MATHEMATICAL ASPECTS OF THE THEORY OF SYSTEMS

Richard Bellman

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The RAND Corporation, Santa Monica, California

1. INTRODUCTION

In attempting to formulate a mathematical theory of systems, various intuitive principles can furnish useful guidance and counsel. A first heuristic principle is that a meaningful theory must be closely related and correlated, in several different ways, with our ideas concerning the roles and general objectives of mathematical model-making in Science. A second maxim is that the terms "problem" and "solution" are most fruitfully regarded as relative, time-varying concepts to be interpreted in terms of the current ability to provide qualitative and quantitative answers. A third, and perhaps most important, principle is a meta-principle which permits, and indeed urges, us to rise above principles if the situation demands. By this we mean that in any type of research it is occasionally important to follow one's intuition and to pursue intriguing leads regardless of immediacy or relevance.

From the foregoing precepts, we are led via familiar syllogisms to several conclusions. The first is that the most important sources of stimuli for a mathematical

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theory of systems will be found in the study of real systems: economic, engineering, physical, military, biological, medical, and so forth. It is certainly conceivable that the armchair philosopher, aided only by the power of pure reason, can conjure up all kinds of meaningful systems as objects of study. In actuality, as the history of mathematics so clearly shows, this has not been the case. It is the consensus of opinion, as well as history, that the imagination of the mathematician intermittently requires the spur of reality. Without this, as in other intellectual areas, what ultimately results is esotericism, sterility, and finally boredom.

Secondly, it is essential to acknowledge that the development of computers—analog, digital, and hybrid types—has significantly, and irrevocably, altered the art of conceiving and using mathematical descriptions of physical phenomena. This, in turn, is merely one aspect of the fact that in the study of systems it is necessary to take account of the manner in which the actual process unfolds in time.

Thirdly, it is important to explore and develop systematically a number of new mathematical theories whose origins lie in the real systems of the world without premature attempts at either application, axiomatization, or confrontation with established methodology. Of particular significance are many nonnumerical theories of algebraic and topological structure.

Taking what has been said into account, we feel that a most appropriate way to inaugurate a mathematical theory of systems is by way of a graduated set of questions (for a general discussion of problems and problem-solving, see [1]). The first major problem towards which we point is that of the identification of systems. Informally, it may be posed in the following terms:

"Given some information concerning the structure of a system, and some observations of inputs, outputs, and internal behavior over time, deduce all of the missing information concerning structure, inputs, and outputs."

With some justice, we may consider it to be a fundamental problem on which many others hinge. As we shall mention again below, the foregoing is part of an attempt to construct a general theory of experimentation. We shall present below a series of questions which indicate a feasible path to the treatment of this general question.

A second problem of major concern involves the simultaneous identification and control of systems. This is a far more complex type of problem than might be imagined, and it requires a certain amount of preliminary discussion before one can obtain a proper perspective. These matters will be discussed below, and a number of references for further reading will be provided.

2. WHAT IS A SYSTEM?

Although this may appear to be a natural first question, it is probably best, at this time, to bypass it and avoid any direct answer. It appears to be far more profitable to begin with the study of a number of important processes and systems in detail, and in this way both to obtain an intuitive feel for the nature and structure of systems and to gain some familiarity with the various means available for treating systems. With experience and understanding, we can undertake axiomatization in a meaningful manner. We must avoid any preliminary "rigor mortis." For a detailed discussion of many of the basic ideas, let us refer to Zadeh and Desoer [2].

3. HOW DO WE DESCRIBE A SYSTEM?

We are so accustomed to a number of powerful traditional techniques that may be used to describe systems that it is easy to overlook the fact that these are only a few of infinitely many ways of superimposing a mathematical structure on a physical process.

Ideally, the choice of state variables should depend upon our analytic and computational expertise, the type of data that is available, and upon the over-all goals. Often, questions of storage, processing, and retrieval of information dominate the more sophisticated analytic techniques when a global view is taken.

