
the \( i \)th specialty. It is the sum of the price of the additional man and
the increase in price multiplied by the number of men to whom it must be
paid. Economists call this the "marginal cost" of the input. For
convenience, we shall abbreviate this expression as $MC_i$. The proportionality conditions (2.12) must now be modified slightly. When the service is allowed to offer a higher price to attract more men, in order to maximize effectiveness, it should adjust the prices it offers and the inputs it obtains until either the marginal product of each input is proportional to the marginal cost of buying it or until the input is not bought at all. Symbolically, for those $x_i$ obeying the first condition in (2.28)

\[
\frac{F_a}{MC_a} = \frac{F_b}{MC_b} = \cdots = \frac{F_c}{MC_c} \quad .
\]

III. A Gradient Approach to Maximization

The development of necessary and sufficient conditions for a constrained maximum has provided us with criteria for the identification of a maximum or, more precisely, for the identification of positions below the maximum. It has not provided us with a method for finding the maximum either in principle or computationally. If we had exact analytic expressions for the $E$ function and the constraints $G$, and if the $x_i$ were not required to be non-negative, we could, at least in principle, use the first order maximum conditions and the constraints to solve for the maximizing values of the $x_i$ and the Lagrange multipliers.\(^{11}\) However, we have not assumed any specific knowledge of the form of $E$, and, by requiring the $x_i$ to be non-negative, we have opened the way for some of the first-order conditions to be inequalities instead of equations. Since the necessary and sufficient conditions do not tell us which $x_i = 0$ at the maximum and hence which first order conditions are inequalities, we are not able to "solve" for the maximizing values. But this is not important
since the theoretical ability to solve for the maximizing values of the variables would have little practical value anyway.

The practical significance of the logic of maximization in military problems is that it indicates the most fruitful directions in which to proceed in the reallocation of funds and it provides us with sufficient conditions for an improvement in military effectiveness. It can be shown that whenever an actual set of inputs departs from the optimum, a reallocation of funds can increase over-all effectiveness. In particular, if

\[
(3.1) \quad \frac{F_b}{MC_b} > \frac{F_a}{MC_a} \quad (x_a > 0)
\]

there will always be a reallocation of funds away from \( x_a \) and to \( x_b \) which will increase \( F \). Furthermore, the largest increase in effectiveness for the reallocation of a fixed sum will result from the transfer of funds from purchase of the input with the smallest ratio of marginal product to marginal cost to the category with the largest, provided that the reallocation is consistent with the inputs being non-negative.

In order to demonstrate this, let the numbers of men actually in the different categories be represented by \( x_1^0, \ldots, x_r^0 \) and assume that this is not an optimum position.\(^{12}\) Expanding \( F \) linearly around this point by Taylor's Theorem, we may write

\[
(3.2) \quad F(x_1, \ldots, x_r) = F(x_1^0, \ldots, x_r^0) + \sum (x_i - x_i^0) F_i^0
\]

where \( F_i^0 \) are the partial derivatives evaluated at the point \( x_i^0 \). If we restrict ourselves to pairwise reallocations, we may hold all but two of the \( (x_i - x_i^0) \) equal to zero, whence we may write
(3.3) \[ F(x_1, \ldots, x_r) = F(x^0_1, \ldots, x^0_r) + (x_a - x^0_a)F^a_a + (x_b - x^0_b)F^b_b \]

where only \( x_a \) and \( x_b \) are permitted to vary. The budget constraint can be written

\[ \sum_{i=1}^r (x_i - x^0_i)MC_i = 0. \]

Again, holding all but two \( (x_i - x^0_i) \) equal to zero, we may write this as

\[ (x_a - x^0_a) = -(x_b - x^0_b) \frac{MC_b}{MC_a}. \]

Substituting (3.5) into (3.3) we find that

\[ F(x_1, \ldots, x_r) - F(x^0_1, \ldots, x^0_r) = MC_b (x_b - x^0_b) \left[ \frac{F^b_b}{MC_b} - \frac{F^a_a}{MC_a} \right]. \]

