SOLVING TWO-MOVE GAMES WITH PERFECT INFORMATION

George B. Dantzig

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SUMMARY

A two-move game with perfect information is considered, such as a move and counter-move situation between two firms or economies. This leads to the problem of finding a global minimum of a concave function over a convex domain and the distressing possibility of local minima at every extreme point. It is shown however that the global minimum can be obtained by solving a linear programming system with side conditions that at least one of certain pairs of variables vanish. The latter problem can be shown to be equivalent to solving a linear programming problem with some integer valued variables.

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Consider a two-move game where player X can engage in any vector $x = (x_1, x_2, \ldots, x_n)$ of activity levels $x_j \geq 0$, consistent with a fixed inventory vector $e = (e_1, e_2, \ldots, e_m)$, say

$$Ex = e \qquad (x \geq 0) \tag{1}$$

where E is an m x n matrix. This constitutes X's move. In so doing he leaves an inventory position $f + \bar{E}x$ for player Y where $\bar{E}$ is a given m' x n matrix and f an m' component vector. This requires that Y chose as his move an activity vector $y = (y_1, y_2, \ldots, y_{n'})$ so that

$$Fy = f + \bar{E}x \qquad (y \geq 0) \tag{2}$$

where F is a given m' x n' matrix. It is assumed that x must be chosen so that an admissible move for Y exists. We remark in passing that a chess or checker game restricted to one move by each player can be cast in this form if there are added side constraints regarding the discrete character of a move. However a competitive situation of a move and a counter-move between two firms or two economies, would be more significant.

Let us suppose the payment to Y by X is given by

$$z = \alpha x - \beta y \tag{3}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$. It is clear that an optimum for X is to chose x so that his payment to Y is

$$\hat{z} = \min_{x} [\alpha x - \min_{y|x} \beta y] \tag{4}$$

where we further assume βy is bounded from below for fixed x.

This is basically a very difficult problem because $\text{Min}_y \beta y$ for y satisfying (2) is a convex function of x but this implies that

$$z' = [\alpha x - \text{Min}_y \, by] \tag{5}$$

is a concave function of x which is to be minimized over a convex domain of x satisfying (1) and (2). This can lead to local optima at one, many, or all extreme points of the convex domain of x.

For example suppose

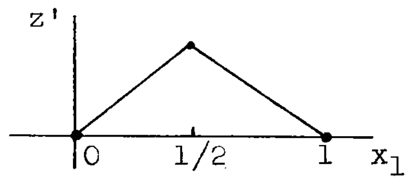
$$\begin{aligned} &x_1 \leq 1 && x_1 \geq 0 \\ &y_1 \leq 1 - x_1 && y_1 \geq 0 \\ &y_1 \leq x_1 && \\ &z = 0 \cdot x_1 - (-y_1) = y_1 \, , && \end{aligned} \tag{6}$$

then the function z' to be minimized is

$$z' = -\text{Min}(-y) = \begin{cases} x_1 & \text{if } 0 \leq x_1 \leq 1/2 \\ 1 - x_1 & \text{if } 1/2 \leq x_1 \leq 1 \end{cases} \tag{7}$$

which has two local minima, one at $x_1 = 0$ and the other at $x_1 = 1$:

(8)



The values of z' at these local minima happen to be equal but a slight perturbation could cause either one to be the global minimum.

By careful application of the duality theorem this problem can be reduced to a linear programming problem subject to a set of n' pairs of linear conditions either $y_j \geq 0$ or $\eta_j \geq 0$ for $j = 1,2,\ldots,n'$; here η_j are the dual variables along with $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ satisfying

(9) $$\pi F_j + \eta_j = \beta_j \qquad \eta_j \geq 0, (j=1,2,\ldots,n')$$

where F_j is the j^{th} column of F. We first remark for any fixed x, there exist an optimum $y = y^*$ satisfying (2) which minimizes βy. Associated with this x is also an optimum solution to the dual of (2) with variables π (unrestricted in sign associated with the m' equations) and non-negative variables $\eta_j \geq 0$ corresponding to y_j satisfying (9). The necessary and sufficient conditions that a solution of the primal and dual systems be optimal is that

(10) either $y_j = 0$ for $j = 1,2,\ldots,n'$

or $\eta_j = 0$.

We now prove the following fundamental theorem:

THEOREM: An optimal solution to the two-move game (1), (2), (3) is found by choosing x and y satisfying (1) and (2), auxiliary variables π and η satisfying (9) and (10), and Min z satisfying (3).

Proof: The proof is along standard lines and immediate. An optimal solution to the game exists at one of the extreme points of the convex of x defined by (1) and (2) say at $x = \hat{x}$ for which there is a $y = \hat{y}$ and $\pi = \hat{\pi}$, $\eta = \hat{\eta}$ that satisfy (2), (9), (10) and yields the value $z = \hat{z}$ defined by (4). Hence

(11) $$\text{Min } z \leq \hat{z}$$

On the other hand we can produce a solution x^*, y^*, π^*, η^* to (1), (2) (9), (10) which minimizes z by devices considered in [1] which shows that this type of problem is equivalent to a linear programming problem with some integer valued variables for which efficient procedure may exist [2], [3]. For the chosen value of $x = x^*$, (10) implies that the y^* is chosen so as to minimize βy. Hence the set of x^*, y^*, chosen this way is an admissible two moves in a game and its $z = \text{Min } z$ must satisfy

(12) $$\hat{z} \leq \text{Min } z \;;$$

whence from (11) we have

(13) $$\hat{z} = \text{Min } z$$

completing the proof.

REFERENCES

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2. Gomory, Ralph, "Essentials of an Algorithm for Integer Solutions to Linear Programs" [communicated to Bull. Amer. Math. Soc. in letter from Princeton April 23, 1958]. This work was directly extended by Gomory at RAND in June 1958 to the case where some variables have integer values.

3. Beale, E.M.L., "A Method of Solving Linear Programming Problems with Some but Not all of the Variables must take Integral Values." Unpublished draft approximately dated May 1958.