A GAME THEORY ANALYSIS OF TACTICAL AIR WAR

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SUMMARY

An important problem in tactical air war is concerned with the allocation at each strike of the tactical forces among such competing air tasks as counter-air, air-defense, and support of ground operations. We formulate a two-person multimove game in which the allocation decisions of the combatants represent the moves of the game. In this game model we assume that counter-air missions destroy enemy forces, air-defense missions reduce the enemy's counter-air operations, and support of ground operations contribute to the payoff. We describe the optimal allocations derived from the game-theoretic analysis of this model. The paper concludes with a discussion of possible implications of our results for operational gaming.
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1. INTRODUCTION

The problem of optimal employment of tactical air forces in the various theater tasks, like many other military questions, can be analyzed as a multimove game between two opposing sides. In this game each side seeks the largest payoff possible in the form of some theater mission.

During the course of a tactical air campaign, the commanders on each side are faced with many decisions which affect the outcome of the campaign. They must decide such things as type of weapon to be used on a particular mission, specific targets to be hit, mission profile, delivery tactics, etc. A most important and basic decision in a tactical air war is the allocation of aircraft among the various theater air tasks. Our concern in this paper will be with the allocation decisions.

We shall present the results of a game-theoretic analysis of a simplified but fairly realistic version of the tactical air war viewed as a multimove game. It almost goes without saying that it is the qualitative aspects of the solution and statements about the nature of the solution, rather than the precise quantitative results, that are the more important.

*Some of the results in this paper were presented to the American Mathematical Society on November 15, 1957 in Los Angeles, California and August 27, 1958 in Cambridge, Massachusetts.
conclusions to be drawn from models of this sort. Besides their relevance to the tactical air problem, our results may have some implications for war gaming or operational gaming. These are discussed in the last section of the paper.

The mathematical proofs of the results to be presented here are given in [1], [2], and [3]. However, the basic idea underlying all of the analyses, together with its proof, is presented in the Appendix. A precise mathematical formulation of the game in normal form is also given in the Appendix.

2. THEATER AIR OPERATIONS

The tactical air war is characterized by a series of air actions or tasks undertaken in order to accomplish some defined theater mission. Among the usual tasks are the following.

Counter Air. These operations are against the enemy's theater air-base complex and organization in order to destroy his aircraft, personnel, facilities, etc.

Air Defense. These represent air-defense operations against the enemy's counter-air operations.

Close Air Support. The targets for close-support operations are concentrations of enemy troops or fortified positions in order to help the ground forces in the battle area. This is accomplished by aerial delivery of fire power against the enemy ground targets.
Interdiction. These operations reduce the enemy's military potential by attacking the transportation facilities.

Reconnaissance. The most important function of these operations is to obtain information about the targets.

Airlift. In this operation the planes are used to transport troops and equipment.

We shall take close air support, interdiction, reconnaissance for the ground forces, and airlift tasks to constitute what is termed "support of ground operations," or "ground support," in this paper. Reconnaissance directed against enemy air will be assumed as part of the counter-air task.

3. FORMULATION OF GAME

We view the tactical air war game as consisting of a series of strikes, or moves, each of which consists of simultaneous counter-air, air-defense, and close-support operations by each side undertaken to accomplish a given theater mission or payoff. Let us assume that at the start of the air operations the Blue side has $p$ planes and the opposing side, Red, has $q$ planes. Let us look at a strike in the campaign, say the initial strike. Suppose that on this strike Blue dispatches $x$ planes on counter-air operations and $u$ planes on air-defense operations, and the remaining amount, $m = p - x - u$ planes, on ground-support operations. Similarly, suppose that
for his first strike Red allocates $y$ planes to counter-air, $w$ planes to air-defense, and the remaining number, $n = q - y - w$ planes, to support his ground forces. For this initial strike and for any future strikes, the above decisions are made by each side in ignorance of the allocation of the opposing side. It is assumed, however, that each side knows the number of planes that he and his opponent have.

Since Red allocates $w$ planes to air-defense we can expect a reduction in the number of Blue's planes that get through to counter-air targets. The number of interceptions by Red will be proportional to $w$, say $cw$, unless Blue's attacking planes are saturated. The proportionality constant, or Red's defense potential, depends on the planes' characteristics and flying altitudes, and on their weapons' characteristics. The number of Blue attacking planes that penetrate Red's defenses is $x - cw$ as long as $cw$ is not larger than $x$. If $cw$ is larger than $x$, no Blue aircraft will penetrate. Hence the number of Blue attacking planes that penetrate Red's defenses is the larger of the two numbers $x - cw$ and 0, or

$$\max(0, x - cw).$$

The objective of Blue's counter-air operations is to reduce the enemy's air force by dropping bombs on certain targets, and the number of aircraft destroyed will vary with the number of attacking planes that penetrate Red's defenses.
If we assume that each of Blue's penetrating planes can destroy b planes of the enemy, then Blue's initial counter-air strike can destroy at most

\[ b \text{ max} (0, x - cw) \]

Red planes. The number of Red planes actually destroyed by Blue's counter-air strike will depend on the number of Red aircraft at risk at the time of the strike. The proportionality constant b, or Blue's kill potential, depends on the target as well as the aircraft characteristics.

We assume that Red's air force is also reduced during the strike by such factors as accidents and antiaircraft fire. Let us assume that these losses are proportional to the number of planes used by Red during the strike, or aq, where a represents Red's accident rate.

Finally, let us assume that Red's air force is increased during the strike by s planes. These replacements are subject to Blue's counter-air attack, but can not be used by Red during this strike.

Thus, the number of Red planes at risk at the time of the strike is \( q - aq + s \). Therefore Blue's initial counter-air strike will destroy

\[ \text{min} [q - aq + s, b \text{ max} (0, x - cw)] \]

Red planes.
The planes used in air-defense are assumed to survive, and the Red aircraft that fail to penetrate the Blue air-defense are assumed to return to base. That is to say, we assume that losses suffered in the air battle are negligible compared to the other losses, and that air-defense aircraft prevent attacking planes from successfully delivering their bombs without necessarily shooting them down. For example, the attacker may be damaged to the extent that he must turn back, he may be chased off course, his bomb run may be disturbed so that he drops his bombs ineffectively, etc.

If we sum the losses and add the replacements, we see that after the initial strike Red's force is reduced to

$$q_1 = q + s - aq - \min[q - aq + s, b \max(0, x - cw)]$$

$$= \max[0, q + s - aq - b \max(0, x - cw)].$$

In exactly the same manner we can analyze the effect of the initial strike on Blue's inventory. We obtain that at the end of the initial strike, Blue's inventory of planes is

$$p_1 = \max[0, p + r - dp - e \max(0, y - iw)],$$

where the coefficients $d$, $e$ and $k$ have the same interpretation as $a$, $b$, and $c$, respectively, and $r$ is the number of Blue replacements.

