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June 1962

P-2591

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PREFACE

This paper was written when the author was working on a research project on the foundations of mathematics supported by the U.S. National Science Foundation (Grants G-6693) and G-14006).

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This paper was prepared for presentation at a Mathematical Colloquium held at The RAND Corporation, Santa Monica, California, March 19, 1962.

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1. INTRODUCTION

This paper contains several remarks and results on the application of general theorems on infinite games with perfect information to some special games of that kind and also the games of pursuit and evasion.

The theory of infinite games with perfect information was started by Gale and Stewart [2]. Some work was also done independently in Poland and preliminary announcements were published [8], [7].* A proof and a natural generalization of the result announced in Sec. 2 of [7] was published by Oxtoby [9]. The result announced in Sec. 1 of [7] was a rediscovery of a theorem of Philip Wolfe [11]. This theorem has been recently refined by M. D. Davis [1] and this will be exploited in the present paper.

By a <u>0—sum 2—person game</u> we mean a triple $\langle A,B,v\rangle$ where A and B are nonempty sets, called sets of strategies, and $v = v(a,b)$ is a real valued function, called pay—off function, defined on $A \times B$. The game is called <u>determined</u>, if

$$\sup_{a\in A} \inf_{b\in B} v(a,b) = \inf_{b\in B} \sup_{a\in A} v(a,b).$$

All the games studied in that paper would have a more special structure. A mapping $\pi : A\times B \to S$ would be

*Secs. 1 and 3 of [7] contain some mistakes.

given (S in some set) and we will consider a class of games $\Gamma(\varphi) = \langle A, B, v_\varphi \rangle$, where $v_\varphi(a,b) = \varphi(\pi(a,b))$, where φ is any real valued function on S. All the theorems of this paper would have the form "...for such and such a class of functions φ the game $\Gamma(\varphi)$ is determined." S will always be a topological space and most of these classes of φ's will have the following form

(1.1) $\varphi \epsilon K_n(S)$ if and only if there exists such a set $R(\varphi)$ of real numbers which is dense in the real line and such that for every $r \epsilon R(\varphi)$ $\{s:\varphi(s)<r\} \epsilon B_n$ or $\{s:\varphi(s)\leq r\} \epsilon B_n$, where $B_0 = F \cup G$, $B_1 = F_\sigma \cup G_\delta$, $B_2 = F_{\sigma\delta} \cup G_{\delta\sigma}$, ...and F [G] denote the classes of closed [open] subsets of S and the subscript σ [δ] denote the closure with respect to denumerable union [intersection].

Let us mention without proof a general theorem of Ryll-Nardzewski on such structures:

If A, B and S are compact topological Hausdorff spaces and π is continuous then the following conditions are equivalent

(i) $\Gamma(\varphi)$ is determined for every continuous function φ.

(ii) $\Gamma(\varphi)$ is determined for every φ which is the characteristic function of an open subset of S.

(iii) $\Gamma(\varphi)$ is determined for every φ which is the characteristic function of a closed subset of S.

(iv) $\Gamma(\varphi)$ is determined for every $\varphi \in K_0(S)$.

Of course K_0 includes upper and lower semicontinuous functions. We do not know whether (i) implies that $\Gamma(\varphi)$ is determined for a class of functions larger than $K_0(S)$, e.g., the first class of Baire (even in the case of compact metric spaces).

The most natural approach for defining games of pursuit and evasion is to take for A and B some classes of vector valued functions such that for every $a \in A$ and $b \in B$ the system of differential equations

$$\dot{p}(t) = a(p(t),q(t),t), \quad \dot{q}(t) = b(p(t),q(t),t)$$

with the initial condition

$$p(0) = p_0, \quad q(0) = q_0$$

has a unique solution (p,q). Here $\pi(a,b) = (p,q)$ and $\varphi(p,q)$ may be defined in many ways, e.g., $\varphi(p,q) = ||p(1) - q(1)||$ or $\varphi(p,q) = \text{arc tg inf}\,\{t : p(t)=q(t)\}$*. But this approach presents some essential difficulties discovered by Zieba [12] (such games are not determined in general and therefore are not adequate for pursuit and evasion).

*Here arc tg is used only because $\inf \emptyset = \infty$, but φ is supposed to be real valued.

