COMPUTATIONAL SOLUTION OF RADIATIVE TRANSFER PROBLEMS IN CLOUD PHYSICS

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1. INTRODUCTION

Multiple scattering of light in clouds is both important and complex. It is necessary to deal with time-varying diffuse light fields in the presence of anisotropic scattering and polarization, the coupling of radiation with heat and moisture transfer (Petrova and Feigel'son, 1962), and other complicating aspects. For the sake of simplicity, we shall view the radiative transfer problem in clouds as that of multiple scattering with an anisotropic phase function, leaving out heat transfer and other effects. Anisotropic scattering processes also take place in lakes and oceans (Tyler and Shaules, 1964) and in glacial ice.

We wish to be able to treat highly anisotropic transfer problems, as one step toward the goal of determining the structure and dynamics of clouds based on experimental measurements. Theory, computation, and experiment must join efforts

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in order to make real progress in this direction.

Anisotropic scattering by an elementary volume in a cloud results in an extraordinary increase in the complexity of the theoretical analysis, (cf. Feigel'son and Dobrova, 1962). An increase of the anisotropy gives rise to the increase of terms in the expansion of the phase function in Legendre polynomials, which in turn causes difficulties in application of the existing techniques of solution. In the case of extreme anisotropic scattering in clouds, the numerical solution requires a tremendous amount of effort.

On the other hand, the computational experience gained thus far has been primarily confined to isotropic scattering in homogeneous and inhomogeneous media, usually with slab geometry and uniform parallel rays of incident radiation. Let us consider the status of the computational solution of isotropic radiative transfer problems, and the possible extensions to the anisotropic case. We cannot hope to be all-inclusive in this brief survey; we can only hope to share some of our experiences and thoughts on these matters.

2. RADIATION FIELDS FOR ISOTROPIC SCATTERING

The standard approaches to the solution of planetary radiative transfer problems have met with grave numerical difficulties. Direct integration of the equation of transfer (Chandrasekhar, 1960),

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\lambda}{2}\int_{-1}^{+1} I(\tau,\mu')d\mu' - \frac{\lambda}{4}e^{-\tau/\mu_0} \tag{1}$$

reveals instabilities in the equations. Successive approximations of the auxiliary equation (Sobolev, 1963),

$$J(\tau) = \frac{\lambda}{2}\int_0^{\tau_1} J(\tau')\, E_1\,(|\tau-\tau'|)d\tau' + \frac{\lambda}{4}e^{-\tau/\mu_0} \tag{2}$$

are slowly convergent for large albedos. The principles of invariance (Chandrasekhar, 1960) lead to ill-conditioned linear algebraic systems. It is clear that we must take a fresh look at the numerical solution of transport problems.

Now, we know that current digital computers are well able to integrate large systems of ordinary differential equations (say of dimension 1000), with given initial conditions, in a few minutes, and with extremely high accuracy. The equations may be linear or nonlinear. We wish to derive new types of equations which make full use of this capability, which will greatly increase in the next few years.

The invariant imbedding method (cf. Bellman, Kalaba, and Prestrud, 1963) is a systematic way of viewing a transport process as an initial value problem, the independent variable generally being the thickness of the medium. The X and Y functions of Chandrasekhar, for example, are solutions of the system of integro differential equations,

$$\frac{\partial X(\tau_1,\mu_0)}{\partial \tau_1} = \frac{\lambda}{2}Y(\tau_1,\mu_0)\int_0^1 Y(\tau_1,\mu')\frac{d\mu'}{\mu'}\ , \tag{3}$$

$$\frac{\partial Y(\tau_1,\mu_0)}{\partial \tau_1} = - Y(\tau_1,\mu_0)\,\frac{1}{\mu_0} + \frac{\lambda}{2}\, X(\tau_1,\mu_0) \int_0^1 Y(\tau_1,\mu')\frac{d\mu'}{\mu'} ,$$

with the initial conditions $X(0,\mu_0) = 1$, $Y(0,\mu_0) = 1$. In the computations, the integrals are approximated as sums via Gaussian quadrature. The system of ordinary differential equations,

