U. S. AIR FORCE
PROJECT RAND

A SIMPLIFICATION OF GAMES IN EXTENSIVE FORM

W. D. Krentel, J. C. C. McKinsey, and W. V. Quine

P-140

Revised 12 February 1951

The RAND Corporation
1500 FOURTH ST. • SANTA MONICA • CALIFORNIA

When a zero-sum two-person game in extensive form is normalized [1], it frequently happens that many of the rows and columns of its matrix are repeated. When this is so, one can greatly simplify the problem of finding the value of the game, and optimum strategies for the two players, by crossing out repetitions of rows and columns. Sometimes, however, the number of repetitions is so great, that it is not feasible even to write out the original matrix at all. In such a case, it becomes desirable to transform the given game in extensive form, so as to reduce the number of repetitions in its matrix. In this paper we develop a method of doing this for a certain class of games in extensive form. The method consists essentially in decreasing the number of strategies available to the players, by the elimination of useless information.

Henceforth by a <u>game</u> we shall mean a finite zero-sum two-person game which satisfies the following conditions: (1) there is a fixed number, n, of moves, which is the same for all plays of the game; (2) none of the moves are chance moves; (3) the i^{th} move, for $i = 1, \ldots, n$, is always made by the same player, independent of the past course of the play; (4) at the i^{th} move, for $i = 1, \ldots, n$, the player making the move chooses an element from a fixed finite set $A(i)$, which does not depend on the past course of the play; (5) the player making the i^{th} move, for $i = 1, \ldots, n$, knows either everything or nothing about each of the past moves (i.e., for each $j < i$, he either knows exactly which element was chosen in the j^{th} move, or else he knows nothing at all about the choice made at the j^{th} move--he does not, for example, know merely that the choice was made from a certain two-element proper subset B of $A(i)$).

These restrictions may seem severe, but it should be noticed that many games which do not comply with them can easily be transformed into games which do. Thus a game can be made to satisfy condition (1) by putting in vacuous moves (moves in which the player has only one alternative available, or moves at which all his alternatives lead to the same outcome). And similarly a game can be made to satisfy condition (5) by putting extra moves into the middle of the game. Such processes as these, however, are artificial and are not even always possible in the case of games which fail to satisfy condition (3). It would therefore be desirable to extend our results directly to a wider class of games.

We now define in a more formal way the class of games with which we shall be concerned. We shall use standard set theoretical notations. The set whose only members are $a_1, a_2, \ldots, a_r$ will be indicated by

$$\{a_1, a_2, \ldots, a_r\} \ .$$

The ordered r-tuple whose first member is a_1, second member is a_2, etc., will be indicated by

$$< a_1, a_2, \ldots, a_r > .$$

We use the sign " $\subseteq$ " to indicate inclusion of one set in another, and " $\in$ " to indicate membership in a set. The set consisting of those members of A which are not in B will be indicated by

$$A - B.$$

We denote the empty set by "Λ". Finally, if A_1, A_2, ..., A_r are sets, then by

$$A_1 \times A_2 \times \cdots \times A_r$$

we shall mean the cartesian product of these sets: i.e., the set of all r-tuples $< x_1, x_2, \ldots, x_r >$ such that $x_1 \in A_1$, $x_2 \in A_2$, ..., $x_r \in A_r$.

We shall denote by I_n the set $\{1, 2, \ldots, n\}$ of the first n positive integers. We shall often write $-S$ instead of $I_n - S$.

By an information function for an n-move game we mean a function F, which is defined over I_n, assumes subsets of I_n as its values, and satisfies the conditions

(1) $$F(1) = \Lambda$$

(2) $$F(i) \subseteq \{1, \ldots, i-1\} \quad \text{for } i = 2, \ldots, n.$$

The intuitive meaning of an information function F is the following: for each i, F(i) is the set of moves whose outcomes are known to the player who makes the i^{th} move [2].

By an n-move game we now mean a system $\Gamma = < A, H, S, F >$ where: (1) A is a function defined over I_n, which assumes finite sets as values; (2) H is a real-valued function defined over the cartesian product $A(1) \times A(2) \times \cdots \times A(n)$; (3) S is a subset of I_n; and (4) F is an information function. The intuitive meaning of F has already been explained; the intuitive meanings of A, H, and S are as follows: for each i, A(i) is the set of alternatives which are available to the player making the i^{th} move; $H(x_1, x_2, \ldots, x_n)$ is the amount to be paid P_1 (the first player) by P_2 (the second player) in case that, for $i = 1, \ldots, n$, the alternative x_i

has been chosen in the i^{th} move; and S is the set of plays which are made by P_1. We shall call H the payoff function of the game Γ.

Now let $\Gamma = < A, H, S, F >$ be a game, and suppose that the elements of S, in order of magnitude, are $i_1, \ldots, i_r$. Let the elements of $F(i_j)$, for $j = 1, \ldots, r$, be, in order of increasing magnitude, $i_{j,1}$, $i_{j,2}, \ldots, i_{j,t_j}$. Then a (pure) strategy for P_1 in Γ is a sequence $< f_{i_1}, \ldots, f_{i_r} >$ where, in case $F(i_j) = \Lambda$, f_{i_j} is simply a member of $A(i_j)$, and otherwise f_{i_j} is a function which maps the cartesian product $A(i_{j,1}) \times A(i_{j,2}) \times \cdots \times A(i_{j,t_j})$ into $A(i_j)$.

Intuitively, a strategy for P_1 can be conceived as a function which, for each i, tells him what to do on the i^{th} move, for every possible state of information he could have regarding the previous course of the play. It is clear that, the more information he has, the greater will be the number of strategies available to him.

In a similar way, by considering the set -S, instead of the set S, we can define a (pure) strategy for the player P_2.

It is easily seen [3] that if we let $N(i)$, for $i = 1, \ldots, n$, be the number of elements in $A(i)$, and if we set, for $i = 1, \ldots, n$,

$$M(i) = \prod_{j \in F(i)} N(j),$$

then the numbers σ_1 and σ_2 of pure strategies available to the first and second players respectively are given by the formulas

$$\sigma_1 = \prod_{i \in S} N(i)^{M(i)}$$

$$\sigma_2 = \prod_{i \in -S} N(i)^{M(i)} \quad .$$

These formulas show how very rapidly the number of strategies increases with the amount of information available to a player; and thus how desirable it is to get rid of unnecessary information.