Therefore, if at \( x^0 \) the ratio of \( F_b \) to \( MC_b \) exceeds the ratio of \( F_a \) to \( MC_a \), then a small increase in \( x_b \), at the expense of a reduction in \( x_a \), will result in an increase in \( F \) above its value at \( x^0 \). Of course, these conclusions, and subsequent ones about reallocations, must be qualified by the existence of the constraint \( x_a \geq 0 \). The reallocation of funds away from \( x_a \) must stop when \( x_a = 0 \). Thus, our first assertion has been proved: a reallocation from an input with a lower marginal product to an input with a higher marginal product, each in relation to its marginal cost, will always increase military effectiveness. The proof of the second assertion follows immediately. If we fix the amount of funds to be reallocated, \( (x_b - x^0_b)MC_b \), and if we consider all pairs of inputs, then clearly, a reallocation between the two inputs for which

\[ \frac{F_b}{MC_b} - \frac{F_a}{MC_a} \]

is the largest will result in the largest positive difference on the left
hand side of (3.6), that is, the greatest increase in military effectiveness.

In this last step, we have deduced a general and practical prescription for military wage policy whose implementation requires only the ability to compare ratios of marginal products and marginal costs. Whenever the proportionality condition is not met, the service is able to obtain an increase in over-all effectiveness at no increase in cost merely by a reallocation of funds and a change in the salaries it offers. Further, the disproportionality in marginal costs and products will always indicate the appropriate direction for reallocation. Of course, practical considerations prevent the armed services from freely moving salaries up and down, but nevertheless there are many possibilities for the useful application of these principles.

It is interesting to observe that there are various ways in which one might proceed to carry out the reallocations. For example, one might use the rule that the marginal products and costs of inputs to be compared should be selected in pairs at random. This would result unambiguously in an improvement, although it would not be the most efficient procedure in that it would not be making use of the steepest gradient—the "fastest way uphill." Also, the random selection would often yield pairs of inputs which, though having unequal marginal product to marginal cost ratios, are both above or below their values at the maximum of $F$. In these cases, the reallocation indicated by the rule "always move uphill" would require changing one of the inputs in a direction which would have to be reversed subsequently if the movement "uphill" were to continue. For example, if both inputs were in excess of their as yet unknown
optimum values, the reallocation rule would call for the increasing of one input. This move would have to be reversed later on. Although the rule "always move uphill" applied randomly should eventually lead one to the maximum, it may do so in an inefficient and oscillatory way.

An alternative procedure would be always to select the steepest gradient and to move along it until it was no longer the steepest or until it no longer yielded increases in effectiveness. This would seem to be very sensible and, for practical purposes, it is probably the method to be preferred. Unfortunately, an analytical discussion of the process of convergence would involve, at best, very complicated mathematics, although the method could be programmed for a high-speed digital computer.

One may imagine yet a third approach to the problem. Consider the following set of rules. Beginning with the current values of the $x_i$ and with any positive value of $u$, preferably near the average or median ratio of $P_i$ to $MC_i$, examine $x_1$, $x_2$, ..., $x_r$ and $u$ in succession. If $F_i - uMC_i$ is positive, increase $x_i$ by a small amount, and if negative, decrease it by a small amount unless doing so would make $x_i$ negative. Similarly, if $B_p - \sum_{i=1}^{r} p_i x_i$ is negative raise $u$, and if positive, lower $u$. A moment's reflection will show intuitively the sense of these rules. The reason for increasing $u$ when $B_p - \sum_{i=1}^{r} p_i x_i$ is negative is that the budget is being exceeded whence the use of all inputs must be curtailed. This is accomplished by raising the average ratio of $F_i$ to $MC_i$. This process can be studied analytically and its convergence demonstrated for the cases which we have discussed.

Returning to the notation of Part II, let us suppose that we are maximizing $E$ subject to $C$. The appropriate Lagrangean expression is of
the form (2.3) with \( k = 1 \). We may model the set of rules described above by supposing that one "round" of adjustments in \( x_1 \) and \( u \) takes place simultaneously in one unit of time. Then, letting \( \dot{x}_1 \) and \( \dot{u} \) denote the first derivatives of \( x_1 \) and \( u \) with respect to "time," we may describe the continuous analogue of the iterative process described in the previous paragraph by the following equations:

\[
\begin{align*}
(3.7) & \quad \dot{x}_i = E_i + uG_i \\
& \quad \dot{u} = -c
\end{align*}
\]

when the variables \( x_1 \) and \( u \) are positive and

\[
(3.8) \quad \dot{x}_i = 0 \\
& \quad \dot{u} = 0
\]

when the variables themselves equal zero.

