ON GAMES OF TIMING

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Summary.

A symmetric game of timing is defined here as a continuous game involving the bilinear functional

$$\int_0^1 \int_0^1 K(x, y)dF(x)dG(y), \qquad K(x, y) = -K(y, x)$$

in which for $x < y$, $K(x, y)$ is a strictly increasing function of x and a strictly decreasing function of y (more precisely defined in §1). If $K(1^-, 1) \leq 0$, there is an optimum pure strategy at 1; if $K(0, 1) \geq 0$, there is an optimum pure strategy at 0. Aside from these trivial cases, it is proved that there is a unique optimal strategy which is either a density from some point a to 1, or is a jump at 0 and a density from a to 1. If the quantity $K(y^-, y)$ varies in sign as y varies, let b be the value such that $K(b^-, b) = 0$, while $K(y^-, y) > 0$ for $b < y \leq 1$. In this case, the optimal strategy is a density from a to 1 where $a > b$. It is shown that the determination of the density function depends on the solution of a certain integral equation with positive kernel, and this is equivalent in a general category of cases to a system of linear differential equations.

1. Preliminaries.

We shall consider a class of symmetric continuous games involving the bilinear functional

$$\int_0^1 \int_0^1 K(x, y)dF(x)dG(y)$$

where x, y range over the real numbers from 0 to 1 inclusive, the symmetry of the game reflecting itself in the skew-symmetry of the kernel K(x, y):

$$K(x, y) = -K(y, x).$$

Concerning the kernel K(x, y), we will suppose that for $x < y$ K(x, y) is a strictly increasing function of x and a strictly decreasing function of y. This property likewise holds for $x > y$ by virtue of the skew-symmetry of K(x, y). But across the main diagonal x = y this property may cease, i.e., there may be a jump of K(x, y) and K(a+b, a) may be smaller than K(a-b, a) for small positive b's.

We shall call such a game a game of timing, by virtue of the following important interpretation. The variables x and y may represent the times at which the players I and II take certain specific actions; and it is profitable for each player to delay action as long as possible, provided his action is prior to his opponent's action. If the times x, y at which players I, II take action are near each other, there is a decided difference in the outcome according as $x < y$ or $x > y$. Each player is thus subject to the following motive: he wishes to delay action so as to increase his reward, but at the same time not to delay so long that his opponent can with effectiveness precede him. An important example of a game of timing is a duel between two opponents.

To be specific, a game will be called a symmetric game of timing if the kernel $K(x, y)$ satisfies the following conditions:

(1 a) $$K(x, y) = \begin{cases} A(x, y) & \text{for } x < y \\ 0 & \text{for } x = y \\ -A(y, x) & \text{for } x > y \end{cases}$$

where $A(x, y)$ is continuous in $x \leq y$.

(1 b) $A(x, y)$ is a strictly increasing function of x and a strictly decreasing function of y.

In what follows, we shall make the following additional hypothesis:

(1 c) $A(x, y)$ has continuous first derivatives in $x \leq y$, and the set of points where $A_x(x, y) = 0$ or $A_y(x, y) = 0$ contains no linear intervals $x = \text{constant}$, $\beta_1 < y < \beta_2$ or $y = \text{constant}$, $\alpha_1 < x < \alpha_2$.*

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* The reader may if desired make the further simplification $A_x(x, y) > 0$, $A_y(x, y) < 0$ for $x < y$.

As a consequence of (1 b), we can assert that

$$A_x(x, y) \geq 0, \quad A_y(x, y) \leq 0 \qquad \text{for } x \leq y.$$

The condition (1 c) makes a mild limitation on the places where either of these derivatives is zero.

We shall show that, aside from trivial cases, the optimal strategy of a game of delayed timing is unique and consists either of a density function from some point a to 1, or consists of a jump at 0 and a density from some point a to 1. The optimum strategy will be obtained as the solution of a certain integral equation with a positive kernel. In a wide category of cases, this integral equation is equivalent to a certain linear differential equation or system of linear first order differential equations.

2. General conditions on an optimum strategy.

It is easy to show the following: if $A(1, 1) \leq 0$, a pure strategy at 1 is the unique optimum strategy; if $A(0, 1) \geq 0$, a pure strategy at 0 is the unique optimum strategy. We exclude these trivial cases and suppose henceforth that

$$(2.1) \qquad A(0, 1) < 0, \qquad A(1, 1) > 0.$$

We shall first suppose that there is an optimum strategy F(x) for the game, and derive necessary conditions satisfied by F(x). Thus

$$(2.2) \qquad V(y) = \int_0^1 K(x, y)dF(x) \geq 0 \quad \text{for all } y,$$

while

$$(2.3) \qquad \int_0^1 V(y)dF(y) = \int_0^1 \int_0^1 K(x, y)dF(x)dF(y) = 0$$

by the skew-symmetry of $K(x, y)$.

Lemma 1. $F(x)$ _cannot have a jump at an interior point_ x_0, $0 < x_0 < 1$, _unless_ $A(x_0, x_0) = 0$. $F(x)$ _does not have a jump at_ $x = 1$.

Proof. If $F(x)$ had a jump at such a point x_0, $0 < x_0 < 1$, we would have

$$V(x_0) = \int_0^1 K(x, x_0)dF(x) = 0$$

by virtue of (2.2), (2.3). Letting $y \longrightarrow x_0^{\pm}$ in (2.1),

$$V(x_0^{\pm}) = \int_0^1 K(x, x_0^{\pm})dF(x) \geq 0,$$

so that

$$\int \left\{K(x, x_0^{\pm}) - K(x, x_0)\right\} dF(x) \geq 0.$$

But $K(x, x_0^{\pm}) - K(x, x_0) = 0$ except for $x = x_0$, and we have

$$\pm A(x_0, x_0) \left\{F(x_0^{+}) - F(x_0^{-})\right\} \geq 0.$$

The first part of the lemma is proved.

The above argument also shows that the following statement can be made at the end-points 0, 1:

(2.4) $A(0, 0)F(0^+) \geq 0, \qquad A(1, 1)\left(1 - F(1^-)\right) \leq 0.$

Because of $A(1, 1) > 0$, the second part of the lemma is proved.

The quantity $V(y)$ _is continuous except at_ $y = 0$ if $A(0, 0)F(0^+) \neq 0$. For, if $x_0 \neq 0$,

$$V(x_0^{\pm}) - V(x_0) = \int \left\{K(x, x_0^{\pm}) - K(x, x_0)\right\} dF(x)$$

$$= \pm A(x_0, x_0)\left\{F(x_0^+) - F(x_0^-)\right\} = 0,$$

while

$$V(0^+) - V(0) = A(0, 0)F(0^+) \geq 0.$$

It follows from (2.1) and (2.2) that

(2.5) $V(y) = 0$ _for all points_ y _in the spectrum of_ $F(x)$.