In these definitions we have supposed that the functions f_{i_j} map $A(i_{j,1}) \times A(i_{j,2}) \times \cdots \times A(i_{j,t_j})$, where $i_{j,1} < i_{j,2} < \cdots < i_{j,t_j}$, into $A(i_j)$. It should be clear, however, that the condition $i_{j,1} < i_{j,2} < \cdots < i_{j,t_j}$ was put in merely to insure uniqueness of the strategies. We could just as well define strategies by taking the cartesian product of the sets $A(i_{j,1})$, ..., $A(i_{j,t_j})$ in some other order; and we shall later make use of this freedom in order to simplify the notation in the proof of one of our lemmas (Lemma 4).

We now prove a lemma which expresses in a formal way the fact that, when P_1 and P_2 have each chosen a strategy, the course of the play is completely determined.

Lemma 1. Let $\Gamma = < A, H, S, F >$ be an n-move game; let the elements of S, in order of magnitude, be i_1, ..., i_r; and let the elements of I_n-S, likewise in order of magnitude, be j_1, ..., j_s (here, of course, $r+s = n$). Let $\alpha = < g_{i_1}, \ldots, g_{i_r} >$ be any strategy for P_1, and let

$\beta = < g_{j_1}, \ldots, g_{j_s} >$ be any strategy for P_2. Then there exists a unique element $< a_1, \ldots, a_n >$ of $A(1) \times \cdots \times A(n)$ such that, for $k = 1, \ldots, n$, we have:

(1) if $F(k) = \Lambda$, then $a_k = g_k$;

(2) if $F(k) \neq \Lambda$, and if the elements of $F(k)$, in order of magnitude, are $k_1, \ldots, k_i$ then

$a_k = g_k(a_{k_1}, \ldots, a_{k_t})$.

Proof. It is obvious that, if such an element exists at all, it is unique. To show that it exists, we make an induction on k.

Definition 1. Where Γ, α, β, and $< a_1, \ldots, a_n >$ are as described in Lemma 1, we call $< a_1, \ldots, a_n >$ the element of $A(1) \times \cdots \times A(n)$ prescribed by $< \alpha, \beta >$, amd we write

$$P_{\Gamma}(\alpha,\beta) = < a_1, \ldots, a_n > .$$

Moreover, we set

$$M_{\Gamma}(\alpha,\beta) = H(P_{\Gamma}(\alpha,\beta)).$$

The function M_{Γ} is called the strategy payoff function (as opposed to H, which is called merely the payoff function).

From the point of view of intuitive interpretation, $M_\Gamma(\alpha,\beta)$ is the amount P_2 pays P_1, when P_1 uses strategy α and P_2 uses strategy β.

Definition 2 Suppose that the strategies available to P_1 in the game Γ are enumerated as

$$\alpha_1, \alpha_2, \ldots, \alpha_u$$

and that the strategies available to P_2 are enumerated as

$$\beta_1, \beta_2, \ldots, \beta_v .$$

Then we call the matrix

$$\begin{pmatrix} M_\Gamma(\alpha_1,\beta_1) & M_\Gamma(\alpha_1,\beta_2) & \cdots & M_\Gamma(\alpha_1,\beta_v) \\ M_\Gamma(\alpha_2,\beta_1) & M_\Gamma(\alpha_2,\beta_2) & \cdots & M_\Gamma(\alpha_2,\beta_v) \\ \vdots & & & \\ M_\Gamma(\alpha_u,\beta_1) & M_\Gamma(\alpha_u,\beta_2) & \cdots & M_\Gamma(\alpha_u,\beta_v) \end{pmatrix}$$

a matrix of the normalized form of the game, or somtimes simply a matrix of the game.

Since in Definition 2 the enumeration of the strategies was done in an arbitrary manner, it is clear that in general a game will have many matrices, each of which can be obtained from each other one by interchanging rows or interchanging columns. It is clear, however, that any theory of how best to play the game will be independent of the order in which the strategies are enumerated: and thus that all these matrices represent the given game equally well. This leads to the following definition.

Definition 3. Two matrices are called equivalent [4] if one of them can be obtained from the other by interchanges of rows and interchanges of columns.

It is clear that the relation of equivalence of matrices is reflexive, symmetric, and transitive. Moreover the set of matrices for the normalized form of a given game is an equivalence class.

As mentioned earlier, the problem of solving a given game is not essentially changed if we cross out repetitions of rows and columns of its matrix. This suggests the following definition [5].

Definition 4. A matrix is said to be in reduced form if it has no two identical rows and no two identical columns. A matrix $\mathcal{N}$ is called a reduced form of a matrix $\mathcal{M}$, if $\mathcal{N}$ is in reduced form, and can be obtained from $\mathcal{M}$ by crossing out repetitions of columns and repetitions of rows.

A matrix can have more than one reduced form; thus the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

has the two reduced forms

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

and

$$\begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix} .$$

However, it is obvious from Definitions 3 and 4 that all the reduced forms of a given matrix are mutually equivalent. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are equivalent matrices, then every reduced form of $\mathcal{M}$ is equivalent to every reduced form of $\mathcal{N}$. We have also the following easy but useful lemma.

Lemma 2. Let $\mathcal{M}$ and $\mathcal{N}$ be matrices which have the same number of columns (rows), and suppose that every row (column) of $\mathcal{M}$ occurs as a row (column) in $\mathcal{N}$, and that every row (column) of $\mathcal{N}$ occurs as a row (column) in $\mathcal{M}$. Then $\mathcal{M}$ and $\mathcal{N}$ have equivalent reduced forms.

After this slight digression into notions about the matrices of games in normalized form, we now return to the extensive form of games.

Definition 5. If F is an information function over I_n, and S is a subset of I_n, then we call the couple $< S,F >$ a pattern of information (over I_n). If $\Gamma = < A, H, S, F >$ is a game, then we call $< S,F >$ the pattern of information of Γ .