Blue now has $p_1$ planes to allocate among the three tasks for the second strike, and Red has $q_1$ planes to allocate for
the second strike. This strike will result in new inventories, \( p_2 \) and \( q_2 \), for the third strike. The process is repeated for the duration of the campaign.

4. PAYOFF

Let us look at Blue's employment of theater air forces during the campaign. We assume that his objective is to assist the ground forces in the battle area, and the results will vary with the number of planes he allocates to ground-support operations. We assume that it is possible to construct for Blue a payoff function, giving the payoff for each strike of the campaign in the form of the distance advanced by the ground forces as a function of the number \( m \) of planes allocated to ground support. This function depends heavily on the characteristics of the ground-support targets — i.e., on the degree of concentration of troops, vehicles, and materiel, and on the fortification of positions. We make no attempt at present to give the explicit form of this function, but merely assume that the payoff, \( f(m) \), is a positive function that increases with increasing allocations.

If Blue's ground forces now must advance while being subjected to Red's ground-support sorties, Blue's yield in ground-support is no longer equal to \( f(m) \) as described above, but is reduced in accordance with the number \( n \) of planes allocated by Red to ground-support missions. If \( g(n) \) is the function that measures the distance gained by Red's ground forces, then the net advance of Blue's ground forces, if he allocates \( m \).
planes to ground-support and Red allocates \( n \) planes to

ground-support, can be written as

\[ Y(m, n) = f(m) - g(n). \]

The foregoing expression represents the payoff to Blue for
this one period or one strike. The payoff for the entire
campaign of \( N \) strikes is the sum of these net yields for each
of the \( N \) strikes, or

\[ M = \sum_{1}^{N} [f(m) - g(n)]. \]

The problem faced by each side is now apparent. At a
given move, Blue would like to allocate a large number of planes
to ground-support missions and thereby increase the value of \( f \),
yet he would like to destroy the Red air force by means of
counter-air operations in order to ensure that \( g \) is small, or
zero, for subsequent moves. Further, if he does not provide
for air-defense he may suffer severe losses to his own air force
if Red elects to mount a large counter-air strike. Each player
has to take the future and the possibilities open to his
opponent into account.

In our model of the tactical air war we shall make the
further simplification that the yield functions \( f(m) \) and \( g(n) \)
are linear, say, \( f(m) = m \), \( g(n) = n \). The payoff in the campaign
then is
(3) \[ M(x, u; y, w) = \sum_{i=1}^{N} [(p - x - u) - (q - y - w)]. \]

Blue wishes to make this payoff as large as possible by properly choosing the \( x \)'s and \( u \)'s during each of the \( N \) strikes, and Red wishes to make the payoff as small as possible by properly choosing the \( y \)'s and \( w \)'s.

5. OPTIMAL STRATEGIES FOR TWO TASKS

We shall begin our discussion of the tactical air war model by considering the case in which the air-defense potentials \( c \) and \( k \) are zero. In effect, we are assuming that there are only two tasks to which aircraft can be allocated, counter-air and ground-support. Equations (1) and (2) now read

(4) \[ q_1 = \max \left[ 0, q + s - aq - bx \right] \]

(5) \[ p_1 = \max \left[ 0, p + r - dp - ey \right] \]

and the payoff (3) reduces to

(6) \[ M(x, y) = \sum_{i=1}^{N} [(p - x) - (q - y)]. \]

This version of the tactical air war game was formulated by Fulkerson and Johnson [4] and was solved by them under the assumption of symmetry in the parameters, i.e., \( a = d \) and
b = e. The complete solution of the asymmetric version was later obtained by Dresher [3]. We shall present his results, and refer the reader to [3] for the complete proofs. A brief description of the method of proof is given in the Appendix.

An outstanding characteristic of the solution of the tactical air game with two tasks is that, independent of the attrition parameters, initial conditions, and relative strengths at a given move, both sides have optimal pure strategies. Although every strike by a player is made simultaneously with his opponent, nevertheless, a player never needs to randomize, or bluff. An optimal strategy for a player can then be specified by giving, for each strike of the campaign, the number of planes he allocates to counter-air operations and the number of planes he allocates to ground-support. These optimal allocations depend on the attrition parameters, a, b, d, e, and on the number of strikes remaining in the campaign.

In order to describe the optimal strategies, let us introduce some notation. From Equation (4) it is clear that if Blue had enough planes he could annihilate Red's air force by allocating

\[ x = \frac{(1 - a)q + e}{b} \]

planes to counter-air. Any allocation to counter-air greater than this is clearly wasteful. Thus it is reasonable to call an allocation in which Blue sends
\[
\bar{x} = \min \left[ p, \frac{(1 - a)q + s}{b} \right]
\]
planes to counter-air and \( p - \bar{x} \) to ground-support, a counter-air tactic, as it does represent maximum effort in counter-air. Such allocations will be denoted by \( A \). Similarly, the allocation in which Red sends

\[
\bar{y} = \min \left[ q, \frac{(1 - d)p + r}{e} \right]
\]
planes to counter-air and the rest, \( q - \bar{y} \), to ground-support could reasonably be labeled as a counter-air tactic, and will also be denoted by \( A \). An allocation in which a player sends all of his planes to the ground-support mission will be denoted by the letter \( G \). Finally, consider an allocation for Blue, say, in which he allocates \( \bar{x} \), where \( \bar{x} < \bar{x} \), planes to counter-air and the rest of his force, \( p - \bar{x} \), to ground-support. In such a tactic Blue does not exert maximum effort in the counter-air mission. We denote this tactic by the symbol \((A, G)\). Similarly an allocation for Red in which he commits \( \bar{y} \), where \( \bar{y} < \bar{y} \), planes to counter-air and \( q - \bar{y} \) planes to ground-support will also be denoted by \((A, G)\).

The optimal strategies depend on the attrition parameters. Let us first suppose that

\[
e - d > 0 \quad \text{and} \quad b - a > 0.
\]
These inequalities say, in effect, that the expected number of planes lost per enemy bomb delivered exceeds the expected number of planes lost as a result of factors such as accidents, antiaircraft fire, etc., per plane flown. It turns out that the optimal strategy for each side in this case requires him to begin the campaign with a series of allocations $A$, and to end with a series of allocations $G$. The points at which the players shift from counter-air to ground-support will, in general, be different for Red and Blue. The precise points of shift depend upon the magnitudes of the attrition parameters.

Suppose now that

$$e - d > 0 \quad \text{and} \quad b - a < 0.$$  

Roughly speaking, Red is less vulnerable to enemy counter-air attack than to accidents. However, the character of the optimal strategy for Red is not changed. He begins the campaign with a series of strikes $A$ and ends the campaign with a series of strikes $G$. The optimal strategy for Blue, on the other hand, is changed somewhat. Blue begins with a series of strikes $(A, G)$. This is followed by a single strike in which his allocation is $A$. For all the remaining strikes, Blue concentrates all of his force on the ground-support mission, that is, uses the allocation $G$. The point at which Red shifts to all-out ground-support occurs later in the campaign than does the point at which Blue shifts to ground-support.
If
\[ e - d \leq 0 \quad \text{and} \quad b - a > 0, \]
the conclusions of the preceding paragraph apply with the roles of Red and Blue interchanged.