We are resigned by now to the fact that in many areas the classical views of space and time and cause and effect must be replaced by the less familiar concepts of relativistic mechanics and quantum mechanics. It is perhaps not so well-known that even within the conventional domains of mathematical physics there is a great deal of freedom as far as analytic formulation is concerned. We shall return to this point below, particularly in connection with the uses of dynamic programming [3,4,5] and invariant imbedding [6,7,8].

Another choice that must be carefully made is that of a deterministic or a stochastic formulation. Furthermore, there is no reason why both versions should not be combined. In general, flexibility and versatility is the goal, a goal that is much more realizable these days with modern computers and other technological devices. It is usually the case that different formulations will be useful for different purposes.

It cannot be sufficiently emphasized that the initial mathematical model determines all of the analytic and computational effort that follows. Once we have replaced the original physical system by its necessarily condensed mathematical image, we have automatically restricted our efforts. So many of the serious difficulties encountered by mathematicians in their efforts in mathematical physics are of their own making, inevitable consequences of a lack of understanding of what is both required and desired by engineers, scientists, and others who must use the language of mathematics for their own purposes. As a game, the difficulties are intriguing; as a tool for scientific research, they are distracting.

4. HOW DO WE PREDICT THE FUTURE BEHAVIOR OF A SYSTEM?

Let us suppose that we have finally decided upon a means of describing a system. It is natural then to ask if we can use the information available concerning the present state to predict its future behavior. This is almost a circular question, since our tendency would be to reject a description which did not, in principle, permit this foretelling of the future. However, we appreciate the fact that there is a vast difference between theoretical and effective prediction. Much of classical analysis is devoted to this problem of cause and effect, of existence and uniqueness of solutions of functional equations. Its importance cannot be over-estimated since the answers to all of our subsequent questions depend upon our ability to treat the problem of obtaining numerical algorithms for solving functional equations of the form

$$\frac{dx}{dt} = g(x(\cdot)), \tag{4.1}$$

where $x(\cdot)$ denotes a dependence upon the present and past history of the system. Leaving aside the numerical aspects, it is essential to possess criteria for determining whether questions are well-posed or not.

The most familiar version of (4.1) is the vector differential equation

$$\frac{dx}{dt} = g(x), \quad x(0) = c. \tag{4.2}$$

Here $x = (x_1, x_2, \ldots, x_N)$, $g(x) = (g_1(x), g_2(x), \ldots, g_N(x))$. Equations of this type can be numerically resolved at the present time in a reasonably routine fashion, provided the dimension of x does not exceed one thousand or so. With the computers available in a year or so, this number will be upped to five thousand, with the computers of ten years hence, we can conservatively think in terms of ten thousand or twenty-five thousand.

Formerly, many ingenious devices were used to circumvent the use of large systems of nonlinear differential equations. They represented formidable, and often insuperable, barriers to progress in understanding. Now, they represent solutions, if by "solution" we mean a simple algorithm for providing numerical answers. This change in viewpoint must reflect itself in the choice and formulation of problems.

In many cases, (4.1) represents a system with an infinite-dimensional state vector. The most common version of this is a partial differential equation, such as

$$u_t = u_{xx} + e^u, \tag{4.3}$$

or

$$\begin{aligned} u_t + uu_x + vu_y &= p_1, \\ v_t + uv_x + vv_y &= p_2. \end{aligned} \tag{4.4}$$

Formerly, partial differential equations required for their solution a great deal of analytic ingenuity, sophistication, and experience. Now, however, we can use very simple and direct methods. Suppose that we write, in (4.1),

$$u = \sum_n u_n(t)e^{inx}. \tag{4.5}$$

Then, (4.3) becomes an infinite system of nonlinear ordinary differential equations,

$$u_n' = n^2u_n + g_n(u_0,u_1,u_{-1},\ldots). \tag{4.6}$$

Suppose we truncate by taking $u_n \equiv 0$ for $|n| > N$. We then have a finite system of the form appearing in (4.2). Taking $N = 100, 200$, or 1000, it is reasonable to expect that we can obtain a numerical solution of (4.3) in a very simple and direct fashion; cf. [9].