Lemma 2. _The spectrum of_ $F(x)$ _either is an interval from_ a _to_ 1, _or is the point_ 0 _and an interval from_ a _to_ 1.

Proof. Denote the closed point set which is the spectrum of $F(x)$ by S. If S is not the entire set from 0 to 1, let $p < y < q$ be one of the open intervals composing the complement of S. For any point y in this interval,

$$V(y) = \int_0^{p^+} A(x, y)dF(x) - \int_{q^-}^1 A(y, x)dF(x),$$

so that, by hypothesis (1 b), V(y) <u>is a strictly decreasing function</u> of y in this interval. By virtue of (2.1) and (2.4), the left-hand end-point p of the interval must be 0. Otherwise p would belong to S, $V(p) = 0$, and by the continuity of V(y) at p, we would then have $V(p + \delta) < 0$ for small positive δ. This contradicts (2.2), so that $p = 0$. The lemma is proved.

The hypothesis (1 b) makes no assertion concerning the behavior of A(x, x), except that for x near 1 we have $A(x, x) > 0$ by (2.1). If there are points x on the main diagonal where $A(x, x) = 0$, let b be the maximum of all such points x. Thus $A(b, b) = 0$, while for $b < x \leq 1$ we have $A(x, x) > 0$. If there are no points x where $A(x, x) = 0$, we set $b = 0$. The interval $b \leq x \leq 1$ will be called the <u>basic interval</u>, by virtue of the following lemma.

Lemma 3. <u>The spectrum of</u> F(x) <u>lies completely in the basic interval</u> $b \leq x \leq 1$.

Proof. This lemma makes an assertion only if $b > 0$. Consider a new game in which the pay-off is K(x, y) but in which x, y are limited to the interval $b \leq x \leq 1$, $b \leq y \leq 1$. Let the solution to this game be $\phi(x)$, $b \leq x \leq 1$, with $\phi(1) = 1$, $\phi(b) = 0$. Extend $\phi(x)$ below b by setting $\phi(x) = 0$, for $x < b$. Set

$$\overline{V}(y) = \int_0^1 K(x, y)d\phi(x).$$

We have: $\overline{V}(y) \geq 0$ for $b \leq y \leq 1$ since $\emptyset(x)$ is the solution to the game in the interval $b \leq x \leq 1$. We have

$$\overline{V}(b^-) - \overline{V}(b) = \int_0^1 \left\{K(x, b^-) - K(x, b)\right\} d\emptyset(x) = 0$$

since $K(b, b) = 0$. Thus $\overline{V}(b^-) = \overline{V}(b) \geq 0$. In the interval from 0 to b, $\overline{V}(y)$ is a strictly decreasing function of y, so that we may assert that

$$\overline{V}(y) > 0 \quad \text{in} \quad 0 \leq y < b.$$

Suppose now that there were points of the spectrum of F(x) in the interval $0 \leq x < b$. Then we would have

$$\int_0^1 \overline{V}(y) dF(y) > 0.$$

But $\int \overline{V}(y) dF(y) = \int_0^1 \int_0^1 K(x, y) d\emptyset(x) dF(y) = -\int_0^1 \int_0^1 K(x, y) dF(x) d\emptyset(y)$

$= -\int_0^1 V(y) d\emptyset(y) \leq 0$ since $V(y) \geq 0$. This contradiction establishes lemma 3.

Lemma 3 shows that we need only consider the game over the basic interval $b \leq x \leq 1$. Henceforth, we consider only the basic interval, and ignore the values before this basic interval. With no loss of generality, we can let this basic interval be the interval from 0 to 1, so that we have

$$(2.6) \qquad A(x, x) > 0 \quad \text{for} \quad 0 < x \leq 1.$$

Lemma 4. <u>In</u> $a < x \leq 1$, $F(x)$ <u>has a continuous derivative</u> $F'(x) = f(x)$.

Proof. Set $F(0^+) = \alpha$, $0 \leq \alpha < 1$. Let y be any point in $a < y \leq 1$. Then (2.5) gives

$$(2.7) \qquad 0 = \alpha A(0, y) + \int_{a^+}^{y} A(x, y)dF(x) - \int_{y}^{1} A(y, x)dF(x).$$

Suppose first that $A(x, y)$ has continuous second derivatives. Then transform (2.7) by integration by parts into

$$0 = \alpha A(0, y) - \alpha A(a, y) - A(y, 1) + 2F(y)A(y, y)$$

$$- \int_{a^+}^{y} F(x)A_1(x, y)dx + \int_{y}^{1} F(x)A_2(y, x)dx$$

where the subscript 1, 2 means differentiation with respect to the first or second argument respectively. This equation can be solved for $F(y)$, since $A(y, y) \neq 0$, and shows that $F(y)$ has a continuous derivative in $a < y \leq 1$.

If $A(x, y)$ does not have continuous second derivatives, we can proceed directly from (2.7). Rewrite (2.7) in the form

$$0 = \alpha A(0, y) + \int_{a^+}^{y} \left\{A(x, y) - A(y, y)\right\} dF(x)$$

$$- \int_{y}^{1} \left\{A(y, x) - A(y, y)\right\} dF(x) + A(y, y)\left\{2F(y) - \alpha - 1\right\}$$

and solve for F(y). The result shows that F(y) has a continuous derivative with respect to y by virtue of the following remarks:

If B(x, y) is continuous and has continuous first derivatives, if B(y, y) = 0, and if F(x) is a continuous distribution function, then

$$\int_a^y B(x, y)dF(x)$$

has a continuous derivative with respect to y. For

$$\frac{1}{h}\left\{\int_a^{y+h} B(x, y+h)dF(x) - \int_a^y B(x, y)dF(x)\right\} =$$

$$\int_a^y \frac{B(x, y+h) - B(x, y)}{h} dF(x) + \int_y^{y+h} \frac{B(x, y+h)}{h} dF(x).$$

The first integral approaches $\int_a^y B_y(x, y)dF(x)$ as $h \rightarrow 0$, while the second integral can be estimated by

$$\left|\int_y^{y+h} \frac{B(x, y+h) - B(y+h, y+h)}{h} dF(x)\right| \leq M\left|F(y+h) - F(y)\right| ,$$

where M is a bound in the first derivatives, and this approaches zero as $h \rightarrow 0$. Thus the derivative exists and

$$\frac{d}{dy}\int_a^y B(x, y)dF(x) = \int_a^y B_y(x, y)dF(x).$$

This shows that the derivative is continuous.