Finally, if
\[ e - d \leq 0 \quad \text{and} \quad b - a \leq 0, \]
then Red and Blue have the same optimal strategy. This strategy is always to allocate all aircraft to the ground-support mission and always neglect the counter-air mission.

The foregoing discussion is summarized in the table below. Moves are numbered from the end of the game in the table. The integers \( f \) and \( g \) which are used in the table determine the first strike (numbered from the end) at which a shift is made from ground to counter-air and are defined as follows: \( f \) is the largest integer for which the inequality
\[
\frac{1}{e} - \frac{1 - (1 - d)^f}{d} \geq 0
\]
holds, and \( g \) is the largest integer for which the inequality
\[
\frac{1}{b} - \frac{1 - (1 - a)^g}{a} \geq 0
\]
## Optimal Allocation of Forces Between Two Tasks

<table>
<thead>
<tr>
<th>Attrition Parameters</th>
<th>Player</th>
<th>Optimal Allocation at Move Number $n$ (counting from end of campaign)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$1 \leq n \leq \min(f+1, g+1)$</td>
</tr>
<tr>
<td>$e - d &gt; 0 \ b - a &gt; 0$</td>
<td>Blue</td>
<td>$G$</td>
</tr>
<tr>
<td></td>
<td>Red</td>
<td>$G$</td>
</tr>
<tr>
<td></td>
<td>Blue</td>
<td>$G$</td>
</tr>
<tr>
<td></td>
<td>Red</td>
<td>$G$</td>
</tr>
<tr>
<td></td>
<td>Blue</td>
<td>$G$</td>
</tr>
<tr>
<td></td>
<td>Red</td>
<td>$G$</td>
</tr>
<tr>
<td>$e - d &gt; 0 \ b - a \leq 0$</td>
<td>Blue</td>
<td>$G$</td>
</tr>
<tr>
<td>$(g = \infty)$</td>
<td>Red</td>
<td>$G$</td>
</tr>
<tr>
<td>$e - d \leq 0 \ b - a &gt; 0$</td>
<td>Blue</td>
<td>$G$</td>
</tr>
<tr>
<td>$(f = \infty)$</td>
<td>Red</td>
<td>$G$</td>
</tr>
<tr>
<td>$e - d \leq 0 \ b - a = 0$</td>
<td>Blue</td>
<td>$G$</td>
</tr>
<tr>
<td>$(f = g = \infty)$</td>
<td>Red</td>
<td>$G$</td>
</tr>
</tbody>
</table>
holds. Observe that if $e - d \leq 0$ then $f$ does not exist, and if $b - a \leq 0$, then $g$ does not exist. We may take $f$ and $g$ to be $+\infty$ in these cases. The integer $t$ denotes the number of the last move (numbered from the end) for which one player chooses $A$ and the other chooses $G$. We have not evaluated $t$ as a function of the attrition parameters.

6. OPTIMAL STRATEGY FOR THREE TASKS

We now return to the more general model with all three tasks — counter-air, air-defense, and close-support — present. We shall see that increasing the number of air tasks to three leads to substantial changes in the character of the optimal tactics.

In order to simplify the analysis we assume that Blue and Red have the same air-defense potential: each plane allocated to defense can prevent one attacking plane from reaching target — that is, we assume that $c = k = 1$. We also assume that each attacking plane that penetrates the defense can destroy one plane in an airfield strike, or $b = e = 1$, and that losses due to aborts, accidents, and antiaircraft fire are negligible. Finally, we assume that replacements are absent, i.e., $r = s = 0$. Then the inventory of planes at the end of a strike will be, for Blue and Red respectively,

$$p_1 = \max [0, p - \max (0, y - u)]$$

$$q_1 = \max [0, q - \max (0, x - w)].$$
We emphasize the fact that these simplifying assumptions have no effect on the general form of the optimal strategies. In the next section we shall show how a change in the value of the air-defense potential influences the specific form of the optimal strategy.

The optimal strategies in the three task model are different from the two task model in the following two important ways: First, the optimal tactics depend upon the relative strengths of the two sides. Second, optimal play requires one player to use a mixed strategy. We shall give a complete description of the optimal employment of tactical air forces in terms of the number of strikes remaining and the relative strengths of the two sides. In our descriptions of the optimal allocations we shall always assume that at the move in question Blue is the stronger side and Red is the weaker, that is to say \( p \geq q \). It should be emphasized that this is merely a convention to facilitate the description of optimal tactics, and is not meant to imply that a side which is the stronger side at a given stage of the game will always remain the stronger for all subsequent moves. Of course, if a player who is initially stronger plays optimally, then he will remain the stronger throughout the campaign.

Our discussion will begin with a qualitative description of the optimal strategies. This will be followed by a numerical table summarizing the optimal tactics for campaigns having at most eight strikes, and finally we shall present an example
to illustrate the importance of proper initial allocations. The mathematical theorem on which the discussion is based can be found in the Appendix.

The optimal tactics have the following properties:

**Campaign ends with ground support.** The campaign always ends with a series of strikes on ground-support — i.e., during the closing period of the campaign both Red and Blue concentrate all their forces on ground-support missions. In this terminal period both sides have the same optimal tactics, regardless of their initial forces.

**Blue (stronger) splits his forces.** At all times other than the closing phase of the campaign, Red and Blue have very different optimal tactics. During any of these early strikes, the stronger side, Blue, has a pure strategy. That is, there exists a best allocation of Blue's air force among the three air tasks. In this connection, there is a critical value (about 2.7) of the ratio of the Blue force size to the Red force size that governs Blue's allocation during the early period in the following manner: If the force ratio is less than this critical value, then the optimal allocation in the early period consists of splitting the stronger air force between two air tasks, counter-air and air-defense, and neglecting the ground-support task. The size of split depends on the relative strengths of the two air forces and the number of strikes left in the campaign. However, if Blue's strength relative to Red's is greater than the critical value, then Blue should divide his
force in a fixed way, regardless of his strength, among the three tasks, counter-air, air-defense, and ground-support. The number of aircraft allocated to each mission, however, is still dependent on the number of strikes remaining.

Red (weaker) mixes his tactics and concentrates his forces. The weaker combatant cannot use a single strategy, but must bluff during all the strikes other than those of the terminal phase. Unlike his opponent, the weaker combatant does not have a single allocation that is best. He must use a mixed strategy and gamble for high payoffs. If he is not too weak — i.e., if the force ratio is less than the critical value — then he concentrates his entire force either on counter-air or on air-defense; but which of these tasks receives the full effort is decided by some chance device. However, if Red is very weak (force ratio larger than critical value), then he allocates his entire air force to any one of the three air tasks with the particular task again chosen at random. In other words, if a player is very weak relative to the opponent, then he takes a chance on an early payoff. Of course, to be most effective, he must bluff correctly — i.e., the random device should select the tasks with the proper relative frequencies.

