DYNAMIC PROGRAMMING AND ADAPTIVE PROCESSES--I: MATHEMATICAL FOUNDATION

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SUMMARY

In many engineering, economic, biological, and statistical control processes, a decision-making device is called upon to perform under various conditions of uncertainty regarding underlying physical processes. These conditions range from complete knowledge to total ignorance. As the process unfolds, additional information may become available to the controlling element, which then has the possibility of "learning" to improve its performance based upon experience; i.e., the controlling element may adapt itself to its environment.

On a grand scale, situations of this type occur in the development of physical theories through the mutual interplay of experimentation and theory; on a smaller scale they occur in connection with the design of learning servomechanisms and adaptive filters.

The central purpose of this paper is to lay a foundation for the mathematical treatment of broad classes of such adaptive processes. This is accomplished through use of the concepts of dynamic programming.

Subsequent papers will be devoted to specific applications in different fields and various theoretical extensions.

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1. Introduction

The purpose of this paper is to lay a foundation for a mathematical theory of a significant class of decision processes which have not as yet been studied in any generality. These processes, which will be described in some detail below, we shall call adaptive.

They arise in practically all parts of statistical study, practically engulf the field of operations research, and play a paramount role in the current theory of stochastic control processes of electronic and mechanical origin. All three of these domains merge in the consideration of the problems of communication theory.

Independently, theories governing the treatment of processes of this nature are essential for the understanding and development of automata and of machines that "learn."

We propose to illustrate how the theory of dynamic programming, [1], can be used to formulate in precise terms a number of the complex and vexing questions that arise in these studies. Furthermore, the functional equation approach of dynamic programming enables us to treat some of these problems by analytic means, and to resolve others, where direct analysis is stymied, by computational techniques.

In this paper, general questions are treated in an abstract fashion. In subsequent papers, we shall apply the formal structure erected here to specific applications.

2. Adaptive Processes

We wish to study multi-stage decision processes, and processes which can be construed to be of this nature, for which we do not possess complete information. This lack of information takes various forms of which the following are typical.

We may not be in possession of the entire set of admissible decisions; we may not know the effects of these decisions; we may not be aware of the duration of the processes and we may not even know the over-all purpose of the process. In any number of processes occurring in the real world, these are some of the difficulties we face.

The basic problem is that of making decisions on the basis of the information that we do possess. An essential part of the problem is that of using this accumulated knowledge to gain further insight into the structure of the processes, using analytic, computational and experimental techniques.

From this intuitive description of the types of problems that we wish to consider, it is clear that we are impinging upon some of the fundamental areas of scientific research. Obvious as the existence of these problems are, it is not at all clear how questions of this nature can be formulated in precise terms.

Particular processes of this type have been treated in a number of sources, such as the works on sequential analysis, cf.

Wald, [14]; the theory of games, cf. von Neumann and Morgenstern, [13]; the theory of multi-stage games, cf. Bellman, [1], Chapter 10; and the papers on "learning processes" of Flood, [5], [6], [7], Robbins, [11], Karlin and Johnson, [8], Bellman, [2], Bellman and Kalaba, [3].

3. The Unfolding of a Physical Process

In order to appreciate the type of process we wish to consider, the problems we shall treat, the terminology we shall employ, and the methods we shall use, it is essential that we discuss, albeit in abstract terms, the behavior of the conventional deterministic physical system.

Let a system S be described at any time t by a state vector p. Let $t_1, t_2, \ldots,$ be a sequence of times, $t_1 < t_2 < \cdots,$ at which the system is subject to a change which manifests itself in the form of a transformation. At time t_1, p_1 is converted into $T_1(p_1)$, at time t_2, $p_2 = T_1(p_1)$ is converted into $T_2(p_2)$, and so on, with the result that the sequence of states of the system is given by the sequence $\{p_k\}$, where

$$(1) \qquad p_{k+1} = T_k(p_k), \quad k = 1,2,\ldots .$$

The state of the system at the end of time t_N is then given by

$$(2) \qquad p_{N+1} = T_N(T_{N-1}(\cdots T_2(T_1(p_1))\cdots)),$$

where p_1 is the initial state of S.