The lemma is proved.

3. The integral equation for f(x).

In (2.5), set $F'(x) = f(x)$:

$$(3.1) \qquad 0 = \alpha A(0,y) + \int_a^y A(x,y)f(x)dx - \int_y^1 A(y,x)f(x)dx, \qquad a \leq y \leq 1.$$

Differentiation of (3.1) gives

$$(3.2) \qquad f(y) = \alpha p(y) + \int_a^1 L(x, y)f(x)dx, \qquad a \leq y \leq 1,$$

where

$$(3.3) \qquad p(y) = -\frac{A_2(0, y)}{2A(y, y)}, \qquad \alpha \geq 0,$$

and

$$(3.4) \qquad L(x, y) = \begin{cases} -\dfrac{A_2(x, y)}{2A(y, y)} & \text{for} \quad x < y \\[2ex] \dfrac{A_1(y, x)}{2A(y, y)} & \text{for} \quad x > y \end{cases}$$

as the fundamental integral equation which f(y) satisfies.

The kernel $L(x, y)$ and the function $\alpha p(y)$ are everywhere ≥ 0 by virtue of hypothesis (1 b) and equation (2.6). Also, in case $A(0, 0) = 0$, we see from (3.1) that $a > 0$; otherwise, by letting $y \longrightarrow 0$ in (3.1) we would obtain a contradiction. Thus, $A(y, y) > 0$ everywhere in the closed interval $a \leq y \leq 1$, and the functions

$L(x, y)$, $\alpha p(y)$ are uniformly bounded in the interval of integration.

Equation (3.2) is the fundamental integral equation satisfied by $f(x)$. But we must also have

$$(3.5) \qquad \int_a^1 f(x)dx = 1 - \alpha$$

and, setting

$$(3.6) \qquad W = \alpha A(0,y) + \int_a^y A(x,y)f(x)dx - \int_y^1 A(y,x)f(x)dx, \qquad a \leq y \leq 1,$$

which must be constant for $a \leq y \leq 1$ by virtue of (3.2), we must have

$$(3.7) \qquad W = 0.$$

Conversely, if for some α and a, $0 \leq \alpha < 1$, $0 \leq a < 1$, there is found a non-negative function $f(x)$ satisfying (3.2), (3.5), (3.7), then

$$(3.8) \qquad F(x) = \begin{cases} 0 & \text{for } x = 0 \\ \alpha & \text{for } 0 < x < a, \\ \alpha + \int_a^x f(\xi)d\xi & \text{for } a < x \leq 1 \end{cases}$$

is an optimal strategy. For, from (3.7), it follows by multiplying (3.6) by $f(y)$ and integrating from a to 1 that

$$(3.9) \qquad 0 = \alpha \int_a^1 A(0, y)f(y)dy.$$

Then we have, for V(y) defined in (2.1),

$$V(y) = \begin{cases} 0 \text{ for } y = 0 \text{ in case } \alpha \neq 0 \\ \text{positive and monotonic decreasing for } 0 < y < a \\ 0 \text{ for } a \leq y \leq 1 \end{cases}$$

Therefore, V(y) satisfies the inequality

$$V(y) \geq 0$$

and this means that F(x) is an optimal strategy. The following theorem has been demonstrated:

Theorem. *There is an optimal strategy* F(x) *if and only if there are numbers* α, a, $0 \leq \alpha < 1$, $0 \leq a < 1$ *and a non-negative function* f(x), *defined in* $a \leq x \leq 1$, *which satisfies the equations* (3.2), (3.5), (3.7). *Then* F(x) *is given in* (3.8).

4. On integral equations with a positive kernel.*

Consider the homogeneous integral equation of the second kind,

$$(4.1) \qquad \int_p^q L(x, y)f(y)dy = \lambda f(x), \qquad p \leq x \leq q,$$

or, in operator form

$$\mathcal{L} f(x) = \lambda f(x)$$

* The theorems obtained in § 4 are related to similar questions studied by Frobenius for finite matrices, and to unpublished work of Bohnenblust and Karlin.

where $\mathcal{L}f(x)$ is the integral operation on the left-hand side of (4.1). We will suppose that the kernel $L(x, y)$ is <u>non-negative</u>, and we are interested in non-negative solution $f(x)$ of (4.1), i.e.,

$$(4.2) \qquad f(x) \geq 0, \qquad 0 < \int f(x)dx < \infty,$$

(throughout this digression, all integral signs will be understood between the fixed limits p, q). The class of functions satisfying (4.2) will be denoted by P_1. To be specific, the assumptions on the kernel $L(x, y)$ are as follows:

$$(4\ a) \qquad L(x, y) \geq 0,$$

the points where $L = 0$ forming on each line x = constant or line y = constant a point set containing no intervals.

$$(4\ b) \qquad L(x, y) \leq M$$

$$(4\ c) \qquad \int \left| L(x_2, y) - L(x_1, y) \right| dy \longrightarrow 0 \quad \text{and}$$

$$\int \left| L(x, y_2) - L(x, y_1) \right| dx \longrightarrow 0 \quad \text{if}$$

$$|x_2 - x_1| \longrightarrow 0 \quad \text{or} \quad |y_2 - y_1| \longrightarrow 0 \text{ respectively.}$$

Weaker assumptions can be made if desired. It follows from (4 a), (4 b), (4 c) that if $f(y)$ is in P_1, then

$$\int L(x, y)f(x)dx \quad \text{and} \quad \int L(x, y)f(y)dy$$

are continuous functions of y and x respectively, and if f(y) is in addition continuous they have a positive minimum.

We shall prove the following theorems.

1. There exists a unique f(x) in P_1, normalized by $\int f(x)dx = 1$, satisfying (4.1) for some positive λ.

2. If μ is a positive number such that

$$\int L(x, y)g(y)dy \geq \mu g(x), \qquad a \leq x \leq b$$

for some g(x) in P_1, then $\mu \leq \lambda$ and $\mu = \lambda$ only if g(x) is a multiple of f(x). (Maximum property of the eigenvalue λ.)

3. The inhomogenous equation

$$\phi(x) = h(x) + \rho \int L(x, y)\phi(y)dy \tag{4.4}$$

where h(x) is a given function in P_1 and $\rho > 0$ has a solution $\phi(x)$ in P_1 if and only if $\rho < \frac{1}{\lambda}$. The solution is then unique and is given by (4.5) $\phi(x) = h(x) + \sum_{\nu=1}^{\infty} \rho^{\nu} \mathcal{L}^{\nu} h(x)$, where $\mathcal{L}^{\nu}$ means the ν^{th} iterate of $\mathcal{L}$.

