USE OF GENERALIZED ACTIVITY NETWORKS IN SCHEDULING

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The past few years have seen an increasing utilization of network models as an aid in scheduling and planning activities. The primary reason for this is the capability to use networks for both communication and analysis purposes. Specifically, networks are easily understood by all levels of personnel in the organizational hierarchy and they can be used as a communication device as they provide a reference point for discussions. Because of this latter feature they promote interaction among the various "types" of people, such as research, planning, and operational personnel. Furthermore, networks facilitate the identification of pertinent data and present a mechanism for data collection. In addition to these characteristics, which can be grossly classified under communication, there are the benefits associated with the ability to analyze the activity. The most common analytical use is found in conjunction with the PERT-CPM type networks, where system parameters, such as duration of the project, estimated completion and arrival times of the lot, part, or assembly, and total project cost, are estimated. There have been many other applications of network representation in the areas of production, planning, and scheduling (e.g. Ref. 3). For the current discussion, the PERT type network is considered as a reference base as it is most commonly associated with the scheduling or planning functions.

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Recently, several articles (Ref. 1 and 2) have pointed out that PERT type networks are limited in terms of the types of logical elements permitted. This refers to the PERT requirements that (1) all activities incident on a node must be completed before the node can be realized, and (2) all arcs emanating from the node must be taken. In many cases it is desirable to represent probabilistic flows from one activity to another. For example, it is necessary that there be two output branches from a maintenance activity where, depending on whether the item passed or failed a test, it would continue normal flow or be routed to appropriate maintenance activities. Another limitation of the PERT type network representation is the restriction to unidirection flow. Referring to the maintenance test example, if an item failed a test one possible decision would be to retest in order to ensure proper test results. This requires a flow in the opposite direction.

THE GENERALIZED ACTIVITY NETWORK

Salah E. Elmaghraby of Yale has developed a basic algebra for the analysis of networks which considers the logical relationships between events and permits flow in any direction (Ref. 2). Utilization of this basic algebra on a complicated network requires the user to consider all of the possible paths from one node to another and the interrelationships of these paths. A.A.B. Pritsker and W. W. Happ of Arizona State University have extended Elmagrahby's work by developing a set of equations which permit the analyses of certain types of networks by direct evaluation. It is, however, only applicable when there are no "and" nodes* in the forward paths of the network loops. We do not intend to delve into the derivation of these equations at this time since this material will be discussed in a forthcoming paper by Pritsker and Happ. The purpose of this discussion is to indicate that a formal technique exists for reducing many types of compound feedback networks. This technique, when used in conjunction with the basic network algebra, provides a strong analytical tool for estimating the system parameters of interest.

*The node classification utilized in this paper is developed in Ref. 2. An "and" node is characterized by the requirement that all activities incident on the node must occur for the node to be realized.

ILLUSTRATION OF THE PROCEDURE

The technique for reducing a generalized activity network consisting of many nodes (all exclusive-or nodes)* and branches into an equivalent network with two nodes and one branch is illustrated below. Consider a hypothetical repair facility modeled after the Heath facility discussed in the study by Ronald Steorts (Ref. 4). The following diagram is the network representation of the flow of a reparable item from receiving to shipping:

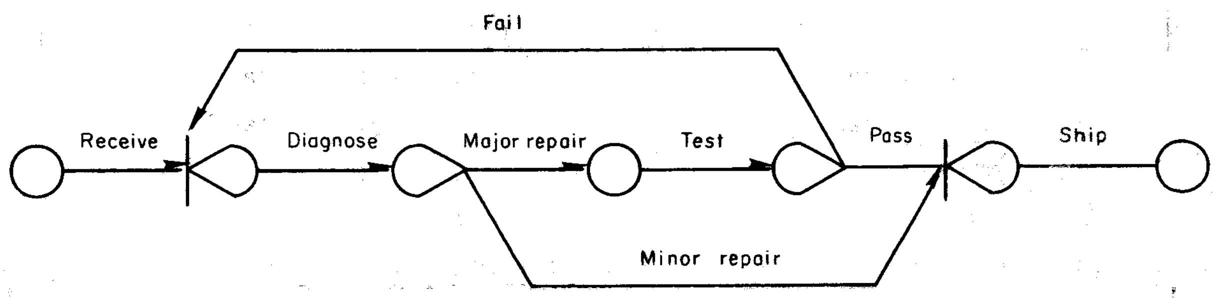

Fig. 1 -- A "Repair Facility"

Suppose we were interested in the expected time it would take for a specific reparable to go through this repair cycle. This quantity will be denoted by ECT. If a probability and a time can be associated with each activity as defined in Fig. 1, then the technique referred to in the above section could be used to determine the time required for an item to go through the system for a given resource configuration. If the resource availabilities are changed, then the time estimates for each of the branches would need to be reconsidered. Each time estimate is composed of three components: transit time, queue time, and operation time. The diagram could just as easily represent each of these components as a separate activity.