There is certainly nothing elegant about this approach, and some mathematicians might complain about this sledge-hammer method. It has one merit: it enables people with very little training in mathematics to solve scientific problems of significance. Moreover, it indicates the strong possibility that the computer may cause an unexpected schism between science and mathematics in many classical areas. Scientists in these areas will no longer require high-powered analytic methods for the treatment of their problems. Let us hasten to add that this does not mean that there will be less over-all need for mathematicians. There will, however, be a considerable shift in their efforts. It is another facet of automation.

Particularly in mathematical biology, we encounter processes involving a dependence upon past history, which is to say, hereditary effects. Then, (4.1) is replaced by differential-difference equations,

$$\frac{dx}{dt} = g(x(t),x(t-\tau_1),\ldots,x(t-\tau_N)), \tag{4.7}$$

where the delays may be constant, or functions of time, or, more generally, themselves be dependent on the state x, (4.8) $\tau_i = \tau_i(x(t),t)$,

cf. [10], [11].

Finally, let us mention the theory of branching processes, a natural outgrowth of the classical theories of iteration and semigroups, cf. Harris [12]. The concept of "point of regeneration" considerably extends the ideas behind (4.1).

It is easy to see that there are unlimited opportunities for mathematical research in the areas we have so briefly examined. Precisely for this reason it is so necessary to examine over-all objectives in order to make sure that the usual clustering phenomenon of science does not occur. We do not want to funnel all of our talent and effort into certain currently fashionable areas at the expense of other, usually more important, areas. This is harmful to science, and particularly destructive to young researchers.

One further point is worth mentioning. Sometimes, in an investigation, we require numerical values in connection with the design and construction of a physical system. In many other cases, only qualitative behavior is needed, information which will culminate in a "yes" or "no." For example, we may want to know whether a system is stable, whether a certain type of steady-state behavior exists, whether there is periodicity or almost-periodicity, and so on. In some situations, this information can be obtained purely by means of analytic techniques. In general, in dealing with complex processes, the only way to obtain the qualitative information is by means of suitably chosen quantitative data.

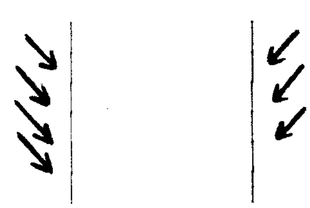
For this reason, it is extremely important to correlate the choice of state variables and the analytic formulation with the numerical capability and the type and accuracy of input data. All of this explains in part the statement made above concerning the effect of the computer upon mathematics. The quality of computational algorithms available must influence the analytic tools used. This is "impedance matching."

5. HOW DO WE DEDUCE THE PAST HISTORY OF A SYSTEM?

So far, we have considered various parts of the problem of determining the future behavior of a system. Let us now consider the case where we catch glimpses of a system from time to time and are then required on that basis to deduce its past, present, and future behavior. By the term "glimpse," we mean that it is not possible at any particular time to obtain values of all of the components of the state vector. A typical example of this occurs in orbit determination, where it is relatively easy to observe the position of an object at any time, but not at all easy to obtain an accurate estimate of its velocity.

Processes of this type give rise to two-point, or in general multipoint, boundary-value problems. These are several levels of difficulty above initial-value problems and in no sense of the word can we regard their analytic or computational solution as routine.

A variety of boundary-value problems arise in mathematical physics. Consider, for example, a transport process associated with a plane-parallel slab.



Given the nature of the flux incident at both boundaries, we wish to determine the reflected fluxes and the internal fluxes.

In connection with the minimization of a functional such as

$$(5.1) \qquad J(u) = \int_0^T g(u,u')dt,$$

we obtain, via the calculus of variations, a differential equation subject to two–point conditions,

$$(5.2) \qquad \frac{d}{dt}\left(\frac{\partial g}{\partial u'}\right) - \frac{\partial g}{\partial u} = 0.$$

What is interesting to point out, and relevant to our principal theme of versatility and flexibility, is that in both of the foregoing cases, there are alternate analytic formulations which avoid the intricacies of two–point boundary–values. In the first case, we can employ invariant imbedding [6,7,8]; in the second case, dynamic programming [3,4,5].