4. If ψ is absolutely integrable, and

$$\psi(x) \leq h(x) + \rho \int L(x, y)\psi(y)dy,$$

then

$$\psi(x) \leq \phi(x)$$

everywhere.

Proof: (1) and (2). Consider the set Ω of μ's satisfying (4.3) for some $g(x)$ in P_1. Setting $g(x) \equiv 1$, shows that there are positive μ's in Ω, namely $\mu = \underset{x}{\text{Min}} \int L(x, y)dy$. Also, integrating (4.3) with respect to x and using condition (b) shows that

$$\mu \leq M.$$

Normalizing the functions $g(x)$ in (4.3) by supposing $\int g(x)dx = 1$, we see that

$$g(x) \leq \frac{M}{\mu} \text{ for every } x.$$

Let $\lambda = \underset{\mu \text{ in } \Omega}{\text{Sup}} \mu$. There is a sequence $\mu_i \longrightarrow \lambda$ with $g_i(x)$, normalized, satisfying

$$\mathcal{L} g_i(x) \geq \mu_i g_i(x). \tag{4.6}$$

But then also,

$$\mathcal{L}\left(\mathcal{L} g_i(x)\right) \geq \mu_i \left(\mathcal{L} g_i(x)\right) . \tag{4.7}$$

The functions $\mathcal{L} g_i(x)$ are positive continuous functions, and are equicontinuous. For,

$$\left|\int L(x_1, y)g_i(y)dy - \int L(x_2, y)g_i(y)dy\right| \leq$$

$$\frac{M}{\mu_i}\int\left|L(x_1, y) - L(x_2, y)\right| dy$$

and equicontinuity follows from hypothesis (4 c) concerning the kernel L(x, y). The theorem of Arzela shows that a subsequence can be found such that $\mathcal{L}g_i(x)$ converge uniformly to a continuous non-negative f(x). From (4.7) we find

$$\mathcal{L}f(x) \geq \lambda f(x), \tag{4.8}$$

while integrating (4.6) shows that $\int f(x)dx \geq \lambda$, so that f(x) is not identically 0. Thus λ belongs to the set Λ. But in fact, equality holds in (4.8). For, both sides of (4.8) are continuous functions, and a strict inequality at any point x would show that

$$\mathcal{L}\big(\mathcal{L}f(x) - \lambda f(x)\big) \geq \delta > 0$$

for some positive δ. But then

$$\mathcal{L}\big(\mathcal{L}f(x)\big) \geq \lambda\mathcal{L}f(x) + \delta \geq \left(\lambda + \frac{\delta}{N}\right)\mathcal{L}f(x)$$

where $N = \max_x \mathcal{L}f(x)$. This shows that $\lambda + \frac{\delta}{N}$ belongs to the set Λ, contrary to the definition of λ. Therefore

$$\mathcal{L}f(x) = \lambda f(x) \tag{4.9}$$

and Theorems (1) and (2) have been established, except for the uniqueness of f(x).

The equality (4.9) shows that f(x) has a positive minimum. The above establishes the existence of a solution g(y) in P_1 to the <u>transposed</u> equation

$$\int L(x, y)g(x)dx = \lambda' g(y).$$

The function g(y) is also continuous and has a positive minimum. Multiplying this equation by f(y) and integrating with respect to y yields

$$\lambda \int f(x)g(x)dx = \lambda' \int f(y)g(y)dy$$

or $\lambda = \lambda'$ since $\int f(x)g(x)dx > 0$. This shows that there cannot be a solution in P_1 of (4.1) for any other value of λ. Suppose now that there is another solution h(x) of (4.9) in P_1. Let

$$c = \operatorname*{Min}_{x} \frac{h(x)}{f(x)} ,$$

so that $h(x) - cf(x) \geq 0$ and = 0 in at least one point. But since

$$\mathcal{L}\big(h(x) - cf(x)\big) = \lambda\big(h(x) - cf(x)\big)$$

it follows that h(x) - cf(x) has a <u>positive</u> minimum (unless it is identically zero). This shows that, indeed,

$$h(x) \equiv cf(x),$$

and the uniqueness in (1) is established.

Remark. The proof we have given depended on Arzela's theorem. Other proofs and other methods can be given depending on notions of weak convergence and applicable to different classes of kernels $L(x, y)$.

(3). Let (4.4) have a solution in P_1. Let $g(x)$ be the positive eigenfunction for the transposed homogeneous equation (with the same eigenvalue λ). Multiplying by $g(x)$ and integrating gives

$$\int \phi(x)g(x)dx = \int h(x)g(x)dx + \rho\lambda \int \phi(x)g(x)dx$$

or

$$(1 - \rho\lambda)\int \phi(x)g(x)dx = \int h(x)g(x)dx.$$

The integrals appearing are both positive and this requires that

$$1 - \rho\lambda > 0 \quad \text{or} \quad \rho < \frac{1}{\lambda} .$$

For $\rho < \frac{1}{\lambda}$, let $S_n(x)$ be the first n+1 terms of the right-hand side of (4.5),

$$S_n(x) = h(x) + \sum_{\nu=1}^{n} \rho^{\nu} \mathcal{L}^{\nu} h(x).$$

Multiplying by $g(x)$ and integrating gives

$$\int S_n(x)g(x)dx = \int h(x)g(x)dx \cdot \left(1 + \sum_{\nu=1}^{n} \rho^{\nu} \lambda^{\nu}\right)$$

or

$$\lim_{n\to\infty}\int S_n(x)g(x)dx = \frac{1}{1-\rho\lambda}\cdot\int h(x)g(x)dx.$$

The sequence $S_n(x)$ is an <u>increasing</u> sequence of (positive) functions which converges to a function $\phi(x)$ (with $+\infty$ admitted as a value). Since $g(x) > 0$ it follows from a standard theorem that $\phi(x)g(x)$ is integrable (so that ∞ is taken as value at most on a set of measure zero) and

$$\int\phi(x)g(x)dx = \frac{1}{1-\rho\lambda}\int h(x)g(x)dx.$$

But, since $g(x)$ has a positive minimum, also $\phi(x)$ is integrable.

Concerning $S_n(x)$ we know that

$$S_n(x) = h(x) + \rho\mathcal{L}S_{n-1}(x).$$

A passage to the limit yields

$$\phi(x) = h(x) + \rho\mathcal{L}\phi(x),$$

and indeed a solution is obtained to (4.4).