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*An exclusive-or node is characterized by the restriction that one and only one of the activities incident on the node can occur. The occurrence of any activity causes the node to be realized.

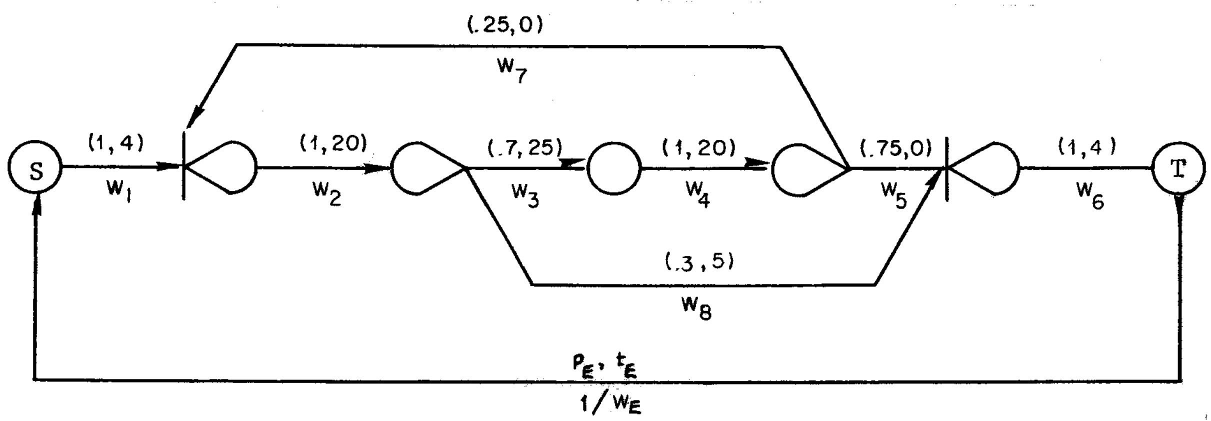


Fig. 2 -- The "Repair Facility" in Analysis Form

Figure 2 depicts the repair facility of Fig. 1 with illustrative probabilities and transfer times. The first element of the ordered pair is the probability of transfer for the activity (p_i); the second element is the estimated mean duration of the activity (t_i). For generality it will be assumed that W_3 is characterized by a Poisson distribution (this knowledge is assumed to be obtained from an analysis of past data).

To determine the expected cycle time we proceed as follows:

(1) Connect the terminal node (T) to the beginning node (S) with an arc, $^1/W_E(S) = {}^1/p_E M_S(t_E)$, where p_E is the probability of transfer from node S to node T; and $M_S(t_E)$ is the moment generating function associated with the transfer time from S to T.

(2) Write the topological equation of the network (H(S))*

H(S) = 1-Σ simple loop product + Σ product of two non-touching loops - Σ product of three non-touching loops + ... = 0.

*This is the topological equation of signal-flow graph theory which describes the relationship between branches for any closed system (see Ref. 5).

In this case, as there are no combinations of non-touching loops, $H(S) = 1-[W_1W_2(W_3W_4W_5+W_8)W_6]\frac{1}{W_E} - W_2W_3W_4W_7 = 0,$

where $W_i = p_i e^{St_i}$ for $i \neq 3$, and $W_3 = p_3 e^{t_3(e^S-1)}$.

NOTE: e^{St_i} is the moment generating function of a constant;

$e^{t_3(e^S-1)}$ is the MGF of the Poisson distribution assuming t_3 is the mean of the Poisson distribution.

(3) Solve the topological equation for $W_E(S)$:

$$W_E(S) = \frac{W_1W_2(W_3W_4W_5+W_8)W_6}{1 - W_2W_3W_4W_7} .$$

Substituting the values of W_i yields:

$$W_E(S) = \Big[p_1p_2p_6p_3p_4p_5 \exp[S(t_1+t_2+t_6+t_4+t_5) + t_3(e^S-1)] + p_1p_2p_6p_8 \exp[S(t_1+t_2+t_6+t_8)]\Big] \Big/ \Big\{1 - p_2p_3p_4p_7 \exp[S(t_2+t_4+t_7) + t_3(e^S-1)]\Big\} .$$