This illustrates the point previously stressed: there is nothing intrinsic about any analytic representation of a physical process. At the risk of being repetitious, let us note that the classical formulations of mathematical physics are of eighteenth and nineteenth century vintage. Even quantum mechanics and relativity theory, of early twentieth century origin, are, expectedly, nineteenth–century in their analytic formulation.

It is reasonable to suspect that many different mathematical interpretations of physical processes exist. This should be the case well within the classical framework, and particularly so (our familiar refrain) when digital computers are available. Mathematics and science alike are determined by the ability to do arithmetic.

From here on out, we are engaged in a perpetual scientific revolution. Unlike the relatively peaceful scientific era of 1700—1950, we can expect that the formulations of theories will constantly change over time to keep parallel with the increase in experimental techniques, the development of computers, and the varying objectives of society.

At the present time we can, without flinching, contemplate the integration of systems of ordinary differential equations of order 1000. In the next few years, this number will be 5000; in ten years or so, 10,000 to 100,000. What happens to so much of classical mathematical physics when these formerly formidable operations become exercises? This question cannot be overstressed, since the answers to it should determine the curriculum in the undergraduate and graduate schools.

6. WHAT ARE THE DUAL SYSTEMS?

We are all familiar with the power and versatility of the geometric concept of duality. One of the advantages of the abstract mathematical point of view is that it permits a ready geometricization of processes far removed from the study of points and lines, and thus a simple transference and translation of existing techniques. Much has been done in the area of duality, but very much remains.

7. HOW DO WE APPROXIMATE A SYSTEM?

The student in school is trained to furnish precise answers to precise questions. This is hardly adequate training for a real world in which the essence of success is that of obtaining reasonable answers to reasonable questions in areas which are vague and imprecise. Even in the academic world, success depends more upon the choice of one's own questions than the supplying of answers to the questions of others. In other words, it is the art of approximation to reality that counts throughout.

A basic part of this is the construction of the original mathematical model. We have discussed some aspects of this above. Let us consider here the simpler matter of obtaining an analytic approximation to an already existing analytic structure. Suppose, for example, that a system is described by an N-dimensional vector nonlinear differential equation

$$\frac{dx}{dt} = g(x), \quad x(0) = c. \tag{7.1}$$

How do we find a linear equation

$$\frac{dy}{dt} = Ay, \quad y(0) = c, \tag{7.2}$$

with the property $x \cong y$ in some prescribed sense. For example, we may wish to minimize $\int_0^T (x - y, x - y)dt$.

Or suppose that a system is governed by the equation

$$\begin{aligned} k(x)u_t &= u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(x,0) &= g(x), \\ u(0,t) &= u(1,t) = 0, \end{aligned} \tag{7.3}$$

and it is desired to find a finite-dimensional system of the type appearing in (7.1) with the property that $x \cong z$ where

$$z = \begin{pmatrix} u(x_1,t) \\ u(x_2,t) \\ \vdots \\ u(x_N,t) \end{pmatrix}.$$

Many other problems of this nature can be posed, and very little has been done in this area; see, however, [13].

These, however, represent only a small part of the questions that arise in connection with the mathematical theory of the approximation of systems. There is the question of what is meant by the approximation of one process by another; the approximation of one set of state variables by another; matters of "lumping" of variables, and of closure and truncation; the mode of calculation of variables. All of these, in turn, are aspects of the choices: discrete versus continuous, stochastic versus deterministic, finite versus infinite, linear versus nonlinear, Markovian versus non-Markovian, static versus dynamic, numerical versus nonnumerical, and variational versus descriptive.

As we shall see below in connection with control processes, this by no means exhausts the decisions which must be made.

8. HOW DO WE DEDUCE THE STRUCTURE OF A SYSTEM?

Let us now consider what are often called "inverse problems." In Sec. 1, we have called them identification problems. Given input-output data, or asymptotic behavior, or, in general, what has usually been considered an answer, the problem is to determine the question. This may be the task of deducing a structure or deducing an equation. It is clear that we cannot expect unique answers to problems of this type, unless we restrict our attention to particular categories of systems. In other words, once we specify the general structure of a system, and reduce the unknown aspects to the determination of constants, then we can expect to determine the particular system within the category whose behavior best approximates the observed behavior.