Another approach is presented by Ryll-Nardzewski [10]. His theory is more strictly dealing with pursuit and evasion (since φ is always of the type $\inf\{t:p(t) = g(t)\}$) but it provides some tools for effective "analytic" solutions of games—while the considerations of the present paper have purely existential character.

In the present paper another approach is given. It is the method of approximation by positional games.* We give also a short exposition of the theory of positional games with perfect information.

2. POSITIONAL GAMES

2.1. The General Theory

We introduce the following definitions:

M is a nonempty set, which we treat often as a discrete Hausdorff space.

S is a nonempty closed subset of the product space $M \times M \times \ldots$

For every $s = (s_0, s_1, \ldots) \in S$, $s|n$ $(n = 0,1,\ldots)$ denotes the finite (or empty) sequence $(s_0, s_1, \ldots, s_{n-1})$.

$J(s|n)$ is a function defined on all $s|n$, taking the values I or II.

$\Phi(s)$ is a real valued function on S.

The system (M,S,J,Φ) is denoted by $\Gamma(\Phi)$.

$\Gamma(\Phi)$ represents a 0-sum 2-person game which is played as follows: The players I and II are constructing a

*Some discussion concerning such approaches is given in [12].

sequence $s \epsilon S$ choosing consecutively $s_0, s_1, \ldots$, the n-th choice being performed by I if $J(s|n) = I$ and by II if $J(s|n) = II$ (where $s|n$ denotes the segment already constructed). Clearly the n-th choice s_n is supposed to be done in such a way that there exists such an $s' \epsilon S$, that $s|n = s'|n$ and $s_n = s'_n$ and hence, since S is closed, they get $s \epsilon S$. Then player II pays to player I the value $\Phi(s)$. A definition of the corresponding triple $\langle A,B,v \rangle$ does not present any difficulties.

The above definition is a slight modification of that of Gale and Stewart [2]. But this modification is essential for the formulation of our results (e.g., Theorem 2 below). In fact, it is obvious how to transform the collection $(x_0, X_I, X_{II}, X, f, S, \Phi)$ of [2] into our (M,S,J,Φ) and vice versa. E.g., the S of [2] corresponds to $\{(s|0, s|1, \ldots): s \epsilon S\}$ in our terminology. It is important that this correspondence induces a homeomorphism of the S of [2] onto our S. This permits a simple translation of the results proved for the formalism of [2] into our terminology (a translation of the proofs would be also easy).

By these remarks a fundamental theorem of Morton Davis [1] can be stated as follows:

(A) <u>If</u> Φ <u>is a characteristic function of a set</u> $X \subseteq S$ <u>such that</u> $X \epsilon B_2$ <u>(see (1.1)) then</u> $\Gamma(\Phi)$ <u>is determined.</u>

The following assertion is a consequence of the axiom of determinateness of Mycielski and Steinhaus [6] (for the proof see [5]):

(B) <u>If</u> M <u>is denumerable and</u> Φ <u>is the characteristic function of any set</u> $X \subseteq S$ <u>then</u> $\Gamma(\Phi)$ <u>is determined</u>.

(Let us recall that this statement is inconsistent with the axiom of choice, since using this axiom one can prove the existence of such an $X \subseteq S$ for which the corresponding $\Gamma(\Phi)$ is not determined see [2],[8]).

The following statements are easy consequences of (A) and (B).

(A^*) <u>If</u> $\Phi \in K_2(S)$ <u>then</u> $\Gamma(\Phi)$ <u>is determined</u>.

(B^*) <u>The axiom of determinateness implies that</u> $\Gamma(\Phi)$ <u>is determined whenever</u> M <u>is denumerable</u>.

$K_2(s)$ is actually the largest class of functions for which (A^*) is known to be valid (see [2],[11],[1]). Any improvement of this result would give improvements of most of the theorems of this paper. It is important for applications that if $\Phi_{ij}(s)$ is a double sequence of continuous bounded functions on S, then the functions

$$\underline{\lim}_{i=\infty} \overline{\lim}_{j=\infty} \Phi_{ij}, \quad \overline{\lim}_{i=\infty} \underline{\lim}_{j=\infty} \Phi_{ij},$$

belong to $K_2(S)$.

By (B^*) all the results of Secs. 2 and 3 admit obvious generalizations on the assumption of the axiom of determinateness and the denumerability of M.

