ON CERTAIN GAMES WITH TRANSCENDENTAL VALUES

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Let \( \Gamma \) be a two person zero-sum game for which the compact pure strategy spaces, \( S_1 \) and \( S_2 \), and the payoff function \( M \), defined over \( S_1 \times S_2 \), are definable in Tarski's system of "elementary algebra" (see [1]). Suppose, also, that \( \Gamma \) has a value which is a transcendental number. We can then conclude that there is no optimal strategy for either player consisting of a step function of finitely many steps (i.e. a distribution in which the probabilities are all concentrated on a finite set of points). For, suppose the contrary for one of the players, say the maximizing one. Then, for some positive integer \( m \), the value of \( \Gamma \) is given by

\[
v = \max_{\langle \alpha_1, \ldots, \alpha_m \rangle \in \mathcal{S}_m} \max_{x_1, \ldots, x_m \in S_1} \min_{y \in S_2} \sum_{i=1}^{m} \alpha_i M(x_i, y),
\]

where \( \mathcal{S}_m \) is the set of all \( m \)-tuples \( \langle \alpha_1, \ldots, \alpha_m \rangle \) such that \( \alpha_i \geq 0 \) for \( i = 1, \ldots, m \), and \( \sum_{i=1}^{m} \alpha_i = 1 \). But, according to [1], \( v \) would be algebraically definable, and it is a principal result of [1] that every algebraically definable number is algebraic.

In particular, our result applies to any game with transcendental value, in which \( M \) is a continuous rational function with integral coefficients.
Example: Take \( M(x, y) = \frac{(1+x)(1+y)(1-xy)}{(1 + xy)^2} \), \( S_1 = \{ x| 0 \leq x \leq 1 \} \), and \( S_2 = \{ y| 0 \leq y \leq 1 \} \). Here, \( \nu = \frac{\lambda}{\pi} \), and a pair of distribution functions yielding this value is given by:

\[
\begin{align*}
F^*(x) &= \frac{\lambda}{\pi} \arctan \sqrt{x} \\
G^*(y) &= \frac{\lambda}{\pi} \arctan \sqrt{y}
\end{align*}
\]

Thus, in this game, there is no optimal strategy consisting of a step function of finitely many steps, for \( \pi \) is a transcendental number.

Reference