If $T_k(p)$ is independent of k, which is to say, if the same transformation is applied repeatedly, then the preceding result can be written symbolically in the form

$$(3) \qquad p_{N+1} = T^N(p_1).$$

The interpretation of the behavior of a physical system over time as the iteration of a transformation was introduced by Poincaré, and extensively studied by G. D. Birkhoff, [4], and others. It furnishes the background for the application of modern abstract operator theory to the study of physical systems, as, for example, in quantum mechanics; cf. von Neumann, [12]. The idea of using this fundamental representation in connection with the formulation of the ergodic theorem is due to B. O. Koopman.

4. Feedback Control

With all this in mind, we are now able to introduce the concept of <u>feedback control</u>.

Supposing that the behavior of the system as described by the foregoing equations is not satisfactory, we propose to modify it by changing the character of the transformation acting upon p. This change will be made dependent upon the state of the system at the particular time the transformation is applied.

In order to indicate the fact that we now have a choice of transformations, we write $T(p,q)$ in place of p. The variable q indicates the choice that is made. Consequently, we shall call it the <u>control variable</u>, as opposed to p, the <u>state variable</u>. To simplify the notation and discussion, we shall assume that the set of admissible transformations does not vary with time.

If q_k denotes the choice of the control variable at time t_k, we have, in place of (3.1), the relation

$$(1) \qquad p_{k+1} = T(p_k, q_k), \quad k = 1, 2, \ldots,$$

with p_{N+1} explicitly determined as in (3.2).

The associated variational problem is that of choosing $q_1, q_2, \ldots, q_N$ so as to make the behavior of the system conform as closely as possible to some preassigned pattern. We wish, however, to do more than leave the problem in this vague format.

5. Causality

Turning back, for the moment, to the deterministic, uncontrolled process discussed in §3, let us note that the state of the system at time t_{k+1} is a function of the initial state of the system, and the number of transformations that have been applied. Consequently, we may write

$$(1) \qquad p_{k+1} = f_k(p_1),$$

where p_1 is the initial state of the system.

For the sake of convenience, let us merely write p in place of p_1. Then, the function $f_k(p)$ is easily seen to satisfy the basic functional equation

$$(2) \qquad f_{m+n}(p) = f_m(f_n(p)), \quad m,n = 1,2,\dots .$$

This is the fundamental semi-group property of dynamical systems.

6. Optimality

With the foregoing as a guide, let us see if we can formulate the feedback control process in the same terms.

To illustrate the applicability of the functional equation technique, let us consider a finite process, of N stages, where it is desired to maximize a preassigned function, ϕ, of the final state of the system, p_N. This is often called a terminal control process.

The variational problem may now be posed in the following terms:

$$(1) \qquad \operatorname*{Max}_{[q_1, q_2, \dots, q_N]} \phi(p_N).$$

This maximum, which we shall assume exists, is again a function of the initial state, p, and the duration of the process. Let us then introduce the function defined for all states p and $N = 1,2,\dots,$ by the relation

$$(2) \qquad f_N(p) = \operatorname*{Max}_{q} \phi(p_N),$$

where q represents the set $[q_1, q_2, \ldots, q_N]$.

Let us now introduce some additional terminology. A set of admissible choices of the q_i, $[q_1, q_2, \ldots, q_N]$, will be called a <u>policy</u>; a policy which maximizes $\phi(p_{N+1})$ will be called an <u>optimal policy</u>.

In order to obtain a functional equation corresponding to (5.2), we invoke the

<u>PRINCIPLE OF OPTIMALITY. An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.</u>

The mathematical transliteration of this statement is the functional relation

$$(3) \qquad f_N(p) = \max_{q_1} f_{N-1}(T(p,q_1)),$$

$N = 2,3,\ldots,$ with

$$(4) \qquad f_1(p) = \max_{q_1} \phi(T(p,q_1)).$$

Further discussion, and various existence and uniqueness theorems for the functions $\{f_i(p)\}$ and the associated policies will be found in [1].

In this way, the calculus of variations is seen to be a part of an extension of the classical theory of iteration, and of semi-group theory.

7. Stochastic Elements

In order to treat questions arising in the physical world in precise fashion, it is always necessary to make certain idealizations. Foremost among these is the assumption of known cause and effect, and, perhaps, even that of cause and effect in itself.