Remark. From Definition 5, together with the definition of a game, it is clear that if $< S_1, F_1 >$ and $< S_2, F_2 >$ are information patterns over I_n, and if the functions A and H are such that $< A, H, S_1, F_1 >$ is a game, then $< A, H, S_2, F_2 >$ is also a game.

Definition 6. If $< S_1, F_1 >$ and $< S_2, F_2 >$ are patterns of information over I_n, we call $< S_1, F_1 >$ and $< S_2, F_2 >$ equivalent (in symbols, $< S_1, F_1 > \cong < S_2, F_2 >$) if, for every A and H such that $< A, H, S_1, F_1 >$ is a game, every reduced matrix of $< A, H, S_1, F_1 >$ is equivalent to every reduced matrix of $< A, H, S_2, F_2 >$.

In terms of the notions just introduced, we can now formulate in a more precise way the main problem with which we are concerned in this paper: we want to find a constructive method for deciding whether two patterns of information are equivalent. So far as regards the useful applications of such a method, we notice the following. For n fixed, there are only a finite number of patterns of information over I_n. Hence if we had such a method, then if we were presented with any game $< A, H, S, F >$ we could find all patterns of information equivalent to $< S, F >$, and then pick out from this finite class the pattern of information $< S', F' >$ which would make the number of strategies of $< A, H, S', F' >$ as small as possible. We are going to see, as a matter of fact, that this "best" pattern of information $< S', F' >$ is independent of A and H.

Lemma 3. If $< S_1, F_1 > \cong < S_2, F_2 >$ then $S_1 = S_2$.

Proof. Suppose that $S_1 \neq S_2$; we want to show that $< S_1, F_1 >$ is not equivalent to $< S_2, F_2 >$. Without loss of generality we can suppose that there is an element λ of S_1 which does not belong to S_2. We now define a function A by setting

$$A(i) = \{0\} \qquad \text{for } i \neq \lambda$$
$$A(\lambda) = \{0,1\} \quad ;$$

and we define a function H by setting, for all $< z_1, \ldots, z_n >$ in $A(1) \times \cdots \times A(n)$,

$$H(z_1, \ldots, z_n) = z_\lambda \quad .$$

It is then clear that $< A, H, S_1, F_1 >$ and $< A, H, S_2, F_2 >$ are games. Our lemma will be established if we can show that they do not have equivalent reduced matrices.

It is readily verified that in the game $\Gamma = < A, H, S_1, F_1 >$, since $\lambda \in S_1$, there is just one strategy, say β, available to player P_2; this strategy makes him pick 0 at each of his moves (he has no other choice, since $A(i) = \{0\}$ for every i in -S). Moreover, there are just two strategies for player P_1, both of which make him pick 0 on every move except the λ^{th}: strategy α_1 makes him pick 0 also on the λ^{th} move; and strategy α_2 makes him pick 1 on the λ^{th} move. We now have

$$M_\Gamma(\alpha_1,\beta) = 0$$
$$M_\Gamma(\alpha_2,\beta) = 1.$$

Hence a matrix for the normalized form of this game is

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

In a similar fashion, since $\lambda \notin S_2$, we see that a matrix for the normalized form of the game $< A, H, S_2, F_2 >$ is

$$(0 \quad 1) \; .$$

Since these two matrices are both in reduced form, and are obviously not equivalent, the proof of our lemma is complete.

In view of Lemma 3, if one wants to find patterns of information equivalent to a given pattern of information $< S_1,F_1 >$, it is useless to examine patterns $< S_2,F_2 >$ where $S_2 \neq S_1$. Hence we can restate our problem as follows: to find a constructive method by means of which one can decide whether two given patterns of information $< S,F >$ and $< S,G >$ are equivalent. In order to formulate our next two lemmas, which give sufficient conditions for such equivalence, it is convenient to introduce some new notions.

Definition 7. By an immediate inflation [6] of an information pattern $< S,G >$ will be meant any information pattern $< S,F >$ for which there are integers λ and μ such that:

(1) λ and μ either both belong to S, or else they both belong to -S;

(2) $G(\mu) \subseteq G(\lambda)$;

(3) $\mu \notin G(\lambda)$;

(4) $F(\lambda) = G(\lambda) \cup \{\mu\}$;

(5) $F(i) = G(i)$ for $i \neq \lambda$.

Definition 8. An information pattern is called completely inflated if it does not possess any immediate inflations. An information pattern $< S,F >$ is said to be a complete inflation of an information pattern $< S, G >$ if $< S,F >$ is completely inflated, and can be obtained from $< S,G >$ by a succession of immediate inflations.

(We shall show later that an information pattern possesses only one complete inflation.)

Lemma 4. If an information pattern $< S,F >$ is an immediate inflation of an information pattern $< S,G >$, then $< S,F > \cong < S,G >$.

Proof. Let λ and μ be integers satisfying the five conditions of Definition 7. Without loss of generality we suppose that λ and μ both belong to S. Let A and H be any functions such that $< A, H, S, F >$ is a game; we are to show that $\Gamma = < A, H, S, F >$ and $\Delta = < A, H, S, G >$ have equivalent reduced matrices.

Let $\mathcal{A}$ be the set of strategies available to P_1 in Γ, and let $\mathcal{B}$ be the set of strategies available to P_1 in Δ. Let $\mathcal{C}$ be the set of strategies available to P_2 in Γ; since $\lambda \in S$, we see that the set of strategies available to P_2 in Δ is also $\mathcal{C}$.

Let $\mathcal{M}$ be a matrix for the normalized form of Γ, and let $\mathcal{N}$ be a matrix for the normalized form of Δ. Since the strategies available to P_2 are the same in the two games, $\mathcal{M}$ and $\mathcal{N}$ have the same number of columns; hence, by Lemma 2, in order to show that $\mathcal{M}$ and $\mathcal{N}$ have equivalent reduced forms it will suffice to show that every row of $\mathcal{M}$ occurs as a row in $\mathcal{N}$, and that every row of $\mathcal{N}$ occurs as a row in $\mathcal{M}$.