The smooth convergence of this process is guaranteed by the following theorem of Hirofumi Uzawa.\(^{14}\) Let \( L(x, u) \) be strictly concave and analytic in the non-negative variables \( x_1, \ldots, x_n \), and convex and analytic in \( u \geq 0 \). Then there is a unique solution of the gradient equations

\[
\begin{align*}
(3.7) & \quad \dot{x}_i = d_{x_i} L_{x_i} \\
& \quad \dot{u} = -d_{u} L_u
\end{align*}
\]

where \( d_{x_i} = 1 \) unless \( x_i = 0 \) and \( L_{x_i} < 0 \), in which case \( d_{x_i} = 0 \), and where \( d_{u} = 1 \) unless \( u = 0 \) and \( L_{u} > 0 \), in which case \( d_{u} = 0 \). This is true for any non-negative set of starting points, \( x_1^0, \ldots, x_n^0 \) and \( u^0 \). Moreover, if \( L(x, u) \) possesses a saddle point under the constraints \( x_i \geq 0 \) and \( u \geq 0 \), the \( x \)-component, \( x_i(t) \), of the solution converges to the \( x \)-component of the saddle point as \( t \to +\infty \).

The appropriate properties of concavity and convexity and the existence of a unique saddle point have been assumed or guaranteed by the
theorem cited in Part II. An important feature of the gradient process from the point of view of the problem under discussion is that it depends entirely upon the concavity of $E$ and $G$ and upon their derivatives. All that is required in order to move towards the optimum is an evaluation of the signs of the derivatives of $L(x,u)$. A complete specification of the $E$ function is not necessary for the application of the logic of maximization.

IV. Application of the Analysis

The theory of maximization of a continuous concave function has a natural application in the personnel problem. Unlike some of the inputs which go into our military establishment, personnel is finely divisible. It is a practical possibility for a service to add a few men in one category and subtract a few from another. Thus, the continuity assumption fits the problem well. The appropriateness of the assumption of concavity is less obvious, but there is good reason to believe that it too fits the problem. Personnel inputs do not appear to exhibit increasing marginal returns or strong positive interactions, at least in the range with which we are concerned.

There are significant differences in supply conditions in different categories of military manpower. Reenlistment rates in the technical specialties are far below those in the non-technical areas. A sample of Air Force first-term reenlistment rates for fiscal 1955 and for the first eight months of fiscal 1957 is shown in Table I. The differences persist in spite of the fact that skilled technicians already have advancement opportunities in the Air Force which are superior to those of the unskilled airmen. These facts are part of the basis for an important
conclusion: an efficient wage policy must take advantage of supply and
demand differences.

Table I
Air Force First-Term Reenlistment Rates by Specialty

<table>
<thead>
<tr>
<th>Fiscal Year</th>
<th>1955</th>
<th>1957</th>
</tr>
</thead>
<tbody>
<tr>
<td>Technical:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Radio-radar system maintenance</td>
<td>7.0</td>
<td>22.7</td>
</tr>
<tr>
<td>Pilotless aircraft control and guidance</td>
<td>5.1</td>
<td>28.8</td>
</tr>
<tr>
<td>Armament system maintenance</td>
<td>7.1</td>
<td>31.8</td>
</tr>
<tr>
<td>Atomic weapons</td>
<td>4.7</td>
<td>21.5</td>
</tr>
<tr>
<td>Non-Technical:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Production control</td>
<td>27.3</td>
<td>41.8</td>
</tr>
<tr>
<td>Firefighting</td>
<td>19.9</td>
<td>39.6</td>
</tr>
<tr>
<td>Transportation</td>
<td>19.1</td>
<td>34.9</td>
</tr>
<tr>
<td>Food service</td>
<td>23.8</td>
<td>38.1</td>
</tr>
<tr>
<td>Over-all</td>
<td>14.5</td>
<td>33.1</td>
</tr>
</tbody>
</table>

