COMPUTING TETRAETHYL LEAD REQUIREMENTS
IN THE LINEAR PROGRAMMING FORMAT

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SUMMARY

There are some types of non-linearities that are difficult to incorporate into a linear programming model. One of these occurs in the scheduling of refinery operations where linear programming methods are widely used. The specific problem is the one of simultaneous determination of the quantities of various components of gasoline and tetraethyl lead to blend to form regular or premium type gasolines. It is believed the general approach can be used for other non-linear situations. Linearization is accomplished by viewing the lead requirements of a blend first as a known general function of the variables each of which is a linear function of the quantities in the blend. The general function is then approximated by a convex linear combination of a mesh of representative points.
1. BACKGROUND

Linear programming models are used to schedule the activities of a refinery, and to determine the quantities of crudes to buy and products to produce in order to maximize profits. Because linear programming models are a system of linear inequalities rather than equations they can be used to express many bottleneck and decreasing payoff situations normally thought of as non-linear.

There are, however, some types of non-linearities that have been difficult to approximate and to incorporate in the general linear programming format. One of these is to determine simultaneously the quantities of tetraethyl lead (TEL) and quantities of components forming the final blend of gasoline. This problem has been studied by Garvin and reported in [1]. Our procedure uses Garvin's underlying approach to obtain, we believe, a simpler result. [See footnote 4.]

The reason that TEL is added to a gasoline (blend or pure component) is to increase its octane rating as measured by one or more standard methods. The exact functional relation between octane and TEL has been the subject of considerable theoretical and empirical research. The results are a set of charts that the industry uses to predict the TEL required to bring various possible blends up to specified octane ratings. Incidentally, once a blend has actually been prepared, TEL is added to it...
until performance in a test engine indicates that the desired octane rating has been achieved—the charts are used only to predict in advance what it would be if the particular blend were produced.

On the basis of this research we shall assume two properties of the effect of TEL on octane rating of gasolines that are approximately true in practice:

1. If two or more gasoline components each have the same amount of TEL per gallon, the octane rating of their blend is the weighted arithmetic average of the octane rating of their components at the specified TEL level.

2. If the octane ratings of two gasolines happen both to be equal to $a$ when the quantity $I$ of TEL per gallon are added to each and they are both equal to $b$ when a different quantity $I'$ per gallon is added, then the octane response curves of the two gasolines will be the same whatever be the amount added per gallon.

Thus if two points on the curve which plots the octane rating of a blend as a function of TEL are known, the rest of the curve is defined. For example, if the octane ratings of the blend at 0 and 5cc of TEL per gallon are known—say these values are $a$ and $b$ respectively—it is possible to use a standard reference chart to determine the octane rating

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1The term "octane rating" included the use of "blending octane rating" of components when the raw octane ratings do not blend volumetrically.
of the blend for any other quantity of TEL (see [2]). In particular, the amount \( t \) of TEL needed to bring the blend up to a specified octane rating for sale on the market as regular or premium gasoline can be determined once the values \( a \) and \( b \) are known. On the other hand, the values of \((a,b)\) of the blend can be determined from the first property for they are equal to the weighted average of the corresponding octane ratings of the components at 0 and 300 TEL per gallon.

It may be wondered by the reader how good an approximation of the TEL might be obtained by simply forming a weighted average of the TEL requirement to bring each component individually up to the specified octane rating? The answer is that the approximation is not possible because some components have so poor an octane rating at 0cc and such poor response to TEL that no amount of TEL would bring them up to the specified standard, at the same time others are so rich that they are already above the standard with no TEL added. Accordingly, the approach we have taken is to assume that the possible \((a,b)\) values of the blend will be in narrower limits than those exhibited by the components.

As pointed out above, the value \( t \) of required TEL to bring a blend up to a fixed octane specification can be determined from a reference chart as a function of the octane rating \( a \) at 0cc of TEL and \( b \) at 300cc of TEL. This can be pictured as a surface in three dimensions when \( t \) is plotted as a function of \( a \) and \( b \). A three-dimensional plot of this surface for 90 octane has been constructed by the authors to see if it is convex. If the surface is not convex it means that sometimes
it might be advantageous not to make one blend of gasoline from
the components but several different kinds of blends each of
which has the required octane rating with varying amounts of
TEL, the sum of which is less than that for a single blend. On
the other hand, if the surface is convex there would be
only one blend. In this case well-known linear programming
methods would indicate that a good procedure (assuming cost
minimization) would be to approximate any point on the surface
by a weighted average of its values on a lattice (mesh) of
near-by points. It turns out, however, that in the range of
(a,b) values of interest convexity does not hold. Fortunately
the surface turns out to be close enough to convexity that
the approximation could be used with little error.

2. THE PROCEDURE

A simple procedure will now be given for computing TEL
requirements in the linear programming format for the case
where there is only one octane specification to be met. This
will then be extended to the case for two different octane
specifications.

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2This discussion assumes that the cost of TEL is propor-
tional to the amount of TEL used and the objective of the pro-
gram is to minimize costs. Convexity of the surface means, by
definition, that the point (a', b') which is the weighted arithmetic
average of two points (a_1, b_1), (a_2, b_2) will not use more TEL
than the weighted arithmetic average of their TEL.

3Octane rating of a gasoline depends on the conditions
used to measure the "pump" characteristic. One gasoline can
appear relatively superior to another using the "research"
method and inferior using the "motor" method. Hence more than
one specification is often used.
Let $a_j =$ octane rating of the $j$-th component with no TEL, 
$j = 1, 2, \ldots, n$.