For example, suppose that we know that $x(t)$, the state vector describing a system, satisfies an equation of the form

$$\frac{dx}{dt} = g(x,a), \quad x(0) = c, \tag{8.1}$$

where a and c are unknown vectors. We wish to determine them so as to minimize the expression

$$\sum_{i=1}^{M} ((x(t_i),b_i) - c_i)^2, \tag{8.2}$$

where the scalars c_i correspond to "observations" at the times t_i. A large number of scientific problems fall in this area [13].

We see from this simple example that the identification problem is again a relative matter, a matter of knowing and parametrizing the set of systems under consideration. We see then the need for a taxonomy of systems which will furnish us a wide variety of choices. Let us repeat the point made before that the solution of identification problems depends on our ability to treat descriptive processes.

9. DISCUSSION

Up to now, we have been taking quite a detached view of system theory, in the sense that we have engaged in the pretense that our only goal was to understand the structure and behavior of systems. In actuality, in the majority of cases, we want to use this understanding for a variety of purposes. It is plausible that greater understanding will enable us to accomplish these purposes in more efficient fashion. Since, however, in many cases we cannot wait for complete understanding (e.g., cancer, inflation, traffic), our objective is again one of approximation: partial control based upon partial understanding, improved control based upon improved understanding.

In what follows, we wish briefly to indicate what we mean by the term "control" and to point out some of the many new questions that have arisen in recent years.

10. HOW DO WE CONTROL A SYSTEM?

A fundamental problem in the world around us is that of making a particular system behave in an acceptable fashion. This ranges from keeping a space vehicle on course, or an economy expanding, to preventing the further growth of a cancerous tumor. Nothing can be more practical than a theory of systems.

One important aspect of control theory is the elucidation of what is meant by the terms "an acceptable fashion." It is not difficult to construct analytic versions of control processes, but these may not have much to do with what is really desired in practice. The price of the introduction of available mathematical techniques is often a strange mixture of oversimplification and overcomplication—oversimplification of the actual process and overcomplication in the application of analysis.

To illustrate these remarks, let us consider a type of mathematical problem which is quite fashionable now, and then, point-by-point, indicate the tacit assumptions that have been made, and how these assumptions weaken the validity of the model and limit its applicability.

Let x be an N-dimensional vector and y an M-dimensional vector related by means of the differential equation

$$(10.1) \qquad \frac{dx}{dt} = g(x,y,t), \quad x(0) = c,$$

the local constraints

$$(10.2) \qquad r_i(x,y,t) \leq 0, \quad i = 1,2,\ldots,k,$$

and the global constraints

$$(10.3) \qquad \int_0^T k_j(x,y,t)dt \leq 0, \quad j = 1,2,\ldots,\ell.$$

It is required to determine x, the state vector, and y, the control vector, so as to minimize the functional

$$(10.4) \qquad J(x,y) = \int_0^T h(x,y,t)dt + \phi(x(T)),$$

cf. [14], [15].

This is a very interesting and difficult problem which contains enough in its particularizations and generalizations to keep mathematicians busy for decades. This, of course, is at once its principal merit and demerit. The mathematician seeing a control process so precisely and neatly posed is apt to closet himself in his study and pay no attention to far more important, but less well-posed, questions. These ill-posed questions are, however, far more significant as far as the development of mathematics itself is concerned.

Let us now analyze the many assumptions adroitly hidden in the preceding. To begin with, there is the explicit criterion function in (10.4). The triple assumption here is first that an objective is known, secondly that it can be expressed in analytic terms, and thirdly that it is a scalar and not a vector criterion.

Let us bypass these exceedingly sticky matters, which can be treated by means of simulation techniques, and continue with our dissection.

Next, there is the assumption implicit in (10.1) that we understand cause and effect well enough to write down an equation of this type. In many important processes, this is not the case. There are several ways of meeting this difficulty, on several levels of sophistication.