Let $\psi(x)$ be any solution to (4.4), so that

$$\psi(x) - \phi(x) = \rho\int L(x, y)\left(\psi(y) - \phi(y)\right)dy.$$

Therefore

$$\left|\psi(x) - \phi(x)\right| \leq \rho\int L(x, y)\left|\psi(y) - \phi(y)\right| dy.$$

Since $\frac{1}{\rho} > \lambda$, we see that $|\psi(x) - \phi(x)|$ must be a null function, or $\psi(x) = \phi(x)$.

(4). Successive iteration of the inequality gives

$$\psi \leq h + \rho\mathcal{L}h + \cdots + \rho^{n-1}\mathcal{L}^{n-1}h + \rho^n\mathcal{L}^n\psi$$

or

$$\psi \leq \phi + \rho^n\mathcal{L}^n\psi.$$

Now

$$|\rho^n\mathcal{L}^n\psi| \leq \rho^n\mathcal{L}^n|\psi|$$

and this approaches zero almost everywhere, by the argument in the proof of 3 above (convergence of the series (4.5)). Thus, $\psi \leq \phi$ almost everywhere. But the inequality then gives

$$\psi \leq h + \rho\mathcal{L}\phi = \phi$$

so that $\psi \leq \phi$ everywhere. This completes the proof of the entire theorem.

5. The dependence on limits of integration.

Suppose that $L(x, y)$ is defined over some rectangle, say $0 \leq x \leq 1$, $0 \leq y \leq 1$, and satisfies conditions (4 a), (4 b), (4 c) there, and that $p(y)$ is defined in $0 \leq y \leq 1$ and is of class $\mathcal{P}_1$. Consider the integral equations in $\mathcal{P}_1$,

$$(5.1) \qquad \int_a^1 L(x, y) f_a(x)\,dx = \lambda(a) f_a(y), \qquad a \le y \le 1,$$

and

$$(5.2) \qquad \phi_a(y) = p(y) + \int_a^1 L(x, y)\phi_a(x)\,dx, \qquad a \le y \le 1,$$

where $0 \le a < 1$. We shall discuss the dependence of the positive eigenvalue $\lambda(a)$, the non-negative normalized eigenfunction $f_a(x)$, and the solution $\phi_a(x)$ of the inhomogeneous integral equation on the parameter a. A similar consideration applies if the upper limit is varied instead of the lower limit.

Theorem 5.1. *The eigenvalue* $\lambda(a)$ *is a strictly decreasing continuous function of* a *with* $\lambda(a) \longrightarrow 0$ *as* $a \longrightarrow 1$.

Proof: For $a' < a$, define

$$h(x) = \begin{cases} 0 \ , & a \le x < a \\ f_a(x), & a \le x \le 1 \end{cases} .$$

Then

$$\int_{a'}^1 L(x, y) h(x)\,dx = \int_a^1 L(x, y) f_a(x)\,dx = \begin{cases} + \ , & a' \le y < a \\ \lambda(a) f_a(y), & a \le y \le 1 \end{cases}$$

and

$$\int_{a'}^1 L(x, y) h(x)\,dx \ge \lambda(a) h(y),$$

so that

$$\lambda(a') \geq \lambda(a).$$

Furthermore, equality can hold only if $h(y)$ is an eigenfunction for the interval $a' \leq y \leq 1$. But $h(y) = 0$ on a subinterval, and an eigenfunction has a positive minimum. Therefore,

$$\lambda(a') > \lambda(a).$$

Over the smaller interval $a \leq x \leq 1$, use the function $f_{a'}(x)$. We have

$$\int_a^1 L(x, y) f_{a'}(x)dx = \lambda(a') f_{a'}(y) - \int_{a'}^a L(x, y) f_{a'}(x)dx$$

$$\geq \lambda(a') f_{a'}(y) - \frac{M^2}{\lambda(a')}(a - a')$$

$$\geq \left\{\lambda(a') - \frac{M^2}{m\,\lambda(a')}(a - a')\right\} f_{a'}(y)$$

where $m = \min_y f_{a'}(y) > 0$, and we have used

$$|f_{a'}(y)| \leq \frac{M}{\lambda(a')}.$$

Therefore

$$\lambda(a) \geq \lambda(a') - \frac{M^2}{m\,\lambda(a')}(a - a'),$$

and this establishes the continuity of $\lambda(a)$.

By integrating (5.1) with respect to y, we obtain

$$\lambda(a) \leq \operatorname*{Max}_{x} \int_{a}^{1} L(x, y)dy \leq M(1 - a).$$

Hence $\lambda(a) \longrightarrow 0$ as $a \longrightarrow 1$. This completes the desired proof.

Theorem 5.2. <u>The eigenfunction</u> $f_a(x)$, $a \leq x \leq 1$, <u>converges uniformly to</u> $f_{a'}(x)$, $a' \leq x \leq 1$, <u>if</u> $a \longrightarrow a'$.

Proof: From (5.1) we find

$$f_a(y) \leq \frac{M}{\lambda(a)} \quad \text{for all } y \text{ in } a \leq y \leq 1. \tag{5.3}$$

Also,

$$\lambda(a) \left|f_a(y_1) - f_a(y_2)\right| \leq \frac{M}{\lambda(a)} \int \left|L(x, y_1) - L(x, y_2)\right| dx.$$

Thus as long as a is bounded away from 1, the functions $f_a(y)$ are equicontinuous, and as $a \longrightarrow a'$ a subsequence can be found which converges uniformly to a function which must coincide with $f_{a'}(y)$ since equation (5.1) for a' is satisfied by the limit function. Since this is true for every infinite set of a's $\longrightarrow$ a', the theorem is proved.

Theorem 5.3. <u>The function</u> $\phi_a(x)$, $a \leq x \leq 1$, <u>converges uniformly to</u> $\phi_{a'}(x)$ <u>as</u> $a \longrightarrow a'$ <u>if</u> $\lambda(a') < 1$. <u>If</u> $\lambda(a_1) = 1$, <u>then</u>

$$\int_a^1 \phi_a(x)dx \longrightarrow \infty \quad \underline{\text{and}} \quad \frac{\phi_a(x)}{\int_a^1 \phi_a(x)dx}$$

<u>converges uniformly to</u> $f_{a_1}(x)$.