Mix and split the same tasks. It is of interest to note that on each strike Red, the weaker side, bluffs with the same tasks that Blue uses in his allocation. Thus if Red is very weak he bluffs with each of the three tasks, and Blue splits his forces among each of the three tasks. However, if Red is
moderately weak, he bluffs with two tasks — counter-air or air-defense — and Blue splits his forces between the same two tasks, counter-air and air-defense.

Blue's defense decreases during campaign. As was noted above, prior to the closing phase of the campaign, Blue splits his forces among his air tasks. The actual split is a function of the force sizes of Blue and Red and the number of strikes left in the campaign. However, as the campaign proceeds, the fraction of Blue's force allocated to air-defense will decrease. At the same time, the fraction allocated by Blue to counter-air will increase. During this time, the chance that Red will attack Blue also decreases, but the chance that Red will defend himself increases.

Blue's defense in a long campaign. In the early stages of a relatively long campaign, the stronger side defends itself against a concentrated attack by the weak side. During this period, Blue dispatches on air-defense a force of planes approximately the size of Red's entire force. Recall that we assumed a particular value for the air-defense effectiveness.

7. SUMMARY TABLE

The attached table summarizes the optimal tactics for campaigns consisting of at most eight strikes. The tabulation gives the optimal allocation for each strike (where the strike number is defined by the number of strikes remaining in the campaign) as a function of the relative sizes of the forces at
### Optimal Tactics in Multistrike Tactical Air Campaign

(Strong Side Having \( p \) Planes and Weak Side Having \( q \) Planes, \( p > q \))

<table>
<thead>
<tr>
<th>Campaign Period</th>
<th>Duration of Campaign (No. of Strikes Remaining in Campaign)</th>
<th>Relative Initial Strengths of Opponent (( \frac{p}{q} ) of Strong Side to Weak Side)</th>
<th>Optimal Initial Allocation by Strong Side (No. of Planes)</th>
<th>Optimal Initial Allocation by Weak Side (Probability of Allocating All Planes)</th>
<th>Expected Value of Campaign (Movement of Front Line)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>1.00 to ( \infty )</td>
<td>Counter-Air: 0</td>
<td>Counter-Air: 0</td>
<td>p - q</td>
</tr>
<tr>
<td>II</td>
<td>2</td>
<td>1.00 to ( \infty )</td>
<td>Counter-Air: 0</td>
<td>Counter-Air: 0</td>
<td>p - q</td>
</tr>
<tr>
<td>III</td>
<td>3</td>
<td>1.00 to 2.00</td>
<td>Counter-Air: ( q )</td>
<td>Counter-Air: 0</td>
<td>p - q</td>
</tr>
<tr>
<td>IV</td>
<td>4</td>
<td>1.00 to 2.33</td>
<td>Counter-Air: 0.5p + 0.5q</td>
<td>Counter-Air: 0.5p - 0.5q</td>
<td>p - q</td>
</tr>
<tr>
<td>V</td>
<td>5</td>
<td>1.00 to 1.70</td>
<td>Counter-Air: 0.5p + 0.5q</td>
<td>Counter-Air: 0.5p - 0.5q</td>
<td>p - 2.5q</td>
</tr>
<tr>
<td>VI</td>
<td>6</td>
<td>1.00 to 1.45</td>
<td>Counter-Air: 0.5p + 0.5q</td>
<td>Counter-Air: 0.5p - 0.5q</td>
<td>p - 2.5q</td>
</tr>
<tr>
<td>VII</td>
<td>7</td>
<td>1.00 to 1.25</td>
<td>Counter-Air: 0.5p + 0.5q</td>
<td>Counter-Air: 0.5p - 0.5q</td>
<td>p - 2.5q</td>
</tr>
<tr>
<td>VIII</td>
<td>8</td>
<td>1.00 to 1.25</td>
<td>Counter-Air: 0.5p + 0.5q</td>
<td>Counter-Air: 0.5p - 0.5q</td>
<td>p - 2.5q</td>
</tr>
</tbody>
</table>

\( p > q \)
the time of that strike. However, the value of the game, which is given in the last column of the tabulation, is for the campaign of given duration.

8. EXAMPLE OF SENSITIVITY TO INITIAL ALLOCATION

The importance of making the proper allocation on the initial strike (and therefore on every strike) can be forcefully illustrated by means of the following example. Suppose that on the initial strike of a five-strike campaign Blue has \( p = 85 \) planes and Red has \( q = 68 \) planes. From the table we see that the value to Blue of the five-strike campaign with these initial conditions is 108. Suppose that on the initial strike Blue concentrates his force of 85 planes on counter-air (which is not an optimal tactic) and then makes optimal allocations during the remaining four strikes. Then if Red dispatches his force of 68 planes on counter-air during the initial strike, Blue's force will be reduced to 17 planes and Red's force to zero. Optimal allocation for the last four strikes would demand dispatching the 17 planes on ground-support each time. In this case the payoff for the campaign of five strikes would be 68, or a reduction of almost 40 per cent from the payoff of 108 if Blue had made an optimal allocation of 75 planes to counter-air and 10 planes to air-defense on the initial strike.

Now suppose that Red, instead of using a mixed strategy on his initial strike, uses the strategy of allocating his entire air force to counter-air on the first strike and then
makes optimal allocations for the remaining four strikes. In this case, an allocation by Blue of 17 planes on counter-air and 68 on air-defense will reduce Red's force to 51 planes and Blue will still have 85 planes. Optimal allocations by both sides for the remaining four strikes will yield a total payoff to Blue of \( \frac{9}{2} (85 - 51) = 153 \), or almost 50 per cent higher to Blue than the expected payoff of 108 that is possible with optimal choice of a mixed strategy by Red.

9. ARBITRARY AIR DEFENSE POTENTIAL

We now assume an arbitrary value \( 0 < k < 1 \) for the air defense potential, and describe the set of optimal strategies as a function of the air defense potential. We retain the assumptions that the parameters \( b = e = 1 \) and \( a = d = 0 \). Thus, setting the air defense potentials \( c \) and \( f \) equal to \( k \), we have that the inventory of planes at the end of a strike will be, for Blue and Red respectively,

\[
p_1 = \max \left[ 0, p - \max (y - ku) \right]
\]

\[
q_1 = \max \left[ 0, q - \max (x - kw) \right] \quad 0 < k < 1.
\]

The optimal allocations in the present instance are still determined by the relative strengths of Blue (the stronger side) and Red (the weaker side) and by the number of strikes remaining. In general, the campaign can be divided into three
time periods, such that in each of these time periods the optimal tactics are similar. These time periods, counting from the end of the campaign, will be referred to as the ground-support phase, counter-air phase, and randomization phase. For these three time periods, the following tactics are optimal:

Optimal tactics during ground-support phase. The ground support phase consists of the last two strikes of the campaign. During this period the optimal strategy for both Red and Blue is to allocate all their resources to ground-support.

