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Much present-day radar performance analysis for fluctuating targets is based on the so-called Swerling Cases I-IV, Swerling[1]; in another paper, Swerling[2], characteristic functions are given for the video integrator output, for a much larger class of probability density functions of the signal fluctuations, and for (to all intents and purposes) arbitrary correlation properties of the fluctuations.

However, the probability distribution functions of the signal fluctuations treated by Swerling[2] were not completely arbitrary. It is the purpose of this note to derive the characteristic function of the video integrator output (assuming pulsed waveforms) for completely arbitrary probability distribution functions of the signal fluctuations, and for completely arbitrary correlation properties; that is to say, for a completely arbitrary joint probability distribution of the individual pulse signal-to-noise ratios. (Subject, however, to the assumption that the signal amplitude and phase do not fluctuate within a single pulse.) The result obtained, while fairly trivial, does not seem to be widely known.

Next, the forms taken by this result will be exhibited in some special cases. Finally, a brief discussion will be given of computational schemes based on the characteristic functions thus obtained.

Let the situation be of the familiar type: IF-filter, square law envelope detector, video integrator, with a pulsed waveform input. Let the individual pulse outputs of the detector be denoted by v_i, $i = 1, \ldots, N$ (normalized with respect to mean-square noise in the usual fashion); let y be the video integrator output:

$$y = \sum_{i=1}^{N} v_i \tag{1}$$

(Swerling[2] considers the case where the video integrator forms an arbitrary weighted sum of pulses; we will make the simplifying assumption that the weights are equal, in return for which the resulting formulas are much simpler. However, this does not imply that the signal-to-noise ratios of the pulse inputs cannot be weighted as for example by a beam shape effect.)

Let $P(y|x_1, \ldots, x_N)$ be the (cumulative) probability distribution function of y, conditional on definite values $x_1, \ldots, x_N$ of the individual pulse signal-to-noise ratios; let $H(x_1, \ldots, x_N)$ be the joint (cumulative) probability distribution function for $(x_1, \ldots, x_N)$; let $P(y)$ be the unconditional (cumulative) probability distribution function of y, that is,

$$P(y) = \int P(y|x_1, \ldots, x_N)\, dH(x_1, \ldots, x_N) \tag{2}$$

Define the characteristic function of y, conditional on $x_1, \ldots, x_N$, by

$$C(p|x_1, \ldots, x_N) = \int_0^\infty e^{-py}\, dP(y|x_1, \ldots, x_N) \tag{3}$$

and the unconditional characteristic function of y by

$$C(p) = \int_0^\infty e^{-py}\, dP(y) \tag{4}$$

Then,

$$C(p) = \int C(p|x_1, \ldots, x_N)\, dH(x_1, \ldots, x_N) \tag{5}$$

It is well known that

$$C(p|x_1, \ldots, x_N) = (1+p)^{-N} \exp\left\{ - \frac{p}{1+p} \sum_{i=1}^{N} x_i \right\} \tag{6}$$

Now, consider the random variable

$$X = \sum_{i=1}^{N} x_i \tag{7}$$

Let $\varphi(p)$ be the characteristic function of X:

$$\varphi(p) = \int e^{-Xp}\, dH(x_1, \ldots, x_N) \tag{8}$$

The basic result is seen immediately from Eqs. (5), (6), and (8):

$$C(p) = (1+p)^{-N} \varphi\left(\frac{p}{1+p}\right) \tag{9}$$

This basic result takes the following forms in special cases:

1. Pulse-to-pulse Fluctuations

Here,

$$H(x_1, \ldots, x_N) = \prod_{i=1}^{N} H_i(x_i) \tag{10}$$

(the individual H_i are not assumed to be necessarily the same.)

Then,

$$\varphi(p) = \prod_{i=1}^{N} \varphi_i(p) \tag{11}$$

where

$$\varphi_i(p) = \int_0^\infty e^{-px} \, dH_i(x) \tag{12}$$

and

$$C(p) = (1+p)^{-N} \prod_{i=1}^{N} \varphi_i\left(\frac{p}{1+p}\right) \tag{13}$$

2. Scan-to-scan Fluctuations

In this case, $X = N x_1$; thus,

$$\varphi(p) = \theta(N p) \tag{14}$$

where

$$\theta(p) = \int_0^\infty e^{-px} \, dH_1(x) \tag{15}$$

$H_1(x)$ in Eq. (15) is the marginal probability distribution function of signal-to-noise ratio for a single pulse.