To treat physical processes in a more realistic way, we must take into account unknown causes and unknown effects. We find ourselves in the ironical position of making precise what we mean by ignorance.

At the present time, there exist a number of approaches to this fundamental conundrum, all based upon the concept of a random variable. Building upon this foundation is the theory of games.

We shall discuss here only the direct application of the concept of stochastic processes, leaving the game aspects for a later date.

The theory of probability in a most ingenious fashion skirts the forbidden region of the unknown by ascribing to an unknown quantity a distribution of values according to certain law. Having taken this bold step, it is further agreed that we shall measure performance not in terms of a single outcome, but in terms of an average taken over this distribution of values. Needless to add, this artifice has been amazingly successful in the analysis of physical processes; e.g. statistical mechanics, quantum mechanics.

Following this line of thought, we begin to take account of unknown effects by supposing that the result of a decision q is not to transform p into a fixed state $T(p,q)$, but rather to transform p into a stochastic vector z whose distribution function is $dG(z,p,q)$, dependent upon both the initial vector p and the decision q. Let us further suppose that the purpose of the process is to maximize the expected value of a preassigned function, ϕ, of the final state of the system.

Before setting up the functional equation analogous to (6.3), let us review the course of the process. At the initial time, an initial decision q_1 is made, with the result that there is a new state p_1, which is observed. On the basis of this information, a new decision, q_2, is made, and so on.

It is important to emphasize the great difference between a feedback control process of this type, in which the q_i are chosen stage-by-stage, and a process in which the q_i are chosen all at once at some initial time.

In the deterministic case, the two processes are equivalent, and it is only a matter of convenience whether we use one or the other formulation.* In the stochastic case, the two processes are equivalent only in certain special situations. We shall be concerned here only with the stage-by-stage choice.

The analogue of (6.4) is then

*This corresponds to the choice we have of describing a curve as a locus of points or as an envelope of tangents.

$$(1) \qquad f_1(p) = \operatorname*{Max}_q \int_z \phi(z)\,dG(z,p,q),$$

and that of (6.3) is

$$(2) \qquad f_N(p) = \operatorname*{Max}_q \int_z f_{N-1}(z)\,dG(z,p,q), \quad N = 2,3,\ldots \,.^*$$

This type of process has been discussed in some detail in [1].

8. Second Level Processes

Fortunately for the mathematician interested in these processes, the tale does not end here! It turns out to be the case that in a number of significant applications, it cannot be safely assumed that the unknown quantities possess known distribution functions.

In many cases, we must face the fact that we are dealing with more complex situations in which far less is known about the unknown quantities. For a discussion of the importance of these processes in the general theory of design and control, see McMillan, [9]; for a discussion of the dangers and difficulties inherent in *any* mathematical treatment, see Zadeh, [15].

A first attempt in salvaging much of the structure already erected is to assume that the unknown quantities possess

*The descriptive version of this equation, when no control is exerted, is, of course, the Chapman-Kolmogoroff equation, the stochastic analogue of (5.2).

fixed, but unknown, distribution functions. Regarding deterministic processes as those of zeroeth-level, and the stochastic processes described in §7 as first-level processes, we shall refer to these new stochastic processes as <u>second-level</u> processes.

Although it is clear that we now possess a systematic method for constructing a hierarchy of mathematical models, we shall restrain ourselves in the remainder of this paper to the discussion of second-level processes.

9. Additional Assumptions

Some further assumptions are required if we wish to proceed from this point to an analytic treatment. These are

I. We possess an à priori estimate for the distribution function governing the physical state of the system, which, until further knowledge is acquired, we regard as the actual distribution.

II. We possess a set of rules which tells us how to modify this à priori distribution so as to obtain an à posteriori distribution when additional information is obtained.

III. We possess an à priori estimate for the distribution functions governing the outcomes of decisions, which, until further knowledge is acquired, we regard as the actual distribution, and, as above, we know how to modify this in the light of subsequent information.

In this paper, we restrict ourselves to the case of known physical states.