$$\text{(4)} \qquad \frac{dX_i(\tau_1)}{d\tau_1} = \frac{\lambda}{2}\, Y_i(\tau_1) \sum_{j=1}^{N} Y_j(\tau_1)\frac{W_j}{\mu_j},\ i = 1,2,\ldots,N ,$$

$$\frac{dY_i(\tau_1)}{d\tau_1} = - Y_i(\tau_1)\,\frac{1}{\mu_i} + \frac{\lambda}{2}\, X_i(\tau_1) \sum_{j=1}^{N} Y_j(\tau_1)\,\frac{W_j}{\mu_j} ,$$

is integrated numerically, with the initial conditions $X_i(0) = 1$, $Y_i(0) = 1$. The computational solution is obtained accurately and rapidly (Bellman, Kagiwada, Kalaba, and Ueno, 1965c).

The invariant imbedding approach has been successfully applied to time-dependent as well as stationary processes, reflection and transmission functions, photon emergence probabilities, the source function, and internal intensities (see Bellman, Kalaba, and Prestrud, 1963; Bellman, Kagiwada, Kalaba, and Prestrud, 1964; and the series of papers by Bellman, Kagiwada, Kalaba, and Ueno, 1964-1965).

3. AN INVERSE PROBLEM: DETERMINATION OF STRATIFICATION IN A SLAB

So far, we have been discussing "direct problems", i.e., we are given the input radiation and the properties of the medium, and we wish to calculate internal and external radiation fields. In "inverse problems", we wish to determine the structure of a medium, given measurements of internal and/or external radiation fields. While inverse problems occur in practically all branches of science, they have been most difficult to treat, both analytically and computationally. Now, however, with the present day computers and computational techniques such as quasilinearization (Bellman and Kalaba, 1965), dynamic programming (Bellman, 1957), and invariant imbedding (Bellman, Kagiwada, Kalaba, and Sridhar, 1964), we are in a position to solve a wide class of inverse problems (Kagiwada, 1965).

Consider a plane-parallel stratified medium, which is isotropic, and which is illuminated by parallel rays of constant net flux π. Using an invariant imbedding derivation, we readily obtain the equation for the diffuse reflection coefficient, $r(\mu, \mu_0, \tau_1)$, for a slab in which the albedo for single scattering, λ, varies with optical height. It is

$$(5) \qquad \frac{\partial R(\mu,\mu_0,\tau_1)}{\partial\tau_1} = -\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)R + \lambda(\tau_1)\left[1+\frac{1}{2}\int_0^1 R(\mu,\mu',\tau_1)\frac{d\mu'}{\mu'}\right] \cdot \left[1+\frac{1}{2}\int_0^1 R(\mu'',\mu,\tau_1)\frac{d\mu''}{\mu''}\right] ,$$

with the initial condition, $R(\mu, \mu_0, 0) = 0$, where $R(\mu, \mu_0, \tau_1) = 4\mu r(\mu, \mu_0, \tau_1)$, τ_1 is the optical thickness of the slab, and μ_0 and μ are the cosines of the incident and reflection angles, respectively. Let us suppose that there are two layers: in the lower layer, the albedo is λ_1, and in the upper layer, the albedo is λ_2. In the direct problem, we would be given λ_1, λ_2, and all other parameters of the problem. A numerical integration (out to thickness T) of the system of ordinary differential equations,

$$(6) \qquad \frac{dR_{ij}(\tau_1)}{d\tau_1} = -\left(\frac{1}{\mu_i} + \frac{1}{\mu_j}\right)R_{ij} + \lambda(\tau_1)\left[1 + \frac{1}{2}\sum_{k=1}^{N} R_{ik}(\tau_1)\frac{W_k}{\mu_k}\right] \cdot \left[1 + \frac{1}{2}\sum_{k=1}^{N} R_{kj}(\tau_1)\frac{W_k}{\mu_k}\right], \quad i = 1,2,\ldots,N, \; j = 1,2,\ldots,N,$$

with the initial conditions, $R_{ij}(0) = 0$, would give a high order approximation to the reflection coefficient. The inverse problem which we wish to treat is the determination of the albedos, the location of the interface, and the total thickness, T, of the slab, when we are given measurements of the reflection coefficient matrix, $b_{ij} \cong r_{ij}(T)$.