Referring back to Definition 1, we see that, in order to prove this it will suffice to show:

(A) For every element α of $\mathcal{A}$, there exists an element α^* of $\mathcal{B}$, such that, for every element γ of $\mathcal{C}$,

$$P_\Gamma(\alpha, \gamma) = P_\Delta(\alpha^*, \gamma);$$

(B) For every element α of $\mathcal{B}$, there exists an element α^* of $\mathcal{A}$, such that, for every element γ of $\mathcal{C}$,

$$P_\Gamma(\alpha^*, \gamma) = P_\Delta(\alpha, \gamma).$$

Let the members of S, in increasing order of magnitude, be $i_1, i_2, \ldots, i_r$; and let the members of -S, also in increasing order of magnitude, be $j_1, j_2, \ldots, j_s$. Let $i_h = \mu$ and $i_k = \lambda$ (by hypothesis, $\mu \in S$ and $\lambda \in S$); since (by Definition 7 (4)) we have $\mu \in F(\lambda)$, we conclude that $\mu < \lambda$, so that $i_h < i_k$, and hence $h < k$. Let the members of $G(i_h)$ be, in order of increasing magnitude, $\mu_1, \mu_2, \ldots, \mu_p$. Remembering that, by Definition 7 (2), $G(i_h) \subseteq G(i_k)$, we let the members of $G(i_k)$ be $\mu_1, \mu_2, \ldots, \mu_p, \mu_{p+1}, \ldots, \mu_q$; these numbers, however, are not necessarily in order of increasing magnitude. Then, by Definition 7 (4), the members of $F(i_k)$ are $\mu_1, \mu_2, \ldots, \mu_p, \mu_{p+1}, \ldots, \mu_q, i_h$.

In order to prove (A), let $\alpha = < f_{i_1}, f_{i_2}, \ldots, f_{i_r} >$ be any member of $\mathcal{A}$. We notice that f_{i_k} is a function which maps $A(\mu_1) \times A(\mu_2) \times \cdots \times A(\mu_p) \times A(\mu_{p+1}) \times \cdots \times A(\mu_q) \times A(i_h)$ into $A(i_k)$. Thus $f_{i_k}(x_1, \ldots, x_q, y)$ is defined whenever $x_i \in A(\mu_i)$ for $i = 1, \ldots, q$ and $y \in A(i_h)$. Moreover, f_{i_h} is a function which maps $A(\mu_1) \times \cdots \times A(\mu_p)$ into $A(i_h)$. Now let g be a function defined by the equation

$$(1) \qquad g(x_1, \ldots, x_q) = f_{i_k}(x_1, \ldots, x_q, f_{i_h}(x_1, \ldots, x_p)).$$

Then it is clear that g maps $A(\mu_1) \times \cdots \times A(\mu_q)$ into $A(i_k)$. Hence the sequence $\alpha^* = < f^*_{i_1}, f^*_{i_2}, \ldots, f^*_{i_r} >$, where $f^*_{i_j} = f_{i_j}$ for $j \neq k$, and $f^*_{i_k} = g$, is a strategy for P_1 in Δ , and hence a member of $\mathcal{B}$. Let $\gamma = < f_{j_1}, f_{j_2}, \ldots, f_{j_s} >$ be any member of $\mathcal{C}$; it is notationally convenient to set, for $i = 1, \ldots, s$, $f^*_{j_i} = f_{j_i}$, so that we also have $\gamma = < f^*_{j_1}, f^*_{j_2}, \ldots, f^*_{j_s} >$. We now wish to show that

$$P_\Gamma(\alpha, \gamma) = P_\Delta(\alpha^*, \gamma);$$

thus, letting

$$P_\Gamma(\alpha, \gamma) = < a_1, \ldots, a_n >$$

and

$$P_\Delta(\alpha^*, \gamma) = < b_1, \ldots, b_n >,$$

we want to show that $a_i = b_i$ for $i = 1, \ldots, n$. We shall do this by an induction on i.

We notice that we cannot have $i_k = 1$, since $F(i_k)$ is not empty. Hence for $i = 1$ we have $a_1 = f_1$, and $b_1 = f^*_1 = f_1$, so that $a_1 = b_1$, as was to be shown.

Suppose now that the statement is true for all $i < \pi$. We distinguish two cases, according as $\pi \neq i_k$ or $\pi = i_k$. In the first case, we have $F(\pi) = G(\pi)$, and thus, supposing that the elements of $F(\pi)$ are $\pi_1, \ldots, \pi_w$,

$$a_\pi = f_\pi(a_{\pi_1}, \ldots, a_{\pi_w})$$

and

$$b_\pi = f^*_\pi(b_{\pi_1}, \ldots, b_{\pi_w})$$

$$= f_\pi(b_{\pi_1}, \ldots, b_{\pi_w}) ;$$

since $\pi_j \in F(\pi)$, however, for $j = 1, \ldots, w$, we have $\pi_j < \pi$, and hence, by the induction hypothesis, $b_{\pi_j} = a_{\pi_j}$; thus

$$b_\pi = f_\pi(a_{\pi_1}, \ldots, a_{\pi_w}) = a_\pi ,$$

as was to be shown. In the second case, the elements of $G(\pi)$ are $\mu_1, \mu_2, \ldots, \mu_p, \mu_{p+1}, \ldots, \mu_q$; and the elements of $F(\pi)$ are $\mu_1, \mu_2, \ldots, \mu_p, \mu_{p+1}, \ldots, \mu_q, i_h$. We have, then,

$$a_\pi = f_{i_k}(a_{\mu_1}, \ldots, a_{\mu_q}, a_{i_h})$$

and, using (1) and the induction hypothesis,

$$b_\pi = g(b_{\mu_1}, \ldots, b_{\mu_q})$$

$$= f_{i_k}(b_{\mu_1}, \ldots, b_{\mu_q}, f_{i_h}(b_{\mu_1}, \ldots, b_{\mu_p}))$$

$$= f_{i_k}(b_{\mu_1}, \ldots, b_{\mu_q}, b_{i_h})$$

$$= f_{i_k}(a_{\mu_1}, \ldots, a_{\mu_q}, a_{i_h})$$

$$= a_\pi ,$$

as was to be shown.