(4) Evaluate $p_E = W_E(0)$:

$$W_E(0) = p_E = \frac{p_1p_2(p_3p_4p_5+p_8)p_6}{1 - p_2p_3p_4p_7} .$$

Substituting for the p_i yields

$$p_E = \frac{(.7)(.75) + .3}{1-(.7)(.25)} = 1 \text{ as expected.}$$

(5) Calculate $\text{ETC} = E\left\{t_E\right\} = \frac{d}{dS}\left\{\frac{W_E(S)}{W_E(0)}\right\}\Bigg|_{S=0}$

$$ETC = \frac{d}{dS}\left\{\frac{W_E(S)}{1}\right\}\Bigg|_{S=0}$$

$$= \Big(\{p_1p_2p_6p_3p_4p_5(t_1+t_2+t_3+t_4+t_5+t_6) + p_1p_2p_6p_8(t_1+t_2+t_6+t_8)\}\Big)$$

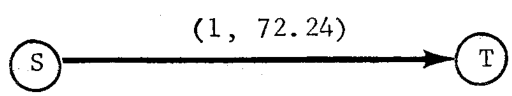
$$[1-p_2p_3p_4p_7]^{-1} + \{p_1p_2p_6p_3p_4p_5 + p_1p_2p_6p_8\}$$

$$\{ \quad p_2p_3p_4p_7(t_2+t_4+t_7+t_3)\}\{1-p_2p_3p_4p_7\}^{-2}.$$

Substituting yields

$E(t_E)$ = ETC = 72.24 time units.

Thus, the equivalent network is:



Two basic formulas were used to find the equivalent network:

(1) $P_E = W_E(0)$,

(2) $ETC = E\left\{t_E\right\} = \frac{d}{dS}\left\{\frac{W_E(S)}{W_E(0)}\right\}\Bigg|_{S=0}$.

In general,

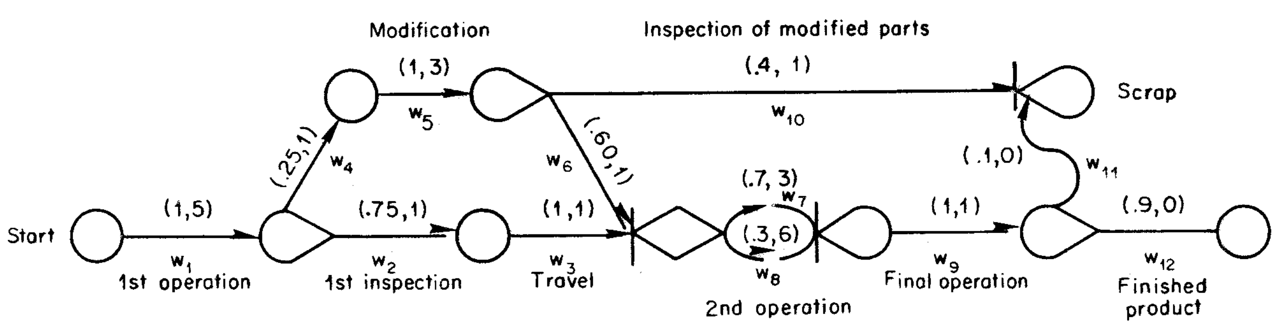
$$E\left\{t_E^n\right\} = \frac{d^n}{dS^n}\left\{\frac{W_E(S)}{W_E(0)}\right\}\Bigg|_{S=0} ,$$

and $\frac{W_E(S)}{W_E(0)}$ is the moment generating function of the transfer time from node S to node T.

The repair cycle time if there was a major repair and no recycle would be 73 hours. The cycle time for minor repair is 33 hours, and the cycle time if there is recycle depends on how many times the reparable item had to be recycled. The estimated cycle time of 72.24 hours takes all these paths into consideration. If this time is felt to be too large, management would be aware of the necessity to take some form of action, such as reallocation of resources.

It would also be of interest to calculate the probability that the cycle time would be $\leq$ some number k. As previously indicated, the moments can be calculated for this distribution and a lower bound can be obtained on this probability by an approximation technique such as Tchebycheff's inequality.

As a second example, consider a production line in which the first operation manufactures a part and the processing time is five hours. Following this operation the part is inspected. Inspection takes one hour and as a result of the inspection 25 per cent of the parts are not accepted and are sent to a modification section. To modify the parts takes three hours. Parts which required modification are reinspected and 40 per cent of these are rejected and scrapped. The other 60 per cent are sent to a second operation with the parts that were accepted after the initial operation. The transit time to the second operation is assumed to be one hour. The second operation takes three hours, 70 per cent of the time, and six hours, 30 per cent of the time. A final inspection is performed which takes one hour and 10 per cent of these items are rejected and scrapped. The network representation of this process is given below.