Thus,

$$C(p) = (1+p)^{-N} \, \theta\left(\frac{N p}{1+p}\right) \tag{16}$$

3. Intermediate Fluctuations

For cases where the fluctuations are neither pulse-to-pulse nor scan-to-scan, there are two difficulties of applying the basic result, Eq. (9).

First is that of specifying the joint probability distribution function $H(x_1, \ldots, x_N)$; second is that of evaluating the characteristic function of X.

Swerling[2] solves this problem for a particular (large) class of functions $H(x_1, \ldots, x_N)$ by assuming, essentially, that the random variables x_i are sums of squares of Gaussian variables. Then, the joint distribution of the Gaussian variables can be specified easily, and the problem solved in these terms.

Computational Methods

For most cases met with in practice, the characteristic function can be used as the basis for computing numerical results, either by

a) inverting it analytically in cases where possible

b) inverting it by machine integration

c) deriving moments (or cumulants, as is usually more convenient) by differentiating the characteristic function, and then using these moments for series expansions.

In Marcum's and Swerling's work[1], all the graphical results were computed by methods (a) or (c), and ninety percent were done by method (c).

Since the moments of a distribution are obtained by evaluating successive derivatives of the characteristic function at $p = 0$, it can be seen by inspection of Eq. (9) that the first n moments of y (the integrator output) can be expressed as functions of the first n moments of X; it is trivial to derive the formulas.

Moreover, in the case of pulse-to-pulse or scan-to-scan fluctuations, the first n moments of X are simple (easily derived) functions of the first n moments of the individual signal-to-noise ratios x_i.

Thus, in the scan-to-scan or pulse-to-pulse cases, one can express the first n moments of y directly as simple combinations of the first n moments of the signal fluctuations. In a great many cases, the first six moments suffice for an accurate series expansion.

In the general case, the first n moments of y are still expressible in the same way in terms of the first n moments of X. However, the latter are now functions (relatively simple ones) of the moments through order n, including cross-moments, of the joint distribution function $H(x_1, \ldots, x_N)$, no longer just of the marginal distributions of the individual x_i.

Provided one can specify the moments through order n of $H(x_1, \ldots, x_N)$, one can then derive the first n moments of y.

For the family of joint distributions considered by Swerling[2], the explicit expressions for C(p) there derived can in many cases be inverted, and in general are suitable for moment expansions, although in either case digital machine rather than desk computation would be required.

The method leading to Eq. (9) can also be applied to other than square law envelope detectors; a similar derivation may be applied whenever the characteristic function of a single pulse out of the detector, having signal-to-noise ratio x, is of the form

$$C_1(p|x) = A(p) \exp\left[x\, B(p)\right] \tag{17}$$

In such cases, Eq. (9) takes the more general form

$$C(p) = \left[A(p)\right]^N \varphi\left[B(p)\right] \tag{18}$$

where $\varphi(p)$ is still given by Eq. (8).

The foregoing is not meant to imply that the method of series expansions based on moments, or even the use of characteristic functions in general, is always the most convenient computational procedure, but merely to show that in a large class of cases these methods provide feasible computational procedures.

There may be cases of interest (e.g., when the fluctuation distributions involved are bimodal or multi-modal) where the series expansion based on moments would require too many terms to be convenient.

In some cases, the fluctuation distributions may have tails which have little effect on the probability of detection curves but which affect the moments strongly. (In general, it can be shown that the percentiles of the signal fluctuation distributions are more stably related to the detection probability curves than are the moments of the fluctuation distributions.)

Special techniques can often be used to attack such cases. For example, truncation of the fluctuation distributions at an appropriate point may in some cases be a useful preliminary to the application of series expansions based on moments.

In other cases, a direct integration (by machine) of $P(y|x_1, \ldots, x_N)$ with respect to $H(x_1, \ldots, x_N)$ may prove to be the most convenient method.

REFERENCES

1. Marcum, J. I. and Swerling, P, "Studies in Target Detection by Pulsed Radar," I.R.E. Trans. Info. Theory, Vol. IT-6, No. 2, April 1960.

2. Swerling, P., "Detection of Fluctuating Pulsed Signals in the Presence of Noise," I.R.E. Trans. Info. Theory, Vol. IT-3, No. 3, September 1957, pp. 175-178.