In order to prove (B), let $\alpha = < g_{i_1}, \ldots, g_{i_r} >$ be any member of $\mathcal{B}$. Let the function f be defined by the equation

$$(2) \qquad f(x_1, \ldots, x_q, y) = g_{i_k}(x_1, \ldots, x_q),$$

for $< x_1, \ldots, x_q, y >$ any member $A(\mu_1) \times \cdots \times A(\mu_q) \times A(i_h)$. Then the sequence $\alpha^* = < g^*_{i_1}, \ldots, g^*_{i_r} >$, where $g^*_{i_j} = g_{i_j}$ for $j \neq k$, and $g^*_{i_k} = f$, is a member of $\mathcal{Q}$. Let $\gamma = < g_{j_1}, \ldots, g_{j_s} >$ be any member of $\mathcal{C}$; as before, it is notationally convenient to set, for $i = 1, \ldots, s$, $g^*_{j_i} = g_{j_i}$, so that we also have $\gamma = < g^*_{j_1}, \ldots, g^*_{j_s} >$. We wish to show that

$$P_\Gamma(\alpha^*, \gamma) = P_\Delta(\alpha, \gamma).$$

Let

$$P_\Gamma(\alpha^*, \gamma) = < a_1, \ldots, a_n >$$

and

$$P_\Delta(\alpha, \gamma) = < b_1, \ldots, b_n >.$$

In order to prove that $a_i = b_i$ for $i = 1, \ldots, n$, we again make an induction on i. For $i = 1$, the proof is as in the proof of (A). Suppose now that $a_i = b_i$ for $i < \pi$. We distinguish two cases, according as $\pi \neq i_k$ or $\pi = i_k$. For the first case, the proof is as in the proof of (A). In the second case we have

$$b_\pi = g_{i_k}(b_{\mu_1}, \ldots, b_{\mu_q})$$

and, using (2) and the induction hypothesis,

$$
\begin{aligned}
a_{\pi} &= f(a_{\mu_1}, \ldots, a_{\mu_q}, a_{i_k}) \\
&= g_{i_k}(a_{\mu_1}, \ldots, a_{\mu_q}) \\
&= g_{i_k}(b_{\mu_1}, \ldots, b_{\mu_q}) \\
&= b_{\pi},
\end{aligned}
$$

as was to be shown.

Lemma 5. If an information pattern $< S,F >$ is a complete inflation of an information pattern $< S,G >$, then $< S,F > \cong < S,G >$.

Proof. Using Lemma 4, we make an induction on the number of immediate inflations necessary to pass from $< S,G >$ to $< S,F >$.

In order to derive a necessary condition for the equivalence of patterns of information, we introduce the notions of mixed strategies, and of the value of a game.

Definition 9. If r is any positive integer, then by $\mathcal{S}_r$ we mean the set of all r-tuples $< x_1, \ldots, x_r >$ of non-negative real numbers which satisfy the condition

$$x_1 + x_2 + \cdots + x_r = 1.$$

The elements of $\mathcal{S}_r$ are called mixed strategies of order r. If Γ

is an n-move game with pure strategies $\alpha_1, \ldots, \alpha_r$ for P_1 and pure strategies $\beta_1, \ldots, \beta_s$ for P_2, and if $X = \langle x_1, \ldots, x_r \rangle$ and $Y = \langle y_1, \ldots, y_s \rangle$ are any members of $\mathcal{S}_r$ and $\mathcal{S}_s$ respectively, then we set

$$M_{\Gamma}(X,Y) = \sum_{i=1}^{r} \sum_{j=1}^{s} M_{\Gamma}(\alpha_i,\beta_j)x_i y_j.$$

If $v(\Gamma)$ is a real number, and if X_o and Y_o are mixed strategies such that, for all X and Y,

$$M_{\Gamma}(X,Y_o) \leq v(\Gamma) \leq M_{\Gamma}(X_o,Y),$$

then we call $v(\Gamma)$ the <u>value</u> of the game, and we call X_o and Y_o <u>optimum mixed strategies</u> for P_1 and P_2 respectively.

In intuitive interpretation, $M_{\Gamma}(X,Y)$ is the average amount P_2 pays P_1 when (over a series of plays) P_1 uses mixed strategy X and P_2 uses mixed strategy Y. The value of the game is the amount P_1 can be sure of obtaining (on the average) by using an optimum strategy X_o, and which P_2 can keep P_1 from exceeding by using an optimum strategy Y_o. The fundamental theorem of game theory [7] tells us that such optimum strategies always exist for a game of the sort with which we are concerned.

It is clear that if two games have equivalent reduced matrices, then they have the same value. Thus we obtain the following necessary condition for the equivalence of information patterns.

Lemma 6. If $< S,F > \cong < S,G >$, then, for every A and H such that $< A, H, S, F >$ is a game,

$$v(< A, H, S, F >) = v (< A, H, S, G >).$$

We now introduce some more special notions connected with information patterns.

Definition 10. If $< S,F >$ is a pattern of information, then by a minerva with respect to $< S,F >$ we mean a sequence $< i_1, i_2, \ldots, i_r >$ of (at least two) elements of I_n such that

(1) $i_1 \in F(i_2)$, $i_2 \in F(i_3)$, $\ldots$, $i_{r-1} \in F(i_r)$;

(2) if $i_1 \in S$, then $i_k \in -S$ for $k = 2, \ldots, r$;

(3) if $i_1 \in -S$, then $i_k \in S$ for $k = 2, \ldots, r$.

We call r the length of the minerva, and i_1 and i_r its terminal elements; i_1 is the first element, and i_r the last. If $< j_1, j_2, \ldots, j_s >$ is a subsequence of $< i_1, i_2, \ldots, i_r >$ which is itself a minerva, then we call $< j_1, j_2, \ldots, j_s >$ a subminerva of $< i_1, i_2, \ldots, i_r >$. A subminerva is called proper if it is not identical with $< i_1, i_2, \ldots, i_r >$. A minerva is called minimal if it has no proper subminervas with the same terminal elements.