Sources: Study of Airpower, op. cit., Part IV, pp. 345-6. The
U. S. Air Force Personnel Report, Military Personnel Retention,

The necessary conditions for an optimum allocation of the personnel
budget, (2.21) or (2.29), provide us with the rationale for salary
differentials for different specialties. It is obvious that at
substantially equal pay scales, marginal products and marginal costs differ
among different specialties. Whereas enough men reenlist in some
unskilled areas, a very small fraction of those in the highly-technical
skills choose to remain in the service beyond their first enlistment.
Consequently the services claim that they face a severe shortage of
skilled technicians.
What is the meaning of a "manpower shortage" in this context? Surely the statements that there is a shortage would be rather trivial if they only meant that the services could make good use of more of these men. In that sense, we all face shortages. If this were the meaning of the statements about shortages, then they would in truth be complaints about budget levels. With greater personnel budgets it is obvious that the services could accomplish more. The statements about shortages of skilled technicians have a more interesting meaning: at the current rates of pay, one extra technician would be more valuable to the services, in proportion to the costs of hiring him, than one extra unskilled man. But this sort of shortage can be alleviated by offering more money to attract more technicians and then effecting offsetting savings by reducing the number of men in areas which are relatively oversupplied. Only in very special and rare circumstances would it be most efficient for the services to offer the same salary scales in all specialties and obviously these conditions do not exist today.

The military services have at their disposal a variety of methods for balancing marginal products with marginal costs. For example, they could make more use of rank, gradually upgrading men in categories in which there is a relative scarcity and promoting fewer in areas in which there is relative over-supply. Such possibilities as more extensive use of the rank of warrant officer should be investigated. Since fringe benefits are a part of the marginal costs of hiring men as well as being a feature which attracts them, they should also be adjusted in such a way as to maximize military effectiveness. This can be done on the same principles as salary adjustments. However, it would be quite difficult administratively
to provide differential fringe benefits not based upon rank and pay. Hence it is likely that pay will remain the dominant variable from this point of view. The mathematical analysis has shown that these variables can be used purposefully to achieve the objectives of the services and it has provided us with a guide to the extent to which differentials should be used.

There are, nevertheless, limitations upon the analysis as a guide to policy and these limitations may indicate some modification of our conclusions in practice. The method used in this paper is static. It does not consider optimum behavior in the face of changing conditions. Development of adequate numbers of some kinds of manpower requires a long time. Further, over time, personnel requirements change as do civilian demands. Consequently, current policies should be based in part upon dynamic and long run considerations. If requirements in some areas will increase over the next decade, then optimum salary scales in those areas will be greater than if requirements will remain constant. However, these considerations do not affect seriously the validity of our conclusions. In practice, they are likely to indicate the desirability of only small adjustments of the static optimum position.

A few practical examples will serve to illustrate the relevance of this logic to the problems which the services face. The Cordier Committee reports that all of the services are experiencing a shortage of officers with between four and fourteen years of service. In fiscal 1956, the combined deficit was estimated to be approximately 37,000 officers. In order to meet numerical requirements, the services have had to take in more junior obligated officers than they would have otherwise preferred to do.
As a result, there is a surplus of a comparable order of magnitude of officers with less than four years' experience. Considering the costs of training the younger men and the losses which result from their inexperience, it would be fair to estimate that the costs to the services of maintaining officers in either category are approximately the same. Nevertheless, the fact that a "shortage" exists in one category and a "surplus" in the other indicates that, with the current force composition, the marginal product of extra officers in the more experienced group is substantially greater than it is in the junior group. Thus our logic applies. If the services were allowed to increase the salaries of officers with four or more years of experience, thereby attracting more of them, they would be able to save an offsetting amount by taking in fewer junior officers. An increase in total effectiveness would result with no increase in total cost.