In formal terms, our state vector is now compounded of a point in phase space, p, and an <u>information</u> <u>pattern</u>, $dG(z,p,q)$. As a result of a decision q_1, there result the transformations

$$(1) \qquad p_0 \longrightarrow p_1 \qquad \text{(observed)}$$

$$dG(z,p^*,q) \longrightarrow dH(z,p^*,q;p_0,G,q_1,p_1) \qquad \text{(hypothesized).}$$

On the basis of these assumptions, and considering a control process which continues in time as described in §7, we wish to pose the problem of determining optimal policies. For the first time, we are considering adaptive processes significantly different from those of the usual deterministic or stochastic control process.

10. Functional Equations for Second-level Processes

As before, we introduce the function

(1) $f_N(p;G(z,p^*,q))$ = the expected value of $\phi(p_N,G_N)$ obtained using an optimal policy for an N-stage process starting in state (p,G).

Depending upon the objectives of the process, only one or the other of p_N and G_N may enter into ϕ. Examples of both extremes abound.

Arguing as in the preceding sections, we see that the basic recurrence relation is

$$(2) \qquad f_N(p;G(z,p^*,q))$$

$$= \operatorname*{Max}_{q_1} \int_w f_{N-1}(w;H(z,p^*,q;p,G,q_1,w))dG(w,p,q_1),$$

for $N = 2,3,\ldots,$ with

$$(3) \qquad f_1(p;G(z,p^*,q))$$

$$= \operatorname*{Max}_{q_1} \int \phi(w,H(z,p^*,q;p,G,q_1,w))dG(w,p,q_1).$$

These equations are quite useful in the derivation of existence and uniqueness theorems concerning optimal policies, return functions, and in ascertaining certain structural properties of optimal policies; cf. [1], [2].

If, however, we treat processes which are too complex for a direct analytic approach, as is invariably the case for realistic models, we wish to be able to fall back upon a computational solution. The occurrence of functions of functions, e.g. the sequence $\{f_N(p;G)\}$, effectively prevents this.

11. Further Structural Assumptions

In order to reduce the foregoing equations to more manageable form, let us assume that the structure of the actual distribution is known, but that the uncertainty arises with regard to the values of certain parameters.

At any stage of the process, in place of an à priori estimate, $G(z,p,q)$, for the distribution function, we suppose that we have an à priori estimate for the distribution function governing the unknown parameters. Again, a basic assumption is that this distribution function exists.

The functional equations that we derive are exactly as above, with the difference in meaning of the distribution functions that we have just described.

12. Reduction from Functionals to Functions

We are now ready to take the decisive step of reducing $f_N(p,G)$ from a functional to a function.

It may happen, and we will give an example in a moment, that the change in the distribution function, from $G(z,p,q)$ to $H(z,p^*,q;p,G,q_1,w)$ is one that can be represented by a point transformation. This will be the case if G and H are both members of a family of distribution functions $K(z;\alpha)$ characterized by a vector parameter α. Thus, if

$$(1) \qquad G(z,p,q) \equiv K(z,p,q;\alpha)$$

$$H(z,p^*,q;p,G,q_1,w) \equiv K(z,p,q;\beta),$$

the change from G to H may be represented by

$$(2) \qquad \beta = \Psi(p,\alpha,q_1,w).$$

Then we may write

(3) $$f_N(p, G(z,p,q)) \equiv f_N(p;\alpha),$$

and (10.2) becomes

(4) $$f_N(p;\alpha) = \operatorname*{Max}_{q_1} \int_w f_{N-1}(w;\beta)dK(w;\alpha).$$

The dependence upon q_1 is by way of (2).

13. An Illustrative Process--Deterministic Version

Let us now show how these ideas may be applied to the study of control processes. Consider a discrete scalar recurrence relation of the form

(1) $$u_{n+1} = au_n + v_n, \quad u_0 = c.$$

Here u_n is the state variable and v_n is the control variable. Suppose that the sequence $\{v_n\}$ is to be chosen to minimize the function

(2) $$|u_N| + b \sum_{k=1}^{N} u_k^2,$$

subject to the constraints

(3) $$|v_i| \leq r, \quad i = 0,\ldots,N-1.$$

Although the precise analytic form of the criterion function is of little import as far as the present discussion is concerned, we have used specific functions to make the presentation as concrete as possible. Furthermore, the defining equation need not be linear.

This is a simple example of a deterministic control process. Introduce the sequence of functions defined by the relation

$$(4) \qquad f_N(c) = \underset{\{v_i\}}{\text{Min}} \left[|u_N| + b \sum_{k=1}^{N} u_k^2 \right],$$

where N takes on the values $1,2,\ldots,$ and c any real value.

