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The old problem of the eight queens is this: how to place eight queens on a chess board so that no queen attacks any of the others. Figure 1 shows one solution. Ball[†] gives the twelve basic solutions, as shown in Fig. 2.

1	2	3	4	5	6	7	8
			Q				
Q							
				Q			
							Q
	Q						
						Q	
		Q					
					Q		

Fig. 1--One of the Solutions to the Eight Queens Problem, Represented by 41582736

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[†]W.W.R. Ball, Mathematical Recreations and Essays, 11th ed., The Macmillan Company, New York, 1962, pp. 165-170.

4	1	5	8	2	7	3	6
4	1	5	8	6	3	7	2
4	2	5	8	6	1	3	7
4	2	7	3	6	8	1	5
4	2	7	3	6	8	5	1
4	2	7	5	1	8	6	3
4	2	8	5	7	1	3	6
4	2	8	6	1	3	5	7
4	6	1	5	2	8	3	7
4	6	8	2	7	1	3	5
4	7	5	2	6	1	3	8
4	8	1	5	7	2	6	3

Fig. 2--The Basic Solutions

The arrangement is standardized because each basic solution has a queen on the fourth square from some corner. Each basic solution has seven other variations. Figure 3 gives the 96 total possible arrangements. The first column of Fig. 3 is the set of basic solutions, as given in Fig. 2. The other seven are derived from the first as follows:

Column 2. Rotate the basic pattern 180° and reflect it through the vertical axis in the middle of the board. In numerical terms, the digits of column 2 are the reverse of those of column 1.

Column 3. Rotate the basic pattern 90° and reflect it.

Column 4. Rotate the basic pattern 270°.

Column 5. Rotate the basic pattern 90°.

	1	2	3	4	5	6	7	8
1	41582736	63728514	25713864	46831752	74286135	53168247	58417263	36271485
2	41586372	27368514	28613574	47531682	71386425	52468317	58413627	72631485
3	42586137	73168524	62713584	48531726	37286415	51468273	57413862	26831475
4	42736815	51863724	72418536	63581427	27581463	36418572	57263184	48136275
5	42736851	15863724	82417536	63571428	17582463	36428571	57263148	84136275
6	42751863	36815724	52814736	63741825	47185263	36258174	57248136	63184275
7	42857136	63175824	62714853	35841726	37285146	64158273	57142863	36824175
8	42861357	75316824	52617483	38471625	47382516	61528374	57138642	24683175
9	46152837	73825164	35714286	68241753	64285713	31758246	53847162	26174835
10	46827135	53172864	64718253	35281746	35281746	64718253	53172864	46827135
11	47526138	83162574	64713528	82531746	35286471	17468253	52473861	16837425
12	48157263	36275184	36814752	25741863	63185247	74258136	51842736	63724815

Fig. 3--The Twelve Basic Patterns and Their Seven Orientations

Column 6. Reflect column 4.

Column 7. Reflect column 1.

Column 8. Rotate the basic pattern 180°.

Figure 4 shows the result of sorting the 96 variations of Fig. 3. The tenth basic pattern of Fig. 2 happens to be symmetric, so that the various orientations on the board yield only three new patterns; this shows up with four sets of duplicates in the sorted array of Fig. 4. Thus, all told, there are only 92 different basic patterns.

Now we come to the overlay problem. How can the 12 basic solutions be shown on one chess board with a minimum of crowding? Figure 5 illustrates a haphazard solution to the overlay problem. The variations come from Fig. 3

according to the pattern 123456788877. That is, the first basic solution is taken from column 1 of Fig. 3; the second from column 2; and so on, the last being taken from column 7.

This solution, however, is crowded: three squares each receive four queens, and 14 squares have none. The theoretical perfect solution would have all squares occupied, 32 of them containing one queen and 32 containing two queens. This perfect solution is probably unattainable, since every pattern shown in Fig. 3 has a queen on the fourth square in from one corner.

	1	2	3	4	5	6	7	8
1	15863724	16837425	17468253	17582463	24683175	25713864	25741863	26174835
2	26831475	27368514	27581463	28613574	31758246	35281746	35281746	35286471
3	35714286	35841726	36258174	36271485	36275184	36418572	36428571	36814752
4	36815724	36824175	37285146	37286415	38471625	41582736	41586372	42586137
5	42736815	42736851	42751863	42857136	42861357	46152837	46827135	46827135
6	46831752	47185263	47382516	47526138	47531682	48136275	48157263	48531726
7	51468273	51842736	51863724	52468317	52473861	52617483	52814736	53168247
8	53172864	53172864	53847162	57138642	57142863	57248136	57263148	57263184
9	57413862	58413627	58417263	61528374	62713584	62714853	63175824	63184275
10	63185247	63571428	63581427	63724815	63728514	63741825	64158273	64285713
11	64713528	64718253	64718253	68241753	71386425	72418536	72631485	73168524
12	73825164	74258136	74286135	75316824	82417536	82531746	83162574	84136275

Fig. 4--The 96 Variations Ordered[a]

[a]Underlines show the duplications caused by the one symmetric pattern.