Considering the path from start to finished product, and using the technique illustrated in the previous example, yields:

$$H(S) = 1 - \left[\frac{W_1 W_9 W_{12}}{W_E} (W_7 + W_8)(W_2 W_3 + W_4 W_5 W_6) \right] = 0$$

$$= 1 - \left[\left(\frac{.9e^{S(5+1+0)}}{W_E} \right) \left(.7e^{3S} + .3e^{6S} \right) \left(.75e^{S(1+1)} + (.25)(.6)e^{S(1+3+1)} \right) \right].$$

Solving for $W_E(S)$,

$$W_E(S) = .4725e^{11S} + .2025e^{14S} + .0945e^{14S} + .0405e^{17S};$$

$$P_E = P_{S-F} = W_E(0) = .81 \qquad \text{(from Eq. 1)};$$

$$E(t_E) = T_{S-F} = \frac{1}{.81} \left. \frac{dW_E(S)}{dS} \right|_{S=0}$$

$$= \frac{11(.4725) + 14(.2025) + 14(.0945) + 17(.0405)}{.81}$$

$$= 12.42 \text{ hours} \qquad \text{(from Eq. 2)}.$$

The equivalent probability and transfer time associated with taking the scrap path can be determined in a similar manner. They are:

$$P_{S-SC} = .19,$$

$$T_{S-SC} = 11.14 \text{ hours}.$$

Thus, the equivalent network of interest can be drawn as

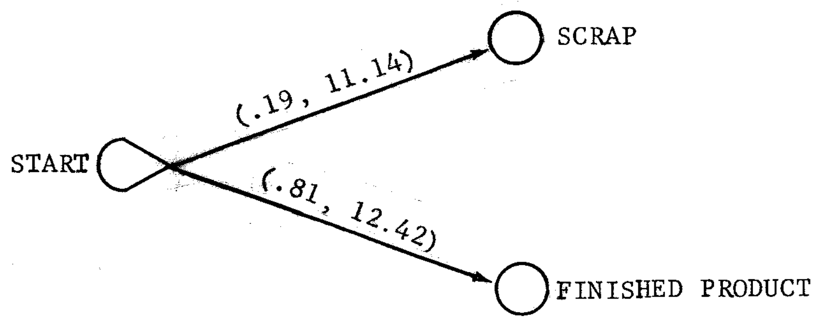


ILLUSTRATION OF THE GENERALIZED NETWORK MODEL APPLIED TO A SCHEDULING PROBLEM

A scheduling problem in which the approach discussed above is being applied is the assessment of the value of alternative countdown plans--schedules of testing activities. We are currently studying the problem at RAND. The following discussion attempts to briefly describe the application.

Countdown is the final validation phase of the assembled launch vehicle and spacecraft prior to launch. Countdown has as its objective the bringing of the vehicle into a launch-ready state. To accomplish a countdown (and countdown planning) involves a large organization structure with many activities and interactions. Furthermore, it brings together diverse groups and treats a highly sophisticated man-machine system.

The key control mechanism of current countdowns is the script. A script is a schedule, formulated prior to countdown, of the operations to be performed during a countdown. A schedule of countdown operations differs from the more common shop schedules in the sense that there is no significant competition for resources and there are no queues in the ordinary sense. The only quantity that is progressing is time. This, of course, makes the problem more applicable to the analytical approach of network representation. The problem in developing a countdown schedule or script is to perform the operations in a sequence that meets the necessary precedence relationships, such as A must be performed before B, and to accurately determine the state of the system--discover and repair malfunctions. This must all be accomplished within a given time constraint--the countdown must be completed within the launch window for a launch to be acceptable.

The "Generalized Activity Network" has been used to model a countdown and performance measure of the system determined through reduction of the network. The performance measure is the probability of successful countdown. This is the probability that the countdown is completed within the launch window and there are no undiscovered malfunctions. Thus, the joint probability of two events is being estimated. One is the time $\leq$ some number k, which is the same concept as discussed in the "Repair Facility" example. The other is the probability of transfer from the start node to a particular terminal state--no undiscovered malfunctions. This is directly determined by the network reduction process. We are now able to compare two alternative schedules to determine which is more desirable in terms of the performance measure.

CONCLUDING REMARKS

The "Generalized Activity Network" model is a technique which possesses all the communication benefits associated with the PERT type networks yet permits the analysis of more complex logical relationships and feedback loops. It is an analytical tool which, when used in conjunction with a technique which estimates activity times, appears to yield a feasible method for short-range evaluations of a given resource configuration. In those cases where resource availability is not an active constraint the network approach provides a mechanism for evaluating alternative schedules.

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