Remarks. It is clear from Definition 10 (1) that if $< i_1, i_2, \ldots, i_r >$ is a minerva with respect to any information pattern, then

$$i_1 < i_2 < \cdots < i_r .$$

It is also clear that $< i_1, i_2, \ldots, i_r >$ is a minerva (or minimal minerva) with respect to $< S,F >$ if and only if it is a minerva (or minimal minerva) with respect to $< -S,F >$. It is easily seen, moreover, that a minerva $< i_1, i_2, \ldots, i_r >$ is a minimal minerva if and only if $i_s \notin F(i_k)$ for $s = 1, \ldots, r - 2$ and $k = s + 2, \ldots, r$. A minerva of length two is always a minimal minerva.

Lemma 7. Let $< S,F >$ and $< S,G >$ be information patterns over I_n, and suppose that, for every A and H such that $< A, H, S, F >$ is a game,

$$v(< A, H, S, F >) = v (< A, H, S, G >) .$$

Then $< S,F >$ and $< S,G >$ have the same class of minimal minervas.

Proof. Suppose that $< S,F >$ and $< S,G >$ have different classes of minimal minervas. We are to show that, for some A and H,

$$v(< A, H, S, F >) \neq v (< A, H, S, G >) .$$

Let p be the smallest integer such that there exists a sequence of length p which is a minimal minerva for one of the two information patterns but not for the other. Without loss of generality we can suppose that there exists a sequence $<\lambda_1, \ldots, \lambda_p >$ of length p which is a minimal minerva for $< S,F >$ but not for $< S,G >$. Using the fact that there is no shorter sequence with the stated property, it can be shown that $\lambda_i \notin G(\lambda_p)$ for $i = 1, \ldots, p - 1$.

We suppose that $\lambda_1 \in -S$, and that $\lambda_i \in S$ for $i = 2, \ldots, p$; the proof is similar in case $\lambda_1 \in S$ and $\lambda_i \in -S$ for $i = 2, \ldots, p$.

Now we define a function A by setting

$$A(i) = \{1,2\} \text{ if } i \in \{\lambda_1, \ldots, \lambda_p\}$$

$$A(i) = \{1\} \quad \text{ if } i \notin \{\lambda_1, \ldots, \lambda_p\} ,$$

and a function H by setting, for $< z_1, \ldots, z_n >$ any member of $A(1) \times \cdots \times A(n)$,

$$H(z_1, \ldots, z_n) = 1 \qquad \text{if } z_{\lambda_1} = z_{\lambda_p}$$

$$H(z_1, \ldots, z_n) = -1 \qquad \text{if } z_{\lambda_1} \neq z_{\lambda_p} .$$

It is then readily verified that

$$v(< A, H, S, F >) = 1$$

and

$$v(< A, H, S, G >) = 0.$$

Since these numbers are unequal, the proof of our lemma is complete.

We return now to the notion of completely inflated patterns of information.

<u>Lemma</u> 8. Let $< S,F >$ be a completely inflated information pattern, and let i_1 and i_2 be integers both of which belong to S, or both of which belong to -S, and such that $i_2 \notin F(i_1)$. Then there are integers

$i_3, \ldots, i_r$ such that $< i_r, i_{r-1}, \ldots, i_2 >$ is a minimal minerva for $< S,F >$, and $i_j \notin F(i_1)$ for $j = 2, 3, \ldots, r$.

Proof. It clearly suffices to show that there is a minerva with the stated properties: for a minerva with the stated properties can be made into a minimal minerva with the stated properties by leaving out enough members of $\{ i_{r-1}, \ldots, i_3 \}$.

Now since $i_2 \notin F(i_1)$, and $< S,F >$ is completely inflated, $F(i_2) \not\subseteq F(i_1)$. Let $i_3 \in F(i_2)$ and $i_3 \notin F(i_1)$. If $i_3 \in -S$, then $< i_3, i_2 >$ is the desired minerva. Otherwise, since $< S,F >$ is completely inflated, and $i_3 \notin F(i_1)$, we conclude that $F(i_3) \not\subseteq F(i_1)$. Let $i_4 \in F(i_3)$ and $i_4 \notin F(i_1)$. If $i_4 \in -S$, then $< i_4, i_3, i_2 >$ is the desired minerva. Continuing in this way we see, since the integers $i_2, i_3, i_4, \ldots$ keep decreasing, that we must finally reach an i_r such that $i_r \in -S$, or else such that $F(i_r) = \Lambda$. The latter possibility is excluded, however, for then we should have $F(i_r) \subseteq F(i_1)$, and hence $i_r \in F(i_1)$, contrary to the way in which i_r was chosen. Thus we finally reach an i_r which belongs to $-S$, and the sequence $< i_r, \ldots, i_2 >$ is then a minerva of the desired sort.

Lemma 9. Let $< S,F >$ and $< S,G >$ be completely inflated patterns of information such that $F \neq G$. Then $< S,F >$ and $< S,G >$ have different classes of minimal minervas.

Proof. Let i_1 be the smallest integer for which F and G assume different values; thus

(1) $$F(i) = G(i) \qquad \text{for } i < i_1$$

(2) $$F(i_1) \neq G(i_1) .$$

We suppose that $i_1 \in S$; the proof is similar in case $i_1 \in -S$.

From (2) we see that there is an integer which belongs to one of the two sets $F(i_1)$ and $G(i_1)$ but not to the other. Let i_2 be the smallest such integer. Without loss of generality we can suppose that $i_2 \in G(i_1)$ and $i_2 \notin F(i_1)$. Thus:

(3) $$\text{for } i < i_2,\ i \in F(i_1) \text{ if and only if } i \in G(i_1);$$

(4) $$i_2 \notin F(i_1);$$

(5) $$i_2 \in G(i_1).$$

If $i_2 \in -S$, then the couple $< i_2, i_1 >$ is a minimal minerva with respect to $< S,G >$ but not with respect to $< S,F >$.