Then

$$(5) \qquad f_1(c) = \underset{|v_0| \leq r}{\text{Min}} \left[|ac + v_0| + b(ac + v_0)^2 \right],$$

and for $N \geq 2$, the principle of optimality yields the relation

$$(6) \qquad f_N(c) = \underset{|v_0| \leq r}{\text{Min}} \left[b(ac + v_0)^2 + f_{N-1}(ac + v_0) \right].$$

14. Stochastic Version

In place of the recurrence relation of (13.1), let us introduce a stochastic transformation

$$(1) \qquad u_{n+1} = au_n + r_n + v_n, \quad u_0 = c.$$

Here $\{r_n\}$ is a sequence of independent random variables assuming only the values 1 and 0. Let

$$(2) \qquad \begin{aligned} r_n &= 1 \text{ with probability } p \\ &= 0 \text{ with probability } 1 - p. \end{aligned}$$

The quantity p is known, and for simplicity taken to be independent of n, although this is not necessary.

We now wish to minimize the expected value of the quantity appearing in (13.2). This is now a stochastic control process of the type described above in general terms. Call the minimum expected value $f_N(c)$. Then, following the procedures of §7, we have the relations

$$(3)\quad f_1(c) = \operatorname*{Min}_{|v_0|\le r}\left[\int_{r_0}\left[|ac + v_0 + r_0| + b(ac + r_0 + v_0)^2\right]dG(r_0)\right]$$

$$= \operatorname*{Min}_{|v_0|\le r}\left[p\left[|ac + v_0 + 1| + b(ac + v_0 + 1)^2\right] + (1 - p)\left[|ac + v_0| + b(ac + v_0)^2\right]\right],$$

and, for general N,

$$(4)\quad f_N(c) = \operatorname*{Min}_{|v_0|\le r}\left[p\left[b(ac + v_0 + 1)^2 + f_{N-1}(ac + v_0 + 1)\right] + (1 - p)\left[b(ac + v_0)^2 + f_{N-1}(ac + v_0)\right]\right].$$

15. Adaptive Control Version

Let us now consider the adaptive control version. We are given the information that the random variables r_n possess distributions of the special type described above, but we do not know the precise value of p.

We shall assume, however, that we do possess an à priori distribution for the value of p, $dG(p)$, and that we possess a known rule for modifying this à priori distribution on the basis of the observations that are made as the process unfolds.

If we observe that over the past $m + n$ stages, the random variables have taken on m values of 1 and n values of 0, we take as our new à priori distribution the function

$$(1) \qquad dG_{m,n}(p) = p^m(1-p)^n dG(p) \Big/ \int_0^1 p^m(1-p)^n dG(p),$$

a Bayes approach.*

Once we have fixed upon a choice of $G(p)$, the à priori distribution function at any stage of the process is uniquely determined, from the foregoing, by the numbers m and n. This simple observation enables us to reduce the information pattern from that of the specification of a number, or vector, in general, plus a function $G_{m,n}(p)$, to that of the specification of three numbers, c and the two integers m and n.

In this way, we reduce the problem from one requiring the use of functionals to one utilizing only functions. This is an essential step not only for computational purposes, but for analytic purposes as well.

Let us then introduce the sequence of functions $\{f_N(c,m,n)\}$ defined once again as the minimum expected value of the quantity in (13.2), starting with the information pattern of m ones and n zeros, and state c.

Then

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*This is an assumption of the type called for in §9. Although reasonable, it is not the only one possible. There are analytical advantages in choosing G to be a beta distribution.