Let the albedo have the form, $\lambda(\tau) = a + b \tanh 10\,(\tau - c)$, so that $\lambda_1 \cong a - b$, $\lambda_2 \cong a + b$, and c is the height of the interface between the two layers. Let the correct values, unknown to us, be $a = 0.5$, $b = 0.1$, and $c = 0.5$. Let the true thickness be $T = 1$. We use $N = 7$ in the quadrature formula, and we seek the values of a, b, c, and T which minimize the expression $S = \sum_{i,j} [r_{ij}(T) - b_{ij}]^2$, where $R_{ij}(T) = 4\mu_i r_{ij}(T)$ is the solution of (6).

The successive approximation technique of quasilinearization is used to solve this nonlinear boundary value problem (Kalaba, 1959; Bellman, Kagiwada, and Kalaba, 1962; Bellman, Kagiwada, Kalaba, and Ueno, 1964a). Two types of numerical experiments are carried out: (i) determine c, the altitude of the interface; (ii) determine a, b, and c, and thereby the two albedos and the position of the interface. The results given in Tables 1 and 2 are indicative of the rapid convergence and accuracy of the method. The solution in Run 3 diverges because of a poor initial approximation.

Table 1. Approximations of c, the level of the interface.

Approximation	Run 1	Run 2	Run 3
Initial approximation	0.2	0.8	0.0
Second approximation	0.5187	0.5024	No
Fourth approximation	0.499990	0.499991	convergence
True value	0.5	0.5	0.5

Table 2. Approximations of λ_1, λ_2, and c.

Approximation	λ_1	λ_2	c
Initial approximation	0.51	0.69	0.4
Third approximation	0.399938	0.599994	0.499878
True values	0.4	0.6	0.5

For the experiment of Table 2, we have to integrate 124 linear differential equations during each of three stages, and to solve a system of linear algebraic equations of order three during each stage. The calculations require about two minutes on an IBM 7044 computer.

Such numerical experiments, using this method or other techniques, enable one to estimate unknown parameters, and to study the effects of the quality and quantity of measurements upon the estimates. This is important from the experimentalist's standpoint. Furthermore, these studies may be useful in the construction of mathematical models of physical situations.

4. ANISOTROPIC SCATTERING

The computational treatment of isotropic transfer problems is well under control for direct and inverse problems, for particle and wave processes (Kagiwada, 1965), for stationary and time-dependent processes, and for various geometries. The above discussion is readily extended to cases of mild anisotropic scattering, when the phase function can be expanded in a Fourier series with only a few terms. The computed spacial distribution of brightness in lower deck clouds is qualitatively correct (Kozlov and Fedorova, 1962). R. Preisendorfer (1965) is able to explain anisotropic radiation fields in the sea via his mathematical model.

For extremely anisotropic phase functions which are appropriate for clouds, of the order of 100 terms are required in the expansion by Legendre polynomials. Up to 48 terms have been included in calculations by Romanova (1962). Enormous amounts of computing times are involved. It appears that we must seek new functions of new parameters, which are much more suitable for anisotropic scattering. One such parameter might be related to the asphericity of the phase function. On the other hand, new computer systems like the IBM—360 or CDC—6600 make even the straightforward approaches feasible (cf. Ueno, Kagiwada, Kalaba, and Bellman, 1965).

If, instead of the uniform rays of the sun, a radar or laser beam is incident on a cloud, we must treat the collimated point source problem (Bellman, Kalaba, and Ueno, 1963). If the beam is pulsed, the time-dependent problem must be treated, involving another degree of complexity. Yet, it is worth investigating the computational aspects of a pulsed laser problem, for the purpose of detecting smog layers, clouds, and even some atmospheric discontinuities not visible to the eye. The superior performance of the pulsed laser may permit its use in studies of cloud development, droplet concentration in various regions of clouds, and light scattering inside the clouds, and in many other inverse problems of terrestrial and planetary clouds.

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