If $i_2 \in S$, on the other hand, then by Lemma 8 there are elements $i_3, \ldots, i_r$ of I_n such that

(6) $$< i_r, \ldots, i_3, i_2 > \text{ is a minimal minerva for } < S,F >;$$

(7) $$i_j \in F(i_1) \text{ for } j = 2, 3, \ldots, r.$$

From (6) we see that

(8) $$i_r < i_{r-1} < \cdots < i_3 < i_2 ,$$

and from (5) we have

(9) $$i_2 < i_1.$$

It follows from (1), (8) and (9) that

(10) $$F(i_j) = G(i_j) \text{ for } j = 2, \ldots, r.$$

From (6) and (10)

(11) $< i_r, \ldots, i_3, i_2 >$ is a minimal minerva for $< S,G >$

From (8), (3), and (7) we have

(12) $i_j \notin G(i_1)$ for $j = 3, \ldots, r$.

From (11) and (5) we see that $< i_r, \ldots, i_3, i_2, i_1 >$ is a minerva for $< S,G >$, and then by (11) and (12) we conclude that it is indeed a minimal minerva for $< S,G >$. But, by (4), $< i_r, \ldots, i_3, i_2, i_1 >$ is not even a minerva for $< S,F >$, which completes our proof.

Lemma 10. Every pattern of information has a unique complete inflation.

Proof. Let $< S,F >$ be any pattern of information, and let $< S,G_1 >$ and $< S,G_2 >$ be two complete inflations of $< S,F >$. By Lemmas 5, 6, and 7 we conclude that $< S,F >$ and $< S,G_1 >$ have the same class of minimal minervas, and that $< S,F >$ and $< S,G_2 >$ have the same class of minimal minervas. Hence $< S,G_1 >$ and $< S,G_2 >$ have the same class of minimal minervas and hence by Lemma 9, since they are completely inflated, $G_1 = G_2$, as was to be shown.

Lemma 11. If two patterns of information have the same class of minimal minervas, then they have the same complete inflations.

Proof. Let $< S,F >$ and $< S,G >$ be two patterns of information which have the same class of minimal minervas. Let $< S,F_1 >$ be the complete inflation of $< S,F >$, and let $< S,G_1 >$ be the complete inflation of $< S,G >$.

By Lemmas 5, 6, and 7 we see that $< S,F_1 >$ and $< S,F >$ have the same class of minimal minervas, and that $< S,G_1 >$ and $< S,G >$ have the same class of minimal minervas. Hence $< S,F_1 >$ and $< S,G_1 >$ have the same class of minimal minervas, and therefore, by Lemma 9, $F_1 = G_1$, as was to be shown.

<u>Theorem</u> 1. Let $< S,F >$ and $< S,G >$ be two patterns of information over I_n. Then the following conditions are all equivalent:

(1) $< S,F > \cong < S,G >$;

(2) for every A and H such that $< A, H, S, F >$ is a game $v(< A, H, S, F >) = v(< A, H, S, G >)$;

(3) $< S,F >$ and $< S,G >$ have the same class of minimal minervas;

(4) $< S,F >$ and $< S,G >$ have the same complete inflation .

<u>Proof</u>. (1) implies (2) by Lemma 6; (2) implies (3) by Lemma 7; (3) implies (4) by Lemma 11; and (4) implies (1) by Lemma 5.

Although we have thus solved the problem of determining when two information patterns are equivalent, we are left with the question how to pick out, for a given pattern of information $< S,F >$, a pattern of information $< S,G >$ which will make the size of the matrix as small as possible. (The complete inflation of an information pattern of course has exactly the opposite property: it makes the matrix as large as possible.) For this purpose we introduce some new notions.

<u>Definition</u> 11. By an <u>immediate reduction</u> of an information pattern $< S,F >$ will be meant any information pattern $< S,G >$ which is equivalent

to $< S,F >$, and for which there exist integers λ and μ such that:

(1) λ and μ either both belong to S, or else they both belong to -S;

(2) $$\mu \in F(\lambda)$$

(3) $$G(\lambda) = F(\lambda) - \{\mu\}$$

(4) $$G(i) = F(i) \quad \text{for } i \neq \lambda \ .$$

An information pattern is called completely reduced if it does not possess any immediate reductions. An information pattern $< S,G >$ is called a complete reduction of an information pattern $< S,F >$, if $< S,G >$ is completely reduced, and can be obtained from $< S,F >$ by a succession of immediate reductions.

Lemma 12. A necessary and sufficient condition that an information pattern $< S,G >$ be an immediate reduction of an information pattern $< S,F >$ is that: (1) every minimal minerva for $< S,F >$ is a minimal minerva for $< S,G >$; and (2) there are integers λ and μ satisfying the four conditions of Definition 11.

Proof. The condition is clearly necessary; for if $< S,G >$ is an immediate reduction of $< S,F >$, then by Definition 11, $< S,G > \cong < S,F >$, and hence, by Theorem 1, $< S,G >$ and $< S,F >$ have the same class of minimal minervas.

Now suppose, if possible, that conditions (1) and (2) of our lemma are satisfied, and that $< S,G >$ is not an immediate reduction of $< S,F >$. By Definition 11 we conclude that $< S,G >$ is not equivalent to $< S,F >$, and hence, by Theorem 1, that there exists a sequence $< \pi_1, \ldots, \pi_p >$ which is a minimal minerva for $< S,G >$ but not for $< S,F >$. Clearly $< \pi_1, \ldots, \pi_p >$ is a minerva for $< S,F >$; hence, since it is not a minimal minerva for $< S, F >$, there are integers h and k such that $k - h > 1$, and $\pi_h \in F(\pi_k)$. Since $< \pi_1, \ldots, \pi_p >$ is a minimal minerva for $< S,G >$, however, we cannot have $\pi_h \in G(\pi_k)$. Hence, from part (4) of Definition 11, we see that $\pi_h = \mu$ and $\pi_k = \lambda$. We suppose now that μ and λ are both members of S; the proof is similar in case they both are members of -S. Then $\pi_h \in S$ and $\pi_k \in S$, and hence $\pi_1 \in -S$, so that $h \neq 1$. Hence $< \pi_1, \ldots, \pi_{h-1}, \pi_h >$ is a minimal minerva for $< S,G >$; since $\mu \notin \{\pi_1, \ldots, \pi_{h-1}\}$ and $\lambda \notin \{\pi_1, \ldots, \pi_h\}$, we conclude that $< \pi_1, \ldots, \pi_{h-1}, \pi_h >$ is also a minimal minerva for $< S,F >$. Moreover, from the fact that $\pi_h \in F(\pi_k)$ we conclude that $< \pi_1, \ldots, \pi_{h-1}, \pi_h, \pi_k >$ is a minimal minerva for $< S,F >$. Since this sequence is not even a minerva for $< S,G >$, however, we have a contradiction of our hypothesis, according to which every minimal minerva for $< S,F >$ is a minerva for $< S,G >$.