Optimal tactics during counter-air phase. The ground-support phase is preceded by a series of strikes in which Red, the weaker side, allocates all of his resources $q$ to counter-air missions. At the same time, Blue, the stronger side, allocates $q$ of his resources to counter-air and the remainder $p - q$ to ground support. The number of strikes in this phase is $\tau - 1$, where $\tau$ is defined as the largest integer less than or equal to $1/k$; i.e.

$$\tau = \left\lfloor \frac{1}{k} \right\rfloor.$$ 

Note that if $k$ is greater than $1/2$, then $\tau = 1$ and the counter-air phase is not present. Also, as $k$ decreases to zero, the length of the counter air phase increases.

Optimal tactics during randomization phase. In general, the counter-air phase is preceded by an initial phase, which we call the randomization phase. If Blue is slightly stronger
than Red he allocates \( q \) planes to counter air and \( p - q \) to air defense. If Blue is moderately stronger he allocates \( x^* \) planes, where \( x^* > q \), to counter air and \( p - x^* \) to air defense. However, if Blue is very much stronger, then he splits his forces among all three tasks; counter-air, air-defense, and close-support. During these strikes Red, the weaker player, does one of the following three things, depending on his relative force weakness:

1. If Red is slightly weaker than Blue, then he concentrates his resources on counter-air.

2. If Red is moderately weaker than Blue, then he concentrates either on counter-air or air-defense, with the particular concentration chosen at random subject to a given probability distribution.

3. If he is very weak he concentrates his resources on one of the three tasks chosen at random subject to a given probability distribution.

The length of the randomization phase is \( N - T - 1 \) moves, where \( N \) is the length of the campaign. For \( N \leq T + 1 \), this phase does not appear. This occurs if the campaign is short or if the air defense potential is very small. For example, a campaign of 21 strikes would need to have an air defense potential of .05 or less for the initial phase to be absent. Or, given an air defense potential of .05, any campaign exceeding 21 strikes would have a randomization phase.
With the exception of the situation in which \( N \leq \tau + 1 \), with the consequent absence of the randomization phase, the optimal strategies require randomization by the weaker player. Thus, they are essentially different from the optimal strategies when the air defense potential is zero. However, if \( N \leq \tau + 1 \), the optimal tactics are essentially those of the case in which \( c = f = 0 \). As was already noted, if \( k > \frac{1}{2} \), then the middle phase is not present. We are then essentially in the case in which the air defense potentials for each side equal one.

The mathematical proofs of these results are given in [2], and an indication of the idea on which they are based is given in the Appendix.

9. IMPLICATION FOR OPERATIONAL GAMING

In their excellent discussion of operational gaming, Thomas and Deemer [5] define operational gaming to be "... the serious use of playing as a primary device to formulate a game, to solve a game, or to impart something of the solution of a game."* Although the primary purpose of our paper is to formulate an air war game and to present a game-theoretic solution of the problem of determining the optimal employment of tactical air forces in theater air tasks, we believe that our analysis also has implications for operational gaming. In our opinion the game we have solved provides further arguments for the proposition that operational gaming is not a helpful device for solving a game or getting significant information about the

*Italics — Thomas and Deemer
solution. In the present discussion we wish to understand the
definition of operational gaming given above to include the
playing of a game by a computer as well as playing by humans.

The tactical air game is representative of the situations
that are often studied by means of operational gaming. It deals
with a fairly realistic problem that can be cast in the form of
a multimove game in which each player, simultaneously with his
opponent, makes a choice from a continuum of alternatives, and
in which the outcome of each move affects all future moves.
To be sure, it is much simpler than most game models studied
by means of operational gaming. This fact, however, merely
serves to strengthen our belief that operational gaming is not
helpful in solving games, or in obtaining significant informa-
tion about the nature of the solution. We shall show that the
nature of the solution of the tactical air game discussed in
this paper is such as to make it highly doubtful that operational
gaming would furnish significant information about the solution
of this particular game. Therefore, since we do not believe
that the more complex the problem, the simpler the solution,
we feel that operational gaming would be even less helpful in
more complex games.

In assessing the usefulness of operational gaming as a
device for solving games, we do not demand that operational
gaming arrive at the analytic solution, i.e., the optimal
strategies. This is primarily because the game itself is a
rather imperfect model, and as such is highly idealized. Thus,
the importance of the analytic solution is not that it enables
the analyst to insert numbers and get back other numbers that
tell him how to act, correct to three (or one) decimal places.
Rather, having the analytic solution, the analyst can see the
major features of the solution and can identify and study the
significant details of the solution, and thereby be guided to
correct actions. Thus, we feel it is fair and proper to ask
the operational gamer whether he can at least determine the major
features of the solution and the more significant details.

Let us then look at the solution of the three-task tactical
air game. Its major features are as follows: (i) the optimal
allocations at a given strike depend on the relative strengths
of the two sides and on the number of strikes left in the
campaign; (ii) the optimal strategy for the weaker player is
mixed, that is, involves randomization. Two other features of
the solution are important, at least as far as operational
gaming is concerned. First, the solutions are not continuous
functions. Second, the significant details of the optimal
strategies are non-intuitive in character. For example, if the
air defense potential \( k \) is such that \( 0 < k < 1 \), the existence
of three phases and the dependence of the length of the random-
ization phase on \( k \) are features that would tax the intuition.
A perusal of Table 1 or of Theorem 1 in the Appendix shows that
the precise numerical or algebraic specifications of the opti-
mal strategies also are not likely to be achieved by intuition.

We now ask about the implications of all this for the
operational gamer. How successful can we expect him to be in getting any of the important aspects of the solution? The first implication for the operational gamer is that he is not likely to make good guesses about the solution. He will therefore have to play the game more than just a few times in order to explore it. This brings up the question of how does one go about exploring the game. What constitutes a sensible exploration procedure, and how many plays of the game are required to carry it through?

A seemingly simple scheme would be to try \( m \) different allocations for each side at each move. The number of plays required to explore all of these possibilities in a game of \( N \) moves is \( m^{2N} \). A very modest desire to test three allocations at each move of a game consisting of five strikes would require \( 3^{10} \), or approximately 59,000, plays. Furthermore, this exploration would be for only one set of initial Blue and Red aircraft inventories. If \( r \) distinct initial values of \( p \) (number of Blue aircraft) and \( s \) distinct initial values of \( q \) (number of Red aircraft) are to be tested, the number of plays becomes \( rs(m)^{2N} \). Clearly the game would have to be played on a computer, and even in that case the large number of plays would pose problems of data interpretation. However, the principal objection to the scheme just outlined does not rest on the difficulties that seem to arise from the large number of plays required, but is based on the following more fundamental reason. The optimal strategies are functions of the strike number (time)
and of the aircraft inventories of the two sides (state variables); that is they have the form \( f(p, q, n) \), where \( n \) is the strike number, \( p \) is the Blue inventory and \( q \) is the Red inventory. On the other hand, the procedure used to explore is essentially one of testing functions of the strike number alone; that is functions of the form \( g(n) \). Thus the procedure does not seem to be very promising.