Proof: Let $g_a(x)$ be the eigenfunction solving the transposed equation to (5.1). Then,

$$\int_a^1 \phi_a(y) g_a(y)dy = \frac{1}{1-\lambda(a)} \int_a^1 p(y) g_a(y)dy.$$

If a lies in the range $a_1 + \delta \leq a \leq 1 - \delta$ for some positive δ, then by the uniform convergence of $g_a(y)$ as a varies there is a positive lower bound m of $g_a(y)$, and the same upper bound as $f_a(x)$ in the proof of theorem 5.2:

$$0 < m \leq g_a(y) \leq \frac{M}{\lambda(a)} \quad .$$

Therefore

$$(5.4) \qquad \frac{m\,\lambda(a)}{M\left(1-\lambda(a)\right)} \int_a^1 p(y)dy \leq \int_a^1 \phi_a(y)dy \leq \frac{M}{m\,\lambda(a)\left(1-\lambda(a)\right)} \int_a^1 p(y)dy.$$

Equation (5.2) gives

$$\phi_a(y) \leq p(y) + \frac{M^2}{m\,\lambda(a)\left(1-\lambda(a)\right)} \int_a^1 p(y)dy \leq M_1$$

and also

$$\left|\phi_a(y_1) - \phi_a(y_2)\right| \leq \left|p(y_2) - p(y_1)\right| + M_1 \int_a^1 \left|L(x, y_2) - L(x, y_1)\right| dx.$$

This proves the equicontinuity of $\phi_a(y)$, and as in the proof of theorem 5.2 the uniform convergence of $\phi_a(y)$ to $\phi_{a'}(y)$ as $a \longrightarrow a'$.

If $a \to a_1$, where $\lambda(a_1) = 1$, the unboundedness of $\int_a^1 \phi_a(x)dx$ is stated in (5.4). Also,

$$\frac{\phi_a(y)}{\int_a^1 \phi_a(x)dx} \leq \frac{p(y)}{\int_a^1 \phi_a(x)dx} + M \leq M_2,$$

and

$$\frac{|\phi_a(y_2) - \phi_a(y_1)|}{\int_a^1 \phi_a(x)dx} \leq \frac{|p(y_2) - p(y_1)|}{\int_a^1 \phi_a(x)dx} + M_2 \int_a^1 |L(x, y_2) - L(x, y_1)| dx.$$

Again this establishes the uniform convergence of

$$\frac{\phi_a(x)}{\int_a^1 \phi_a(x)dx}$$

to the solution $f_{a_1}(x)$ of the homogeneous equation.

It also follows from theorem 4 of §4 that $\phi_a(y) \geq \phi_{a'}(y)$ if $a < a'$.

Theorem 5.4. <u>For</u> $a_1 < a < 1$, <u>there is a non-negative function</u> $\psi_a(x)$, $a \leq x \leq 1$ <u>such that</u>

$$\int_{\bar{a}}^1 \left| - \frac{\phi_{a+h}(x) - \phi_a(x)}{h\, \phi_a(a)} - \psi_a(x) \right| dx \to 0 \quad \text{as } h \to 0,$$

<u>where</u> $\bar{a}$ <u>is the maximum of</u> a, $a+h$. <u>This function</u> $\psi_a(x)$ <u>satisfies the integral equation</u>

$$(5.5) \qquad \psi_a(y) = L(a, y) + \int_a^1 L(x, y)\,\psi_a(x)\,dx, \qquad a \le y \le 1.$$

<u>If</u>

$$\frac{1}{h}\int_a^{a+h} L(x, y)\,dx \longrightarrow L(a, y) \quad \underline{\text{as}} \quad h \longrightarrow 0 \quad \underline{\text{for each}}\ y,$$

<u>then</u> $\frac{\partial}{\partial a}\phi_a(x)$ <u>exists</u> <u>and</u>

$$\psi_a(x) = -\frac{1}{\phi_a(a)}\,\frac{\partial}{\partial a}\,\phi_a(x).$$

Proof: Subtraction of the equations satisfied by $\phi_a(y)$, $\phi_{a+h}(y)$ gives

$$-\frac{\phi_{a+h}(y) - \phi_a(y)}{h\,\phi_a(a)} = \int_{\bar{a}}^1 L(x, y)\left(-\frac{\phi_{a+h}(x) - \phi_a(x)}{h\,\phi_a(a)}\right)dx$$

$$+\frac{1}{h}\int_a^{a+h} L(x, y)\,\frac{\phi_{\tilde{a}}(x)\,dx}{\phi_a(a)}$$

where $\tilde{a} = \min(a, a+h)$. Define $\psi_a(y)$ as the non-negative solution of (5.5), and set

$$-\frac{\phi_{a+h}(y) - \phi_a(y)}{h\,\phi_a(a)} - \psi_a(y) = \chi(y),$$

$$\frac{1}{h}\int_a^{a+h} L(x, y)\,\frac{\phi_{\tilde{a}}(x)}{\phi_a(a)}\,dx - L(a, y) = \eta(y).$$

We have

$$\chi(y) = \eta(y) + \int_{\bar{a}}^{1} L(x, y)\chi(x)dx$$

or

$$|\chi(y)| \leq |\eta(y)| + \int_{\bar{a}}^{1} L(x, y)\,|\chi(x)|\,dx.$$

Theorem 4 of §4 and (5.4) gives

$$\int_{\bar{a}}^{1} |\chi(y)|\,dy \leq \frac{M}{m\,\lambda(a)\left(1 - \lambda(a)\right)} \int_{\bar{a}}^{1} |\eta(y)|\,dy.$$

But

$$\int_{\bar{a}}^{1} |\eta(y)|\,dy \leq \left(\frac{\frac{1}{h}\int_{0}^{h} \phi_a(x)dx}{\phi_a(a)} - 1\right) \int_{\bar{a}}^{1} L(a, y)dy$$

$$+ \frac{1}{h}\int_{a}^{a+h} \frac{\phi_a(x)}{\phi_a(a)}\left[\int_{\bar{a}}^{1} |L(x, y) - L(a, y)|\,dy\right]dx.$$

By virtue of hypothesis (4 c) and the uniform convergence of $\phi_a(x)$, as in theorem 5.3, it follows from this estimate that

$$\int_{\bar{a}}^{1} |\eta(y)|\,dy \quad \text{and also} \quad \int_{\bar{a}}^{1} |\chi(y)|\,dy \rightarrow 0 \quad \text{as} \quad h \rightarrow 0,$$

and this proves the main portion of the theorem. If

$$\frac{1}{h}\int_{a}^{a+h} L(x, y)dx \rightarrow L(a, y) \text{ for each } y,$$

this means that $\eta(y) \rightarrow 0$ for each y, and therefore from

$$|\chi(y)| \leq |\eta(y)| + M \int_{a}^{1} |\chi(x)| \, dx$$

we see that $\chi(y) \longrightarrow 0$. q.e.d.

