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Robin B. S. Brooks
John Y. Lu

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Robin B. S. Brooks and John Y. Lu*

The RAND Corporation, Santa Monica, California

Under suitable assumptions Hadley and Whitin [1] give the average number of backorders incurred per year under an economic order policy as

$$B(r,Q) = \frac{\lambda}{Q}\int_r^\infty \varphi(x)(x-r)dx, \tag{1}$$

where Q is the order quantity, r is the reorder point, λ is the average number of demands per year ($\lambda \geq 0$), and φ is the probability density for the number of demands during the lead time (assumed to be a continuous random variable). Unfortunately this function, while convex in r and convex in Q, is not in general convex in both r and Q. As pointed out by Veinott [2], this nonconvexity can lead to a failure of the optimization technique suggested in [1]. In this note we show that if r is restricted to be suitably large, then B(r,q) is, within such a region, convex. We assume φ differentiable. Our result is stated precisely as follows:

THEOREM. <u>Suppose</u> μ <u>a number such that</u> $\varphi(x)$ <u>is nonincreasing for</u> $x \geq \mu$. Then B(r,Q) <u>is convex in the region described by</u> $\mu \leq r < \infty$ <u>and</u> $0 < Q < \infty$.

Thus, in particular, when φ is normal B(r,Q) is convex when the reorder point r is in excess of the mean demand during the requisitioning lead time.

<u>Proof of the Theorem</u>. If we let X be a random variable with density function φ, then the integral in (1) represents the expectation of the random variable X_r defined by

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$$X_r = \begin{cases} 0 & \text{if } X < r \\ X-r & \text{if } X \geq r \end{cases}$$

Now for positive x the probability that X_r exceeds x is simply $\Phi(x+r)$, where $\Phi(y) = \int_y^\infty \varphi(t)dt$. Thus, using the fact that the expectation of a nonnegative random variable may be obtained by integrating the "tail" of its distribution, we have

$$\int_r^\infty \varphi(x)(x-r)dx = E(X_r) = \int_0^\infty \Phi(x+r)dx = \int_r^\infty \Phi(x)dx.$$

Thus, we may rewrite (1) as

$$B(r,Q) = \frac{\lambda}{Q} \int_r^\infty \Phi(x)dx. \tag{2}$$

This expression is easily differentiated to obtain the following matrix of second partial derivatives:

$$\begin{pmatrix} \frac{\partial^2 B}{\partial r^2} & \frac{\partial^2 B}{\partial r \partial Q} \\ \frac{\partial^2 B}{\partial Q \partial r} & \frac{\partial^2 B}{\partial Q^2} \end{pmatrix} = \lambda \begin{pmatrix} \frac{1}{Q}\varphi(r) & \frac{1}{Q^2}\Phi(r) \\ \frac{1}{Q^2}\Phi(r) & \frac{2}{Q^3}\int_r^\infty \Phi(x)dx \end{pmatrix} \tag{3}$$

In order for B to be convex within a region, it suffices that the above matrix be positive semi-definite within that region. The matrix in (3) will be positive semi-definite when its diagonal terms and its determinant are nonnegative. The diagonal terms are clearly nonnegative whenever $Q > 0$. Thus it suffices to examine the determinant of (3). This determinant may be written as $\frac{\lambda}{Q^4} h(r)$, where

$$h(r) = 2\varphi(r) \int_r^\infty \Phi(x)\,dx - [\Phi(r)]^2. \tag{4}$$

Hence it suffices to show that $h(r) \geq 0$ whenever $r \geq \mu$.

The derivative of h is given by

$$h'(r) = 2\varphi'(r) \int_r^\infty \Phi(x)\,dx. \tag{5}$$

Now $\int_r^\infty \Phi(x)\,dx \geq 0$, while $\varphi'(r) \leq 0$ for $r \geq \mu$ ($\varphi(r)$ is nonincreasing for $r \geq \mu$). Thus, $h'(r) \leq 0$ for $r \geq \mu$. It follows that $h(r)$ is nonincreasing for $r \geq \mu$. But it is obvious from (4) that $\lim_{r\to\infty} h(r) = 0$, so this together with the fact that $h(r)$ is nonincreasing for $r \geq \mu$, implies that $h(r) \geq 0$ for $r \geq \mu$.

REFERENCES

1. Hadley, G. and T. M. Whitin, Analysis of Inventory Systems, Prentice-Hall, Inc., 1963.

2. Veinott, A. F., Review of Hadley and Whitin in Journal of the American Statistical Association, Vol. 59, March 1964, pp. 283-285.