The problem of incorporating the dependence of the strategies on \( p \) and \( q \) is one which the gamer must solve if he is to proceed in a sensible manner. We are unaware of any satisfactory solution to the problem. The simplest method that we can think of would be to consider regions of the \((p, q)\) plane and make the allocations depend on the regions as well as on the move. This, however, can quickly lead to the consideration of an astronomical number of pure strategies. For example, if at each move we have \( t \) regions of the \((p, q)\) plane and wish to consider \( m \) allocations for each of the regions, then for an \( N \)-move game we have \( m^tN \) possible pure strategies for each side. With \( m = 3 \), \( t = 9 \), and \( N = 5 \), we get \( 3^{45} \), or approximately \( 2.9 \times 10^{21} \) pure strategies for each side.

It might be argued that by "judicious choices" the number of alternatives can be decreased. This, we feel, begs the question. For, in view of the complex, non-intuitive nature of the solution, how is one to know what constitutes a "judicious choice" before one knows the solution. There are other reasons for objections to the procedure that we have outlined.
We shall not go into them here, as many of them are common to all gaming techniques, and will be discussed below.

In essence, the use of operational gaming to learn about the solution of a game, whether done by humans or computers, or both, is an attempt to determine the functions \( f(p, q, n) \) that give the optimal strategies by a rather restricted sampling of the pure strategies. Although the large number of plays required for an even modest sampling of the strategies appears to us to be a serious obstacle to the success of operational gaming, we feel that there are more fundamental reasons for being pessimistic about gaming as a device for finding the solution of the game. These reasons, which are connected with the nature of the solution, will now be discussed.

Our pessimism stems, first of all, from the fact that the optimal strategies are functions \( f(p, q, n) \) of inventory and strike numbers which are not always continuous. Some of the difficulties arising from the fact that the strategies are functions of \( p \) and \( q \) as well as \( n \) have already been discussed. We now assert that a fundamental difficulty arises from the fact that the solution is discontinuous. Operational gaming, at best, can give information only about a finite number of strategies at a discrete set of points \( (p, q) \), for in view of the discontinuities it is not possible to say anything with any degree of assurance about the intermediate points \( (p, q) \). Even if we increase the number of discrete points considered, we still can have discontinuities between points, and so be
unable to interpolate meaningfully. Actually, it can be shown that one cannot really be sure that he even has correct information about the points he samples. This is due to two things. First, the gamer is only considering a finite number of allocations from among a continuum. Second, during the course of play he is led to inventory positions \((p, q)\) different from the ones that he is sampling. He must then treat such a point as one of the points that he is considering, or interpolate between them.

Perhaps the most serious objection to operational gaming as a device for studying a game arises from the fact that the solutions of many games involve mixed strategies. Since we are not dealing with a one move matrix game, we do not see how repeated plays would lead one to the mixed strategy solution. Any attempt to form a matrix of results of playing one strategy against another would not lead to anything significant. Such a matrix would merely represent the results of playing a finite number of pure strategies from the infinite set of pure strategies of one player against a similar set for his opponent. Thus, the solution of this matrix game would indicate nothing.

This inability to discover the need for randomization, or bluffing, is clearly a major stumbling block in the path of operational gaming. Its effect is not only to keep the players in ignorance of precisely what their best strategy is, but may even keep them in ignorance of the nature of their best strategy. The fallacy, to say the least, of using a technique to discover
the best method of action in a situation, when the technique
is often incapable of making such discoveries is obvious. If
one is serious about applying operational gaming to decision
making, the neglect of mixed strategies could lead to rather
serious consequences. As an illustration, we refer to the
second paragraph of Section 8. There, Red used a pure strategy
of all out counter-air on the first strike when he should have
randomized, with a resulting fifty percent increase in payoff
possible to Blue. Furthermore, this particular pure strategy
might very easily be considered as "judicious" or "good" by a
Red commander on the basis of doctrine or experience. To get
a rough, non-technical idea of the effects of always playing a
pure strategy when one should randomize, one should think of
what would happen to a poker player who never bluffs.

In conclusion, we wish to summarize the principal reasons
why we believe that operational gaming would not be a helpful
device for learning about the solution of the game presented.
First, the complexity of the solution seems to indicate an
excessively large number of plays for the exploration of the
game. Secondly, even if the exploration is carried out, the
fact that the solution is a function of position and time, and
may be discontinuous makes it appear that at the very best,
fairly incomplete ideas about the solution are to be expected.
Thirdly, it seems unlikely that the very important randomization
features can be discovered by means of gaming. The authors
believe that this must cast doubt on the use of gaming to solve
more complicated games, for there is no reason to believe that as problems become more complex, solutions become simpler.
APPENDIX

The following theorem for the three task model has been proven for the case \( c = f = 1 \). Its proof is given in [1], but a short indication of the method is given below. Let \( N \) denote the length of the campaign, and \( m \) the number of a particular move, counted from the end of the campaign.

**Theorem.** If \( N = 1 \) or \( 2 \), the value of the game is given by

\[
V_N(p_N, q_N) = N(p_N - q_N).
\]

**Blue has an optimal pure strategy:**

\[
x_m = u_m = 0 \quad \text{for} \quad m \leq N.
\]

**Red has an optimal pure strategy:**

\[
y_m = w_m = 0 \quad \text{for} \quad m \leq N.
\]

If \( N \geq 3 \) the value of the game is given by the \((N - 2)\) piecewise linear function:

\[
V_N(p_N, q_N) = A_N p_N - B_N q_N, \quad i = 1, 2, \ldots, N - 2,
\]

\[
V_N(p_N, q_N) = \lambda_{N-1} q_N, \quad \text{for} \quad i = N - 1.
\]
where the constants $A_N^1$ and $B_N^1$ are positive and monotone decreasing in $i$ for fixed $N$; the value of the superscript $i$ is determined by the ratio $p_N/q_N$.

The optimal strategies for the two players are as follows:

(i) At move $m = 1$, 2 (counting from the end) the players choose

$$x_m = u_m = y_m = w_m = 0.$$  

(ii) At move $m = 3$, if $p_3 \geq q_3$, then Blue chooses $x_3, u_3$ such that

$$q_3 \leq x_3 \leq \min\left(\frac{p_3 + q_3}{2}, \frac{3q_3}{2}\right),$$

$$u_3 = x_3 - q_3.$$  

Red chooses either $y_3 = q_3$ or $w_3 = q_3$, each with probability $1/2$.