$\beta_j =$ octane rating of the $j$-th component with 3cc TEL, 
$j = 1, 2, \ldots, n$.

$a =$ octane rating of the blend with no TEL (unknown).

$b =$ octane rating of the blend with 3cc TEL (unknown).

$i =$ cc of TEL per gallon of blend to bring the blend to specification octane (unknown).

Plot the various pairs $(a_j, \beta_j)$ as coordinates of points on a two-dimensional graph. Form a convex hull of these points, the shaded region in Figure 1.
Let the rating of a blend at 0 and 3cc TEL be $a$ and $b$. It will be assumed that the octane rating of a blend at 0-TEL is the arithmetic average of the $\alpha_j$ rating of its components and at 3cc TEL it is arithmetic average of the $\beta_j$ rating of its components. If the $(a,b)$ value of the blend is plotted as a point on the graph, it must fall in the shaded region of Figure 1. By suitable weighting of the components, any point in the shaded area can be obtained.

Associated with each given blend point $(a,b)$ is the lead requirement, $L$, per gallon which is necessary to bring the

![Diagram](image.png)

*Fig. 2 — Possible blend characteristics*
blend up to the required octane rating.  

Select a mesh of possible \((a, b)\) values. These are shown as points connected by dotted lines in Figure 2. Let \((x_k, y_k, l_k)\) be the values \((a, b, l)\) for the \(k\)-th point in the mesh. The mesh should be sufficiently fine so that given any possible blend point \((a, b, l)\) we can always find three points in the mesh that surround it such that the weighted average of these three points that yield \((a, b)\) will also give a weighted TEL requirement that approximates closely enough the TEL requirement of the point \((a, b)\). The actual blend is viewed in two ways as a blend of components and an average of "standard" blends represented by the points \((x_k, y_k)\).

Let \(x_j\) represent the quantity of the \(j\)-th component in the blend and \(y_k\) represent the quantity of the \(k\)-th type standard blend, i.e., the mesh point \((x_k, y_k, l_k)\) where \(k = 1, 2, \ldots, p\). Let \(l\) be the total amount of TEL required. The variables \(x_j\) form a part of a larger model; they in turn determine indirectly \(l\) via the auxiliary variables \(y_k\) by means of the relations.

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Up to this point this method is similar to one described in a paper, "Application of Linear Programming in the Oil Industry" by W. W. Carlin, H. W. Grindall, J. B. John and H. R. Nellis, presented at the Third Annual Meeting of the Institute of Management Science, Los Angeles, California, October 16-19, 1956 (see [2]). These authors represent the surface in terms of octane number with \(Q\) and TEL as above and lead susceptibility. The broken-line fits to certain curves are analogous to our use of \(a\) mesh; however, their use of incremental variables leads to more relations than our approach.
Volume: \[ \sum_{j=1}^{n} x_j = \sum_{k=1}^{p} y_k \]

Octane: 0-TEL \[ \sum_{j=1}^{n} a_j x_j = \sum_{k=1}^{p} a_k y_k \]

Octane: 300 TEL \[ \sum_{j=1}^{n} b_j x_j = \sum_{k=1}^{p} b_k y_k \]

TEL: \[ L = \frac{1}{L} \sum_{k=1}^{p} y_k \]

The first relation states that the volume of the components equals the volume of the different possible blends. The second relation (if divided by the common volume) states that the average "a" value of the components equals the average "a" value of the possible blends. The third relation is similar for the average "b" value. The fourth relation states that the total TEL is the weighted sum of the amounts required per unit for each blend.

If it is desired to develop a blend which meets two different octane specifications simultaneously, a second system of constraints are developed similarly which we will denote by primes. A TEL balance equation must be included since the same TEL affects both types of octane ratings. The enlarged subsystem would then be:
First Octane Specification:

Volume: \[ \sum_{j=1}^{n} x_j = \sum_{k=1}^{p} y_k \]

Octane: \[ \sum_{j=1}^{n} a_j x_j = \sum_{k=1}^{p} a_k y_k \]

Octane: \[ \sum_{j=1}^{n} b_j x_j = \sum_{k=1}^{p} b_k y_k \]

TEL: \[ L = \sum_{k=1}^{p} l_k y_k \]

Second Octane Specification:

Volume: \[ \sum_{j=1}^{n} x_j = \sum_{k=1}^{p'} y_k' \]

Octane: \[ \sum_{j=1}^{n} a_j x_j = \sum_{k=1}^{p'} a_k y_k' \]

Octane: \[ \sum_{j=1}^{n} b_j x_j = \sum_{k=1}^{p'} b_k y_k' \]

TEL balance: \[ \sum_{k=1}^{p'} l_k y_k = \sum_{k=1}^{p'} l_k y_k' \]
Note that we do not force the selected mesh points under the second way of measuring octane to be in one to one correspondence with those used for the first way since there could be a range of possible $a', b'$ values for a given $a, b$ value. The correlation is obtained automatically through the equating of the average blend values with the averages for the components.

The above equations do not include slack variables which can be added to the octane equations to allow the blend octanes to exceed specifications.

The optimal solution will usually choose three mesh points for each octane specification whose weighted averages will be interpreted as the true blend point.

It is possible that the three mesh points finally selected can be widely separated (due to a lack of convexity in the underlying TEL requirements function). This can be interpreted to mean that it may pay to make up two or three types of standard blends. Empirically there appears in these cases to be only a slight gain over the single blend.
REFERENCES