2.2. Special Games

Let M be a metric space and let every sequence $s \in S$ be convergent in M (we do not suppose that S is closed in $M \times M \times \ldots$ with respect to the nondiscrete topology in M). Let φ be a real valued function over M, and $\Phi(s) = \varphi(\lim_{n=\infty} s_n)$. The game $\Gamma(\Phi)$ is denoted by Γ_φ.

Theorem 1. If $\varphi \in K_1(M)$ then Γ_φ is determined.

Proof. The mappings $l_n : S \to M$ defined by $l_n(s) = s_n$ are continuous (even if M is discrete). Therefore the mapping $l(s) = \lim_{n=\infty} s_n$ is of the first class of Baire. We have

$$\{s : \Phi(s) < (\leq)\ r\} = \{s : \varphi l(s) < (\leq)\ r\} = l^{-1}\{x : \varphi(x) < (\leq) r\} .$$

Since $\varphi \in K_1(M)$ then for every $r \in R_{(\varphi)}$ (see (1.1)) one of the sets $\{x : \varphi(x) < (\leq)\ r\}$ belongs to B_1. Therefore, since S is a metricable space, for every $r \in R_{(\varphi)}$ one of the sets $l^{-1}\{x : \varphi(x) < (\leq)\ r\}$ belongs to B_2. Hence $\Phi \in K_2(S)$ and the theorem follows by (A^*).

Corollary 1. If φ is a characteristic function of a set $X \subseteq M$ such that $X \in B_1$ then Γ_φ is determined.

Now we formulate an additional condition on the space S.

(C) For every $s \in S$ there exists a sequence of real positive numbers $a_0, a_1, \ldots$ such that $\lim_{n=\infty} a_n = 0$ and for every m and every $s' \in S$ such that $s'|m = s|m$ there is $\rho(\lim_{n=\infty} s_n, \lim_{n=\infty} s'_n) \leq a_m$, where ρ is the distance in M.

This condition implies a slightly weaker one.

(C') The mapping $l: S \to M$, where $l(s) = \lim_{n=\infty} s_n$, is continuous.

Theorem 2. If (C') holds and $\varphi \in M_2(M)$, then Γ_φ is determined.

The proof is analogous to that of theorem 1 (using the continuity of l).

Corollary 2. If (C') holds and φ is the characteristic function of a set $X \subseteq M$ such that $X \in B_2$ then Γ_φ is determined.

2.3. Examples

1. A subset X of the real line is given. The players I and II choose consecutive real numbers $x_0, x_1, \ldots$; the even choices are done by player I and the odd choices by player II. The choices are supposed to satisfy the condition

$$(*) \qquad x_n < x_{n+1} \quad \text{and} \quad x_{n+1} - x_n \leq x_n - x_{n-1}$$

Player I wins if $\lim_{n=\infty} x_n$ exists and $\lim_{n=\infty} x_n \epsilon X$; otherwise II wins.

<u>This game is determined if</u> $X \epsilon B_1$. (This follows from Corollary 1; here the space M is the real line completed with the point $+\infty$).

The same result holds for various modifications of (*) e.g., those considered by Hanani [3].

2. Replace (*) in Example 1 by

$$(**) \qquad |x_{n+1} - x_n| < 1/2^n .$$

<u>This game is determined if</u> $X \epsilon B_2$. (This follows from Corollary 2; the essential point of (**) is that the series $\Sigma\, 1/2^n$ is convergent, which implies (C)).

3. Replace (*) in Example 1 by

$$x_n < x_{n+1} < x_{n-1} \quad \text{or} \quad x_{n-1} < x_{n+1} < x_n \quad (n = 1,2,\ldots)$$

and let player I win if $\lim_{n=\infty} x_{2n} \epsilon X$ and player II win if $\lim_{n=\infty} x_{2n} \notin X$.

<u>This game is determined if</u> $X \epsilon B_2$. To prove this let us remark that player II can force the condition

$$|x_n - x_{n+1}| < 1/2^n$$

without weakening his possibilities, i.e., this condition gives a game which is determined if and only if the former game was. But this condition implies (C), hence the result holds by Corollary 2; (We have to take for M the real plane and the consecutive choices are to be interpreted as modifications of one of the coordinates (depending on the parity of the choice) of a point in M).