(iii) At the $(m + 1)$-st move, where $3 \leq m \leq N - 1$, if $p_{m+1} \geq q_{m+1}$, then the ratio $p_{m+1}/q_{m+1}$ determines an integer $i$, $1 \leq i \leq m - 1$, and Blue chooses

$$x_{m+1} = \frac{(2m - A_m^1)p_{m+1} - (m - 2B_m^1)q_{m+1}}{m + B_m^1},$$

$$u_{m+1} = p_{m+1} - x_{m+1}, \text{ for } i = 1, 2, \ldots, m - 2,$$
\[ x_{m+1} = \left( 2 - \frac{1}{B_m^{m-2}} \right) q_{m+1} \]
\[ u_{m+1} = \left( 1 - \frac{1}{m} \right) q_{m+1}, \quad \text{for } i = m - 1, \]

where the constants \( A_m^i \) and \( B_m^i \) are those associated with a game of length \( m \) and initial condition \( p_m, q_m \). Red chooses either \( y_{m+1} \) or \( w_{m+1} = q_{m+1} \) with probabilities \( \alpha_m^i = B_m^i/(m + B_m^i) \) and \( \beta_m^i = m/(m + B_m^i) \), respectively for \( i = 1, 2, \ldots, m - 2 \); however if \( i = m - 1 \), Red chooses \( y_{m+1} = q_{m+1} \) with probability \( \alpha_m^i = 1/m \), or \( w_{m+1} = q_{m+1} \) with probability \( \beta_m^i = 1/B_m^{m-2} \), or \( y_{m+1} = w_{m+1} = 0 \) with probability \( 1 - 1/m - 1/(B_m^{m-2}) \).

The constants \( A_N^i \) and \( B_N^i \) of the theorem are defined inductively as follows:

\[ A_3^i = 3 \]
\[ A_n^{n+1} = (A_n^{n-2} + 1) \]
\[ n \geq 3 \]
\[ B_3^i = 3 \]
\[ B_n^{n+1} = \left( 4 - \frac{1}{A_n^{n-2}} - \frac{1}{B_n^{n-2}} \right) \]
\[ n \geq 3 \]
\[ B_n^0 = A_n^0 = 0 \]
\[ n \geq 3 \]
\[ A_n^{i+1} = \left( \frac{A_n^{n-2}(2B_n^i + A_n^i)}{B_n^i + A_n^{n-2}} \right) \]
\[ \quad \text{for } i \geq 1; \quad n = 1+2, 1+3, \ldots \]
\[ B_n^{i+1} = \left( \frac{3A_n^{n-2}B_n^i}{B_n^i + A_n^{n-2}} \right) \]
The superscript 1 is associated with the ratio \( p_N/q_N \) by means of a step function \( \Phi(p_N/q_N) \). For each \( N \), the jump points of \( \Phi \) are determined by a sequence \( \lambda^1_N, i = 1, \ldots, N - 2 \) to be defined presently. We have

\[
\Phi(p_N/q_N) = 1 \quad \text{whenever} \quad \lambda^1_N \leq p_N/q_N \leq \lambda^{1+1}_N.
\]

The sequences \( \lambda^1_N \) are defined as follows:

\[
\begin{align*}
\lambda^1_N &= 1 \\
\lambda^{n-1}_n &= +\infty & n &= 3, 4, 5, \ldots \\
\lambda^{n-2}_n &= B^{n-2}_n - 1 & n &= 4, 5, 6, \ldots \\
\lambda^1_n &= \frac{B^{n-1}_n - B^1_n}{\lambda^{n-1}_n - \lambda^1_n} & n &= 4, 5, 6, \ldots \\
& & i &= 1, 2, 3, \ldots, m - 3.
\end{align*}
\]

The basic idea used to analyze the tactical air game is the same for each of the three cases considered. It consists of showing that if the game of length \( N - 1 \) can be solved, then the game of length \( N \) can be reduced to consideration of the initial, or \( N \)-th, move. Thus the problem is reduced to solving what is essentially a one move game. The details of the solution of the resulting one move game, however, vary from case to case, and are quite complicated. For these details we
refer the reader to the papers [1], [2], and [3] where complete proofs are given; here we shall show how the reduction to a consideration of one move is accomplished for the cases in which air defense is present; i.e., $c = f = k$, $0 < k \leq 1$. The modifications necessary for the case $k = 0$ are simple.

First, it is necessary to define strategies precisely for the game in the normal form. In what follows, moves will be numbered from the end of the game; i.e., the $n$-th move means $n$ moves to the end of the game. It is also assumed that each player knows the manner in which the game proceeds from stage to stage and that at each stage both players know the state variables and the entire past history of play.

Pure strategies of the game in normal form are defined inductively on the number of moves. First, a strategy for Blue in a one move game is a point $X_1 = (x_1, u_1)$, where $x_1 \geq 0$, $u_1 \geq 0$, and $x_1 + u_1 \leq p_1$. Similarly, a strategy for Red in a one move game is a point $Y_1 = (y_1, w_1)$ where $y_1 \geq 0$, $w_1 \geq 0$, and $y_1 + w_1 \leq q_1$. Now let $\sigma_N$ be a strategy for Blue in an $N$-move game. Of course, $\sigma_N$ is a function of $p_N$ and $q_N$. Then, in a game of $(N + 1)$ moves, at the $(N + 1)$-st move Blue chooses a point $X_{N+1} = (x_{N+1}, y_{N+1})$ in the triangle $\Delta_{N+1}$ defined by

$$x_{N+1} \geq 0, \quad u_{N+1} \geq 0, \quad x_{N+1} + u_{N+1} \leq p_{N+1},$$

and simultaneously Red chooses a point $Y_{N+1} = (y_{N+1}, w_{N+1})$ in
the triangle \( D_{N+1} \) defined by

\[
y_{N+1} \geq 0, \quad w_{N+1} \geq 0, \quad y_{N+1} + w_{N+1} \leq q_{N+1}.
\]

The choices yield the state variables \( p_N \) and \( q_N \), given by the equations

1. \( p_N = \max \left[ 0, \ p_{N+1} - \max \left( 0, \ y_{N+1} - ku_{N+1} \right) \right] \).

2. \( q_N = \max \left[ 0, \ q_{N+1} - \max \left( 0, \ x_{N+1} - kw_{N+1} \right) \right] \).