The first is to replace a deterministic process by a stochastic process in which uncertainty is represented by stochastic functions with known distributions. It is not, however, sufficient to assume that this ingenious device disposes of uncertainty. On the contrary, it forces us to examine a number of problems which did not arise in classical control theory because of the low order of complexity of the systems considered. Essentially, these are the problems of the storage, processing, and retrieval of information.

In a deterministic process, it is inherent that knowledge of the state at one time determines the state at any time in the future. The analogue of this for the usual stochastic process is that a knowledge of the probability distribution for current states determines the probability distribution at any subsequent time. Since this is seldom sufficient information for effective control of a system, we face the problem of observation of a system. In this way enter "feedback control," "on line" control, and so on.

As the theory of dynamic programming shows, one can use the concept of feedback control to construct a theory of deterministic control processes [3,4,5]. The classical theory is dual to this, and each possesses its own analytic and computational advantages. In dealing with stochastic control processes, this duality breaks down. There is now a continuum of varieties of control processes between the two poles of complete information at each time, and of no information once the process has started. As mentioned above, the processes of the real world furnish us with important prototypes. The concepts of information and policy are now dominant.

In the preceding paragraph, we used the term "policy." A policy is a rule for decision-making. It determines what to do in terms of where one is and what one knows. It has several advantages over the usual prescription of a control vector as a function of time. Among these are its more intuitive flavor, its applicability in simulation processes where optimization may not be meaningful, its high degree of compatibility with the capabilities of computers, the many new types of approximation it makes available, and the uniform way in which deterministic, stochastic, and adaptive processes can be treated.

Furthermore, the emphasis upon feedback control concepts has the desirable effect of focussing attention on state variables, means of observing state variables, the accuracy of these observations, interruption of communication, and, in general, all aspects of the acquisition, processing, storing, and use of information.

It is important to point out that the recognition of the fact that time and other resources are required for data-processing leads to an analogue of the classical uncertainty principles for quantum mechanics. We can obtain approximate information quickly and cheaply, or more accurate information slowly and expensively. In either case, we incur a cost of control. There are many interesting questions associated with the choice of variables to observe and the cost involved in obtaining information of various degrees of accuracy.

11. HOW DO WE IMPROVE AN OPERATING SYSTEM?

It is not sufficiently emphasized that the design and control of a new system is a different problem from that of improving the performance of a system which has been operating for some time. Leaving aside the inevitable psychological aspects of change, there is the standard problem of transient effects. How does one change from the current policy to an optimal policy without dangerous or costly side effects due to transients? The associated mathematical questions are quite difficult. For example, starting with a given policy, how does one improve monotonically?

All of this is connected with the vital problem of "on-line" control.

12. HOW DO WE LEARN ABOUT AND CONTROL A SYSTEM?

What has preceded is preliminary to the basic problem of controlling a system whose properties are not completely known. In order to do this, we must divide our energies between the tasks of observing, experimenting, and learning, and the tasks of exerting control influences. Although these matters are of paramount importance, only recently have problems in these areas been precisely formulated. They constitute part of the new field of adaptive control.

It is most likely the case that the current formulations are too complex for convenient analytic or applied use. What is needed is work in the direction of Monte Carlo methods, stochastic approximation, and particularly experimentation along the lines of simulation. What the most fruitful directions are is certainly not clear at this time.

13. CONCLUSION

There are, of course, no such things as objective questions. The choice of questions, and even their framing, consciously or unconsciously reflects the philosophical attitudes of the questioner. We have tried our best to keep the questions verbal and thus avoid any hardening of the analyses or ossification of the issues. Nonetheless, from time to time we have weakened and indicated possible analytic approaches.

We feel that a great deal of significant work can be done following conventional lines, and that it should be done. We also feel that it is worthwhile to spend a good deal of time and energy searching for new routes. It is probably best to mix strategies and to pursue both activities, not only within research groups but within the individual himself.

In any case, we hope that we have made it clear that the theory of systems is one of the most exciting of modern scientific fields, with application and inspiration everywhere. A young mathematician can hardly go astray in this domain.

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