(2) $$f_1(c,m,n) = \min_{|v_0| \leq r} \Big[p_{m,n}\big[b(ac + v_0 + 1)^2 + |ac + v_0 + 1|\big] + (1 - p_{m,n})\big[b(ac + v_0)^2 + |ac + v_0|\big]\Big],$$

where $p_{m,n}$ is the expected probability using the probability distribution in (1), i.e.,

(3) $$p_{m,n} = \frac{\int_0^1 p^{m+1}(1 - p)^n dG(p)}{\int_0^1 p^m(1 - p)^n dG(p)} .$$

For $N \geq 2$, we have the recurrence relation

(4) $$f_N(c,m,n) = \min_{|v_0| \leq r} \Big[p_{m,n}\big[b(ac + v_0 + 1)^2 + f_{N-1}(ac + v_0 + 1, m + 1, n)\big] + (1 - p_{m,n})\big[b(ac + v_0)^2 + f_{N-1}(ac + v_0, m, n + 1)\big]\Big].$$

In this fashion, we obtain a computational approach to processes with general criteria and an analytic approach to processes with criteria of particular type. A thoroughgoing discussion of the analytic aspects of the solution of processes of this nature described by linear equations and quadratic criteria will be found in a forthcoming doctoral thesis by Marshall Freimer.

Previous applications of these techniques may be found in [2] and [3].

16. Sufficient Statistics

The fact that the past history of the process described in the preceding paragraphs can be compressed in the indicated fashion, so that functions rather than functionals occur, is a particular instance of the power of the theory of "sufficient statistics;" cf. Mood, [10].

Many further applications of this important concept will be found in the thesis of Freimer mentioned above.

In a number of cases, this compression of data occurs asymptotically as the process continues; e.g. the central limit theorem. A number of quite interesting questions arise from this observation.

17. Discussion

In the foregoing pages, we have attempted to construct a mathematical foundation for the study of the many fascinating aspects of the field of adaptive control. In further papers, we shall discuss a number of complex problems which arise from this approach.

From the purely mathematical point of view, we are now able to contemplate a theory of continuous control processes of adaptive type, obtained as a limiting form of the theory of discrete control processes. A variety of significant convergence questions are encountered in this way.

Furthermore, we can on the same foundations construct a theory of multi-stage games.

Finally, the problem of computational solution is by no means routine, and there are a variety of interesting approaches based upon approximations in function space and approximations in policy space to be explored.

From the conceptual point of view, we must face the fact that there are many further uncertainties to be examined, in the state of the system, in the observation of the random effect, in the transmission of the control signal, in the duration of the process, and even in the criterion function itself.

REFERENCES

1. Bellman, R., Dynamic Programming, Princeton University Press, Princeton, New Jersey, 1957.

2. Bellman, R., "A problem in the sequential design of experiments," Sankhya, vol. 16, 1956, pp. 221-229.

3. Bellman, R., and R. Kalaba, "On communication processes involving learning and random duration," 1958 IRE National Convention Record, part 4, July, 1958, pp. 16-21.

4. Birkhoff, G. D., Dynamical Systems, American Mathematical Society Colloquium Publications, vol. IX, 1927.

5. Flood, M. M., "The influence of environmental nonstationarity in a sequential decision-making experiment," Decision Processes, edited by R. M. Thrall, C. H. Coombs, and R. L. Davis, John Wiley and Sons, Inc., New York, 1954.

6. Flood, M. M., "On game-learning theory and some decision-making experiments," Decision Processes, edited by R. M. Thrall, C. H. Coombs, and R. L. Davis, John Wiley and Sons, Inc., New York, 1954.

7. Flood, M. M., "On stochastic learning theory," Trans. New York Acad. Sci., February, 1954.

8. Karlin, S., R. Bradt, and S. Johnson, "On sequential designs for maximizing the sum of n observations," Ann. Math. Stat., vol. 27, 1956, pp. 1061-1074.

9. McMillan, B., "Where do we stand," IRE Trans. on Information Theory, vol. IT-3, 1957, pp. 173-174.

10. Mood, A. M., Introduction to the Theory of Statistics, McGraw-Hill Book Company, Inc., New York, 1950.

11. Robbins, H., "Some aspects of the sequential design of experiments," Bull. Amer. Math. Soc., vol. 58, 1952, pp. 527-535.

12. von Neumann, J., Mathematische Grundlagen der Quantenmechanik, Dover Publications, Inc., New York, 1943.

13. von Neumann, J., and O. Morgenstern, The Theory of Games and Economic Behavior, Princeton University Press, Princeton, New Jersey, 1948.

14. Wald, A., Sequential Analysis, John Wiley and Sons, Inc., New York, 1947.

15. Zadeh, L., "What is optimal," IRE Trans. on Information Theory, vol. IT-4, 1958, p. 3.