A strategy \( \sigma_{N+1} \) for Blue in the \((N+1)\)-move game is then defined as a choice \( x_{N+1} \) in \( \Delta_{N+1} \) and a function \( \Xi_N \) that associates with each point \( (x_{N+1}, y_{N+1}) \) in the product space \( \Delta_{N+1} \times D_{N+1} \) a strategy \( \sigma_N \) in the \( N \)-move game. Thus \( \sigma_{N+1} \) can be written as

\[
\sigma_{N+1} = (x_{N+1}; \ \Xi_N).
\]

In a like manner, a strategy \( \tau_{N+1} \) for Red in the \((N+1)\)-move game is defined as a choice \( y_{N+1} \) and a function \( \Psi_N \) that associates with each \( (x_{N+1}, y_{N+1}) \) a strategy \( \tau_N \) in the \( N \)-move game. Thus we have

\[
\tau_{N+1} = (y_{N+1}; \ \Psi_N).
\]
Mixed strategies for the players can now be defined inductively in a similar manner. In a game of one move a mixed strategy for Blue is a probability distribution \( g_1 \) over \( \Delta_1 \) and a mixed strategy for Red is a probability distribution \( h_1 \) over \( D_1 \). Suppose now that mixed strategies for games of length \( N \) have been defined. Let \( g_N \) be a mixed strategy for Blue in an \( N \)-move game. A mixed strategy \( g_{N+1} \) in a game of \((N+1)\) moves is a probability distribution \( \varepsilon_{N+1} \) over \( \Delta_{N+1} \) and a function \( \phi_N \) that associates to each \((x_{N+1}, y_{N+1})\) a mixed strategy in the \( N \)-move game. Thus the mixed strategy in the \((N+1)\)-move game can be written as

\[ g_{N+1} = (\varepsilon_{N+1}, \phi_N). \]

Mixed strategies \( h_{N+1} \) for Red are defined similarly by a distribution \( h_{N+1} \) on \( D_{N+1} \) and a function \( \psi_N \), and can be written as

\[ h_{N+1} = (h_{N+1}, \psi_N). \]

Suppose that in the game of length \( N \) there exist strategies \( g_N^* \) for Blue and \( h_N^* \) for Red with the following properties:

1. If Blue plays \( g_N^* \) and Red plays \( h_N^* \), the expectation \( E(g_N^*, h_N^*) \) exists.

2. For all Red pure strategies \( \tau_N \), \( E(g_N^*, \tau_N) \) exists.
and
\[ E(G_N^*, T_N) \geq E(G_N^*, H_N^*). \]

(iii) For all Blue pure strategies \( \sigma_N \), \( E(\sigma_N, H_N^*) \) exists, and
\[ E(\sigma_N, H_N^*) \leq E(G_N^*, H_N^*). \]

In this event the game is said to have a value \( V_N(p_N, q_N) \) given by
\[ V_N(p_N, q_N) = E(G_N^*, H_N^*), \]
\( G_N^* \) is said to be an optimal strategy for Blue, and \( H_N^* \) is said to be an optimal strategy for Red. The value, as indicated by the notation, is a function of the initial conditions. For the remainder of this section we shall suppress the subscript N, and write \( E(G_N, H_N^*) \), say, merely as \( E(G, H) \), and write \( E(G_{N+1}, H_{N+1}) \), say, as \( E(G_1, H_1) \).

Suppose that the game of length N has value \( V(p, q) \) which is continuous in p, q and Blue and Red have optimal strategies \( G^* \) and \( H^* \) respectively. Define
\[ L_1(x_1, y_1) \equiv (p_1 - x_1 - u_1) - (q_1 - y_1 - w_1) \]
and

\[ M_1(x_1, y_1) = L_1(x_1, y_1) + V(p, q), \]

where \( p, q \) are obtained from \( p_1, q_1 \) by means of the choices \((x_1, y_1)\) and equations (1) and (2). Let

\[ g_1^* = (g_1^*, g^*) \quad H_1^* = (h_1^*, H^*), \]

where \( g_1^* \) is a probability distribution over \( A_1 \) and \( h_1^* \) is a probability distribution over \( D_1 \). We shall derive sufficient conditions for \( g_1^* \) and \( h_1^* \) to satisfy in order that \( G_1^* \) and \( H_1^* \) should be optimal in the \((N+1)\)-move game.

From the various definitions it follows that

\[
(3) \quad E(G_1^*, H_1^*) = \int \int L_1(x_1, y_1) dg_1^* + \int \int E(G^*, H^*) dg_1^* dh_1^*
\]

\[ = \int \int L_1(x_1, y_1) dg_1^* + \int \int V(p, q) dg_1^* dh_1^* \]

\[ = \int \int M_1(x_1, y_1) dg_1^* dh_1^*. \]

Furthermore, if \( \tau_1 = (y_1, \Psi) \), with \( \Psi(x_1, y_1) = \tau \),

\[
(4) \quad E(G_1^*, \tau_1) = \int L_1(x_1, y_1) dg_1^* + \int E(G^*, \tau) dg_1^*
\]

\[ \geq \int L_1(x_1, y_1) dg_1^* + \int V(p, q) dg_1^* \]

\[ = \int M_1(x_1, y_1) dg_1^*, \]
for all \( \tau_1 \).

Similarly, if \( \sigma_1 = (X_1, \emptyset) \)

\begin{equation}
E(\sigma_1, H_1^*) \leq \int M_1(X_1, Y_1) dh_1^*,
\end{equation}

for all \( \sigma_1 \). Since we have shown in (3) that \( E(G_1^*, H_1^*) \) exists, in order to show that \( G_1^* \) and \( H_1^* \) are optimal strategies, we must show that

\[ E(\sigma_1, H_1^*) \leq E(G_1^*, H_1^*) \leq E(G_1^*, \tau_1), \]

for all Blue pure strategies \( \sigma_1 \) and Red pure strategies \( \tau_1 \).

It now follows from (3), (4) and (5) that sufficient conditions that \( G_1^* \) and \( H_1^* \) be optimal are

\[ \int M_1(X_1, Y_1) dg_1^* \geq \int M_1(X_1, Y_1) dg_1^* dh_1^*, \quad \text{for all } Y_1 \]

\[ \int M_1(X_1, Y_1) dh_1^* \leq \int M_1(X_1, Y_1) dg_1^* dh_1^*, \quad \text{for all } X_1. \]

What we have just proved can also be stated in a slightly different manner. Given that the game of length \( N \) has been solved and has value \( V(p, q) \), then the solution of the \( (N+1) \)-move game is obtained by solving the one move game \( \Gamma_1 \) with payoff

\[ M_1(X_1, Y_1) = L_1(X_1, Y_1) + V(p, q), \]
where $V(p, q)$ is obtained from $p_1, q_1$ by the choices $X_1, Y_1,$ and equations (1) and (2). The value of the $(N+1)$-move game equals the value of $\Gamma_1$. If the optimal strategies for Blue and Red in $\Gamma_1$ are $g_1^*$ and $h_1^*$ respectively, and are $G^*$ and $H^*$ respectively in the $N$ move game, then the optimal strategies in the $(N+1)$-move game are $(g_1^*, G^*)$ for Blue and $(h_1^*, H^*)$ for Red.
REFERENCES