Lemma 13. Let $< S,F >$ be a completely reduced information pattern, and suppose that μ and λ are integers such that $\mu \in F(\lambda)$. Then there is a minimal minerva with respect to $< S,F >$ which has λ for its last element, and μ for its next-to-the-last element.

Proof. If $\mu \in S$ and $\lambda \in -S$, or if $\mu \in -S$ and $\lambda \in S$, then $< \mu, \lambda >$ is itself a minimal minerva. Hence we suppose that λ and μ are either both in S or both in -S.

Now suppose, if possible, that there is no minimal minerva $< \pi_1, \ldots, \pi_r >$ for $< S,F >$ such that $\pi_{r-1} = \mu$ and $\pi_r = \lambda$. We easily conclude that there exists no minimal minerva $< \pi_1, \ldots, \pi_s >$ for $< S,F >$

such that, for some $r \leq s$, $\pi_{r-1} = \mu$ and $\pi_r = \lambda$. Hence if we define a function G by the equations

$$G(i) = F(i) \qquad \text{for } i \neq \lambda$$
$$G(\lambda) = F(\lambda) - \{\mu\},$$

then we see that every minimal minerva for $< S,F >$ is also a minimal minerva for $< S,G >$. We conclude by Lemma 12 that $< S,G >$ is an immediate reduction of $< S,F >$, contrary to the hypothesis that $< S,F >$ is completely reduced.

Lemma 14. If $< S,F >$ and $< S,G >$ are completely reduced information patterns such that $< S,F > \cong < S,G >$, then $F = G$.

Proof. We wish first to show that, under the hypothesis of the lemma, we have, for $i = 1, \ldots, n$,

$$F(i) \subseteq G(i).$$

Suppose, then, that

$$j \in F(i).$$

Let $< \lambda_1, \ldots, \lambda_r >$ be a minimal minerva for $< S,F >$ such that $\lambda_{r-1} = j$ and $\lambda_r = i$, (such a minimal minerva exists, by Lemma 13). Since $< S,G >$ is equivalent to $< S,F >$, we see by Theorem 1 that $< \lambda_1, \ldots, \lambda_r)$ is a minimal minerva for $< S,G >$ also and hence that $\lambda_{r-1} \in G(\lambda_r)$, or

$$j \in G(i),$$

as was to be shown.

Similarly it is seen that, for $i = 1, \ldots, n$,

$$G(i) \subseteq F(i),$$

so that

$$F(i) = G(i),$$

as was to be shown.

Theorem 2. Every information pattern has a unique complete reduction.

Proof. It is clear that an information pattern has at least one complete reduction. On the other hand, if an information pattern $< S,F >$ has complete reductions $< S,G >$ and $< S,H >$, then $< S,G >$ and $< S,H >$ are both equivalent to $< S,F >$, and hence to each other; by Lemma 14, therefore, since they are completely reduced, they are identical.

It is clear from Lemma 12, together with the first part of the proof of Theorem 2, how one can go about finding the complete reduction of a given information pattern $< S,F >$: one first writes down all the minimal minervas for $< S,F >$, and then, for $i = 1, \ldots, n$, deletes from $F(i)$ each element j such that none of the minimal minervas ending in i has j for its next-to-the-last element.

Our final theorem shows (in view of the formulas given earlier for σ_1 and σ_2) that the complete reduction of a pattern of information will (independently of A and H) make the matrix for the game at least as small as the matrix corresponding to any other equivalent pattern.

Theorem 3. Let $< S,F >$ be a completely reduced information pattern (over I_n), let $< S,G >$ be any information pattern equivalent to $< S,F >$, and let i be any member of I_n. Then

$$F(i) \subseteq G(i).$$

Proof. Let j be any member of $F(i)$. Then, using Lemma 13, we see that j is the last element but one, and i is the last element, of some minimal minerva for $< S,F >$. Since $< S,G >$ is equivalent to $< S,F >$, we see by Theorem 1 that this minimal minerva is also a minimal minerva for $< S,G >$, and hence that $j \in G(i)$, as was to be shown.

FOOTNOTES

1. 1. For an explanation of the basic concepts of the theory of games, see Theory of Games and Economic Behavior by J. von Neumann and O. Morgenstern, especially Chapter II, or "Extensive Games" by H. W. Kuhn, Proceedings of the National Academy of Sciences, vol. 36 (1950), pages 570-576.

2. Condition (2) of this definition means that the player making the i^{th} move does not know the outcome of the j^{th} move, for any $j > i$; thus this condition can be regarded as meaning that the moves are temporally ordered according to increasing size of their numbers. It should be noticed, however, that, for a given game in extensive form, there may be more than one way of numbering the moves so as to satisfy conditions (1) and (2).

3. These formulas were called to our attention by Mr. L. S. Shapley, to whom we are also indebted for many other helpful suggestions.

4. In the usual algebra of matrices, the word "equivalence" is of course used in a wider sense. Since we shall never be concerned with matrix theory proper, however, no confusion should arise from our more restricted understanding of the term.

5. The notion of the reduced form of a matrix was introduced by H. W. Kuhn, loc. cit.

6. The idea of formulating our results in terms of inflations was suggested to us by L. S. Shapley. We had previously used a certain

recursively defined operation which made our proofs considerably more difficult.

7. For a proof of the fundamental theorem, see page 153 of <u>Theory of Games and Economic Behavior</u> by J. von Neumann and O. Morgenstern.

je