6. The optimal strategy.

The theory of §§4, 5 will now be applied to the particular integral equation (3.2) with kernel $L(x, y)$ given by (3.4). The conditions (4 a), (4 b), (4 c) are easily verified. The parameter a is to vary over the range $0 \leq a < 1$, except that of $A(0, 0) = 0$ the case $a = 0$ is to be excluded. As in §§4, 5 denote the positive eigenvalue of the homogeneous equation by $\lambda(a)$, and the positive eigenfunction by $f_a(x)$, $a \leq x \leq 1$.

The integral equation (3.2) has a non-negative solution for $\alpha = 0$ only when $\lambda(a) = 1$, and for $\alpha > 0$ only when $\lambda(a) < 1$. There are two cases: either there is a value a_1 such that $\lambda(a_1) = 1$, or $\lambda(a) < 1$ for all a.

Case I. $\underline{\lambda(a_1) = 1}$. When $a = a_1$ and $\alpha = 0$, the function $f_{a_1}(x)$ satisfies the integral equation (3.2) and the condition (3.5). The corresponding quantity W defined in (3.6) vanishes, since this is obtained by multiplying (3.6) by $f_{a_1}(y)$ and integrating from a_1 to 1, the double integral on the right-hand side vanishing by the skew-symmetry of the kernel. The theorem of §3 shows that we have an optimum strategy. That this is a unique optimum strategy follows from the discussion in Case II below.

Case II. $\underline{\lambda(a) \leq 1 \text{ for all } a}$. We consider here also Case I, but then limit the range of a to $a_1 < a < 1$. For the positive solution of (3.2), set

$$f(x) = \alpha \phi_a(x)$$

where $\phi_a(x)$ satisfies

$$(6.1) \qquad \phi_a(y) = p(y) + \int_a^1 L(x, y)\phi_a(x)dx, \qquad a \leq y \leq 1.$$

The condition (3.5) becomes

$$(6.2) \qquad \int_a^1 \phi_a(x)dx = \frac{1-\alpha}{\alpha}$$

which gives a unique determination $\alpha(a)$, $0 < \alpha(a) < 1$, in terms of a. Denote the quantity defined in (3.6) by W(a), and set

$$\frac{W(a)}{\alpha(a)} = U(a)$$

where

$$(6.3) \quad U(a) = A(0, y) + \int_a^y A(x, y)\phi_a(x)dx - \int_y^1 A(y, x)\phi_a(x)dx,$$

$$a \leq y \leq 1,$$

the value being independent of y when y varies in the indicated range. The parameter a must be determined by the remaining condition (3.7).

Lemma. U(a) <u>is a strictly decreasing, continuous function of a with the following properties</u>:

$$\lim_{a \to 1} U(a) = A(0, 1) < 0$$

$$\lim_{a \to a_1^+} U(a) \leq 0 \quad \text{in case } \lambda(a_1) = 1$$

$$U(0) = \frac{A(0, 0)}{1 + \int_0^1 \phi_0(y)dy} > 0 \quad \text{in case } \lambda(a) < 1 \text{ for all } a \text{ and } A(0, 0) > 0.$$

Proof. The continuity of U(a) is a consequence of theorem 5.3. The first limiting relation is clear. To obtain the second and third relations, use two equivalent forms of (6.3). First multiply (6.3) by $\phi_a(y)$ and integrate with respect to y from a to 1:

$$(6.4) \qquad U(a)\int_a^1 \phi_a(y)dy = \int_a^1 A(0, y)\phi_a(y)dy$$

the double integral terms vanishing because of the skew-symmetry of the kernel. Also, set y = a in (6.3).

$$(6.5) \qquad U(a) = A(0, a) - \int_a^1 A(a, x)\phi_a(x)dx.$$

Divide (6.4) by $\int_a^1 \phi_a(y)dy$, let $a \to a_1^+$, in case $\lambda(a_1) = 1$, and use theorem (5.3). There results

$$\lim_{a \to a_1^+} U(a) = \int_a^1 A(0, y)f_{a_1}(y)dy.$$

But

$$\int_{a_1}^{1} A(0, y) f_{a_1}(y)dy \leq \int_{a_1}^{1} A(a_1, y) f_{a_1}(y)dy,$$

and this second integral is zero since it is the negative of the corresponding quantity W in Case I, which was proved to be zero in Case I.. This proves the second limiting relation.

To obtain the third relation set a = 0 in (6.4) and (6.5). The two integrals on the right-hand sides are then identical, and their elimination gives

$$U(0) = \frac{A(0, 0)}{1 + \int_0^1 \phi_0(x)dx} .$$

There remains to establish the strictly decreasing character of U(a). We have by differentiation of (6.3):

$$(6.6) \qquad U'(a) = -A(a, y)\phi_a(a) + \int_a^y A(x, y) \frac{\partial \phi_a(x)}{\partial a} dx$$

$$- \int_y^1 A(y, x) \frac{\partial \phi_a(x)}{\partial a} dx, \qquad a \leq y \leq 1,$$

which is independent of y in the indicated range. Again, obtain two expressions for U'(a), first by multiplying by $\frac{\partial \phi_a(y)}{\partial a}$ and integrating from a to 1, and second by setting y = a:

$$U'(a) \int_a^1 \frac{\partial \phi_a(y)}{\partial a}\, dy = -\phi_a(a) \int_a^1 A(a, y) \frac{\partial \phi_a(y)}{\partial a}\, dy$$

$$U'(a) = -A(a, a)\phi_a(a) - \int_a^1 A(a, x) \frac{\partial \phi_a(x)}{\partial a}\, dx.$$

The two integrals on the right-hand side are identical, and their elimination gives

$$U'(a) = \frac{-A(a, a)\phi_a(a)}{1 - \frac{1}{\phi_a(a)} \int_a^1 \frac{\partial \phi_a(y)}{\partial a}\, dy} < 0,$$

since $\frac{\partial \phi_a(y)}{\partial a} \leq 0$ as in theorem 5.4. This completes the proof of the lemma.

This basic lemma allows us to complete our discussion. In Case I, the lemma shows that $U(a) < 0$ for all a in the range $a_1 < a < 1$, and establishes the uniqueness of the optimal strategy obtained in Case I.

In Case II, under the supposition $A(0, 0) > 0$, we see from the lemma that there is a unique value of a which causes $U(a)$ to vanish. This gives an optimal strategy and also establishes its uniqueness.

There remains the case when $A(0, 0) = 0$, which is not yet covered by the lemma because of the unboundedness of the kernel. The following lemma completes the theory.

Lemma. <u>If</u> $A(0, 0) = 0$, <u>there is a positive</u> a_1 <u>with</u> $\lambda(a_1) = 1$.