1	2	3	4	5	6	7	8
5	2 8 9	6	1 10	7 11 12	3 4		
1 12	3 11	4	8		6 9 10	2 5 7	
7 9	6	2	11	1 4 5	8	3	10 12
3	10		7 12	6	2	9 11	1 4 5 8
4	1 5 7 12	3 8 11	9			10	2 6
6 8 10			4 5	2 3		1 12	7 9 11
2	4	1 9 10 12			5 7 11	6 8	3
11		5 7	2 3 6	8 9 10	1 12	4	

8 Queens

Solution 123456788877

$\Sigma X^3 = 630$

14 zeros

21 ones

15 twos

11 threes

3 fours

Fig. 5--A Haphazard Solution to the Overlay Problem

Exploring the overlay problem completely would require examining 8^{12} (= 69 billion) permutations, plus a criterion by which to judge minimum crowding. The latter requirement is easy: we can take as our index the sum of the cubes of the number of queens on each cell. By this criterion, the solution of Fig. 5 has an index of 630; the perfect solution would have an index of 288. Figure 6 shows a solution with an index of 396, which may be the lowest possible. The purpose of this paper is to explain the heuristic by which the solution of Fig. 6 was obtained.

1	2	3	4	5	6	7	8
11	1	4 12	3 7 8	5 10	6 9	2	
2	7 8	6 10	9	1	4 12	5 11	3
6 10	5 9 12	2	4 11	3		1	7 8
1 4		3		7	5 8 11	10 12	2 6 9
3 8	10	1 5	6	9 12	2	7	4 11
5 7 12	6 11	8	2	4		3 9	1 10
9	2 3	7	5	8 11	1 10	4 6	12
	4	9 11	1 10 12	2 6	3 7	8	5

Fig. 6--The Solution 35437811262 for the Overlay Problem[a]

[a]The index is 396.

The process advances in twelve stages. For Stages 1 and 2, two of the variations of Fig. 3 are chosen arbitrarily and applied together to a chess board. This board is then replicated eight times. For example, having chosen the variations 3 and 5 arbitrarily, the basic patterns 25713864 and 71386425 would be applied and replicated, giving eight boards in storage with the pattern of Fig. 7.

At Stage 3, all eight variations in the third row of Fig. 3 are applied to these eight boards, one per board. For each board, the sum of cubes of the cell counts is calculated, and the smallest of these sums is selected; it turns out in this case to be the sum for board 4. The solution up to this point, then, is 354.

1	2	3	4	5	6	7	8
	1					2	
2				1			
		2				1	
1							2
		1			2		
			2				1
	2				1		
			1	2			

Fig. 7--Stages 1 and 2 Chosen as 3, 5

Board 4 is now replicated to the other seven working boards. All the variations of the fourth row of Fig. 3 are applied, and the best configuration selected again. This procedure continues through 12 stages, resulting in the pattern of Fig. 6.

For the input arrangement of Fig. 3, 64 such computer runs were made, testing every possible combination of the choices for Stages 1 and 2. Three of these 64 tries produced patterns with an index of 396 (the other two being 463587226151 and 781264558323). The highest value for the index during these 64 tries was 462.

That this procedure does not yield the optimum pattern is evident by making another computer run with the input data (the 12 basic solutions from Ball) reversed. The table of variations in storage is then Fig. 3 read from bottom to top. The 64 arbitrary choices for the first two patterns is made as before, and the procedure works its way down the new array. For this input, the smallest index found is 402.

The overlay problem was stated vaguely; namely, to display all 12 solutions on one board with the least crowding. "Least crowding" could be defined in many ways; we might call for the least number of squares containing three queens, for example.

As far as that goes, why select the sum of cubes as a criterion? (The time-honored sum of squares would not discriminate strongly enough against a cell count of four.) We could just as well decide on the sum of fourth powers. Would that make a difference?

It would indeed. With the sum of fourth powers as the criterion for choosing, the theoretical perfect solution would have an index of 544; the best solution found by the sum of cubes criterion (26 cells of one queen; 26 of two; 6 of three) would then have an index of 928. A computer run was made with the input data of Fig. 3, but with fourth powers calculated. Not only did this result in new patterns, but no pattern yielded an index below 942.

Figure 8 presents a pattern produced by the fourth power criterion. Seven of its cells are vacant; 24 cells have one queen; 27 have two; 6 have three. The pattern has an index of 942 for fourth powers, or 402 for cubes.

Like all heuristics, the decision-making process used in this study provides no guarantee of a best solution (unless, of course, all possible combinations were explored). The ordering of the input data and the criterion used both affect the results.

The computations for this paper were performed on an IBM 1620 at Los Angeles Valley College.

The overlay problem is a multi-stage decision process. The scheme suggested here reduces it to a series of single-stage decisions, for each of which a simple criterion of success is given.

1	2	3	4	5	6	7	8
11	8	5 7	2 4	1 6 10	9 12	3	
	4 6	3 10 12	8		5 11	2 7	1 9
3 10 12	7 9		1 5	2	8	4	6 11
1 6	5	2 4 11	9		3	10	7 8 12
2 9	10	8	6	7 12	4	1 11	3 5
7 8	1 12		11	3 5	2	6 9	4 10
4	3 11	6	7 12	9	1 10	5 8	2
5	2	1 9	3 10	4 8 11	6 7	12	

8 Queens

Best Solution, $\Sigma X^4 = 942$

24 ones
27 twos
6 threes

$\Sigma X^3 = 402$

Fig. 8--A Pattern Produced by the Criterion of Fourth Powers