To illustrate further, the Committee reports that among men completing their first enlistment in 1956, there were wide divergences in reenlistment rates between men in technical skills and unskilled men. In principle, the men in both categories are now paid according to the same schedules, although in fact advancement opportunities are somewhat better for the men with technical skills. However, only 13 per cent of the men trained in electronics reenlisted. The services have estimated the optimum sustaining rate for men in this area to be about 33 per cent.\(^{16}\) On the other hand, the reenlistment rate for vehicle drivers, military police and men in food service was slightly in excess of 25 per cent, and just above the optimum sustaining rate.\(^{17}\) This problem and its cure are analogous to the shortage of experienced officers. It is clear that the marginal product of one more
electronics expert would far exceed the marginal product of one more vehicle driver. Nevertheless, the services are required to offer approximately the same salary scales in both categories. If the services were free to offer more money to men in electronics, it is clear that a reallocation of funds could again increase over-all effectiveness.

There are practical problems involved in this prescription. Reducing the salaries of men already in the services during their enlistment is difficult and undesirable in that they are not free to change jobs. Consequently, it would be unrealistic to advocate a reduction in the salaries of individual men whose skills are in oversupply. Also, one must be cautious about recommending a salary increase on the basis of supply and productivity conditions which may be temporary because it may have to be reversed. These difficulties need not be serious. First of all, periodic salary revisions can be accompanied by a "savings clause" stipulating that no individual is to have his earnings reduced. The Cordiner recommendations include such a clause. Within a few years of the downward revision, new men will have replaced the old and the pay cut will have become effective for most of the men in the category. Secondly, the problem of salary rigidity could be solved by a system of flexible reenlistment bonuses. For example, the Congress could provide each service with a budget for bonuses which could be varied, within limits, by administrative decision. Accordingly, each year, each service could review its experience and decide upon appropriate readjustments in the allocation of bonuses among different groups of men. The allocation of the bonus budget could be carried out on the basis of the principles explained in Part II of this paper.
As a first example, we may compare the marginal costs of training products to the services of skilled men and career service in the technical specialties. To take a specific example, the Air Force spends about $29,000 on the training and maintaining of a bombing system repairman during his first four-year enlistment. In exchange, the man provides the Air Force with a year and a half of fully productive service. Only about a quarter of these men return for a second enlistment. The Air Force has paid for the skilled work of those who do not reenlist at an annual rate of $15,000. On the other hand, the men who reenlist provide nearly four full years of productive service. Further, their added experience makes their services more valuable, per hour, than those of the first termers. The Air Force pays approximately $11,000 a year on the average for the services of these men, about half of which is salary. Thus we have shown directly an inequality in the respective ratios of marginal product to marginal cost. At current rates of pay, the last man to enlist as a bombing system repairman costs the Air Force about three times as much, per year of productive services, as the last man to reenlist for a second term. On the other hand, because of his inexperience, his services were less valuable, per year, than those of the career man. Simple calculations can be made to show that even a large bonus offered to these technicians to induce them to reenlist would result in a net saving for the service.
competitive, it can be shown that the policy of making marginal products proportional to marginal costs, instead of to prices, would lead to an inefficient allocation of resources. Even though the services face rising supply prices, they could continue to follow proportionality conditions of the form (2.12) or (2.21). The latter policy would not lead to as high a level of military effectiveness, for any given budget, but it would lead to a more efficient use of resources. At a given real cost, the economy would be able to give the service a sufficiently higher budget that it could, in turn, reach a higher level of effectiveness. However, in fact, our economy is not perfectly competitive and it is not clear which of the two policies would lead to a greater total real output. In practice, it probably does not make much difference.

11. There would be one first order maximum condition corresponding to each \( x_i \) and one constraint corresponding to each Lagrange multiplier, whence an equal number of equations and unknowns. Of course, an equal number of independent equations and unknowns is necessary and sufficient for a unique solution only in the case of linear equations. However, the existence and uniqueness of the maximum would indicate that at least in ordinary cases the equations would possess a unique solution.

12. It is of course assumed that the entire budget is spent in all cases.

13. A "small increase" in this sense is defined as any increase which does not destroy the inequality (3.1). Of course, any increase in \( x_b \) will tend to decrease \( P_b \) and increase \( MC_b \), and conversely for a decrease in \( x_a \). The argument is only valid, strictly speaking, in the neighborhood of the point \( x^0 \). It indicates the steepest gradient of the constrained effectiveness function at that point.


16. Ibid., pp. 47-51.

17. Loc. cit.

18. Career men are those who have reenlisted at least once.