Proof. Suppose that the lemma were false, so that $\lambda(a) < 1$ for all a in $0 < a < 1$. Formula (6.4) shows that $U(a) < 0$ since $A(0, y) < 0$ for all $y > 0$ because of $A(0, 0) = 0$. There is therefore no solution of the game with $a > 0$. But we have already established in a discussion near the beginning of §3, that a cannot be 0 in the case $A(0, 0) = 0$. There would therefore be no optimal strategy and if we suppose that in this case an optimal strategy must exist, the resulting contradiction would establish the lemma.*

We now give another proof of the lemma independent of the general theory of games. Again suppose that the lemma were false, so that $\lambda(a) < 1$ for all a in $0 < a < 1$. Take a positive ε, and consider the game with the same kernel $K(x, y)$ but with x, y limited to the range $\varepsilon \leq x \leq 1$, $\varepsilon \leq y \leq 1$. For this range the whole preceding theory is valid since $A(\varepsilon, \varepsilon) > 0$. Also, the kernel in the new integral equation corresponding to (3.2) is the same, so that the eigenvalues are the same, but the term $p(y)$ in (3.2) and (3.3) is now given by

$$p(y) = \frac{-A_2(\varepsilon, y)}{2A(y, y)} .$$

That is, all previous equations remain valid with 0 in the argument always replaced by the new lower limit ε.

Let the solution of the game, corresponding to $f(x)$ in (3.2) be $\psi_\varepsilon(x)$, with the values a_ε, α_ε of a and α. We have

$$\int_{a_\varepsilon}^{1} \psi_\varepsilon(x)dx = 1 - \alpha_\varepsilon$$

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* It is possible to show from very general considerations in the theory of games that in the present case, with $A(0, 0) = 0$, an optimal strategy exists. In these general considerations, difficulties occur at the end points 0, 1. Here, no difficulties occur at 0 because of $A(0, 0) = 0$ (continuity at 0), nor at 1 by the argument in lemma 1.

and since the corresponding W and U are zero, we have corresponding to (6.4)

$$(6.7) \qquad \int_{a_\varepsilon}^{1} A(\varepsilon, y)\,\psi_\varepsilon(y)\,dy = 0$$

and

$$(6.8) \qquad \alpha_\varepsilon A(\varepsilon, 1) + \int_{a_\varepsilon}^{1} A(x, 1)\,\psi_\varepsilon(x)\,dx = 0$$

corresponding to (6.3) with $y = 1$. From (6.7), since $A(0, y) < 0$ for $y > 0$, we see that $a_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Select a fixed $b > 0$ so that $A(b, 1) < 0$. We have, from (6.7) for sufficiently small ε,

$$\int_{a_\varepsilon}^{b} A(\varepsilon, y)\,\psi_\varepsilon(y)\,dy + \int_{b}^{1} A(\varepsilon, y)\,\psi_\varepsilon(y)\,dy = 0$$

or

$$\int_{a_\varepsilon}^{b} \big(A(\varepsilon, y) - A(\varepsilon, a_\varepsilon)\big)\,\psi_\varepsilon(y)\,dy + \int_{b}^{1} \big(A(\varepsilon, y) - A(\varepsilon, b)\big)\,\psi_\varepsilon(y)\,dy$$

$$= -A(\varepsilon, a_\varepsilon)\Big(1 - \alpha_\varepsilon - \int_{b}^{1} \psi_\varepsilon(y)\,dy\Big) - A(\varepsilon, b)\int_{b}^{1} \psi_\varepsilon(y)\,dy.$$

The left-hand side is ≤ 0, while the right-hand side approaches $-A(0, b) \lim_{\varepsilon \to 0} \int_{b}^{1} \psi_\varepsilon(y)\,dy$ which is > 0 unless the limit expression is zero. Therefore $\lim_{\varepsilon \to 0} \int_{b}^{1} \psi_\varepsilon(y)\,dy = 0$.

From (6.8), we now have

$$\alpha_\varepsilon A(\varepsilon, 1) + \int_{a_\varepsilon}^{b} \{A(x, 1) - A(b, 1)\} \psi_\varepsilon(x)dx + A(b, 1)\left(1 - \alpha_\varepsilon - \int_b^1 \psi_\varepsilon(x)dx\right)$$

$$+ \int_b^1 \{A(x, 1) - A(1, 1)\} \psi_\varepsilon(x)dx + A(1, 1) \int_b^1 \psi_\varepsilon(x)dx = 0.$$

Take the limit as $\varepsilon \longrightarrow 0$. All the integrals above are ≤ 0, with $\int_b^1 \psi_\varepsilon(x)dx \longrightarrow 0$, and $A(0, 1) < 0$, $A(b, 1) < 0$. If $\alpha_\varepsilon \longrightarrow \alpha_0$, $0 \leq \alpha_0 \leq 1$, we see that the limit of the left-hand side of the above equation is definitely < 0, contrary to the equation. This is a contradiction and the lemma is established.

This completes the proof of the main theorem.

7. Reduction to linear differential equations.

In a wide category of cases, the integral equation (3.2) is equivalent to a system of ordinary linear differential equations. This occurs if the function $A(x, y)$ has the following special form:

$$A(x, y) = \sum_{i=1}^{n} p_i(x)q_i(y).$$

Set

$$\int_y^1 p_i(x)f(x)dx = \xi_i(y), \qquad \int_y^1 q_i(x)f(x)dx = \eta_i(y)$$

or

$$(7.1)\qquad \begin{cases} \xi_j'(y) = -p_j(y)f(y), & \xi_j(1) = 0, \quad j = 1, \cdots, n \\ \eta_j'(y) = -q_j(y)f(y), & \eta_j(1) = 0, \quad j = 1, \cdots, n . \end{cases}$$

The integral equation becomes

$$(7.2)\qquad 2A(y, y)f(y) = -\alpha \sum_i p_i(0)q_i'(y) - \sum_i q_i'(y)\left(\xi_i(a) - \xi_i(y)\right)$$

$$+ \sum_i p_i'(y)\,\eta_i(y).$$

Temporarily set $\alpha p_i(0) + \xi_i(a) = c_i$, where c_i are arbitrary coefficients. Substitution of (7.2) into (7.1) gives a system of ordinary linear differential equations for $\xi_j(y)$, $\eta_j(y)$, depending linearly and homogeneously on the parameters c_i. The solutions of this system also depend linearly and homogeneously on c_i. The equation

$$c_i = \alpha p_i(0) + \xi_i(a)$$

gives a system of n linear equations for the determination of c_i in terms of a, α and the conditions (3.5), (3.7) then determine α, a. The question of the unique solvability for the parameters is answered by our main theorem, provided the monotonicity conditions of the theorem are satisfied.