2.4. Some Win—Lose Games

Let M be a complete metric space and S any subset of M x M x ... closed with respect to the topology induced by the discrete topology in M. Let $S^{(c)} = \{s : s \epsilon S$ and s is convergent in the metric of M$\}$. A subset $X \subseteq M$ is given. Player I wins if $\lim_{n=\infty} s_n$ exists and $\lim_{n=\infty} s_n \epsilon X$; otherwise II wins. Let Γ_X denote this game.

Theorem 3. If $X \epsilon G_\delta$ then Γ_X is determined.

Theorem 4. If $S^{(c)}$ satisfies (C') (i.e., the mapping $1 : S^{(c)} \to M$ is continuous) and $X \epsilon F_{\sigma\delta}$, then Γ_X is determined.

Proofs. Player I wins if and only if $s \epsilon 1^{-1}(X)$, where 1 is the limit mapping confined to $S^{(c)}$. Since $S^{(c)} \epsilon F_{\sigma\delta}$ (since M is complete this follows by a standard analysis of the Cauchy definition of convergence) it follows in both cases that $1^{-1}(X) \epsilon F_{\sigma\delta} \subseteq B_2$. Hence the theorems follow from (A).

3. APPROXIMATION OF CONTINUOUS GAMES BY POSITIONAL GAMES

A game of pursuit and evasion or any other continuous game with perfect information (we suppose only an intuitive knowledge of such games) can be in some sense represented by the games considered in the previous sections. Here follows a brief description of such a presentation of a special game of pursuit and evasion.*

P is a complete metric space with the metric $\rho(\cdot,\cdot)$. $p_0, g_0 \epsilon P$ are two distinct points (the initial positions). V is a positive real number (the ratio of velocities). Player I (the pursuer) chooses any point $p_1 \epsilon P$ such that $\rho(g_0,g_1) \leq V \cdot \rho(p_0,p_1)$. Then I chooses any $p_2 \epsilon P$ and II chooses any $g_2 \epsilon P$ with $\rho(g_1,g_2) \leq V \cdot \rho(p_1,p_2)$ etc.

Player I wins if and only if both limits $\lim_{n=\infty} p_n$, $\lim_{n=\infty} g_n$ exist and are equal. Otherwise II wins. Let Γ denote that game.

<u>Corollary 3.</u> Γ <u>is determined</u>.

<u>Proof</u>. We put $M = P \times P$ and we interprete the consecutive choices in the same way as in Example 3 (Sec. 2.3). Let X be the diagonal of $P \times P$. We have $X \epsilon G_\delta$. Therefore the corollary follows from Theorem 3.

Let us consider another game which differs from Γ in that way that a pay—off function is introduced

$$\Phi(s) = \sum_{n=0}^{\infty} \left[\rho(p_n,p_{n+1}) + \rho(g_n,g_{n+1})\right] + \lim_{m=\infty} \overline{\lim_{n=\infty}} \, m \cdot \rho(p_n,g_n) .$$

Φ could be interpreted as a length of the play and we suppose that the pursuer wants to make it small while the

*One approach of this kind was formulated also by G. Choguet (unpublished).

evader has an opposite tendency.* Since $\Phi \in K_2(S)$, then by (A^*) such a game is determined.

Given the space P, the value of this game, i.e., the number sup inf v(a,b)= inf sup v(a,b), could be considered as a function of the initial positions $p_0, g_0 \in P$ and V. Let us denote this function by $f(p_0, g_0, V)$. Of course $f(p,p,V) = 0$.

Let P be the unit circle and suppose $p_0 \neq g_0$. It is clear that

$$f(p_0, g_0) \begin{cases} < \infty \text{ if } V < 1 \\ = \infty \text{ if } V > 1 \end{cases}$$

but no reasonable way of calculating f(for $V < 1$) is known. It has been proved.** that $f(p_0, g_0, 1) = \infty$.

We could generalize the above games supposing that the two players are picking their choices in two different spaces P and Q and taking some pay-off functions Φ. Also various modifications of the condition $\rho(g_n, g_{n+1}) \leq V \cdot \rho(p_n, p_{n+1})$ could be introduced together with some conditions on the choices of player I.

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*We have accepted (Sec. 1) a definition of games according to which player I wants to get large values while player II small values and moreover the pay-off functions are supposed to be real valued. To get all this we should take arc $t_g \Phi$ rather than Φ.

**The argument of Besicovitch described by Littlewood [4] can be modified to get a proof of this fact.

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