OPTIMAL TWO- AND THREE-STAGE PRODUCTION SCHEDULES WITH SETUP TIMES INCLUDED

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SUMMARY

A collection of production items and two machines or stages are given. Each item must pass through stage one, then stage two. Each machine can handle only one item at a time. There are two arbitrary (positive) numbers given for each item representing the setup plus work time for that item to pass through each stage.

A simple decision rule leads to the optimal scheduling of the items minimizing the total elapsed time for the entire operation. For example, the decision rule permits one to optimally arrange twenty production items in about five minutes by visual inspection. A three-stage problem is also discussed and solved for a restricted case.

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§1. Two-stage production schedule.

Let us consider a typical multistage problem formulated in the following terms by R. Bellman:

"There are n items which must go through one production stage or machine and then a second one. There is only one machine for each stage. At most one item can be on a machine at a given time.

"Consider 2n constants A_i, B_i, $i = 1, 2, \cdots, n$. These are positive but otherwise arbitrary. Let A_i be the setup time plus work time of the i-th item on the first machine, and B_i the corresponding time on the second machine. We seek the optimal scheduling of items in order to minimize the total elapsed time."

A simple decision rule leads to an optimal scheduling of the items minimizing the total elapsed time for the entire operation. For example, the decision rule permits one to optimally arrange twenty production items in about five minutes by visual inspection.

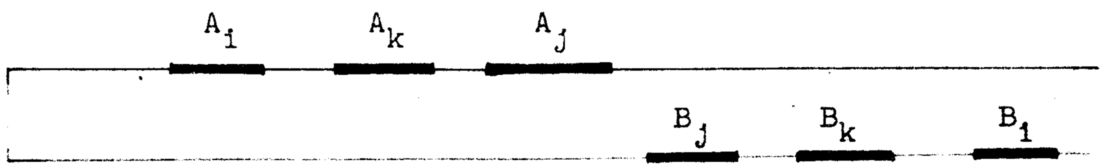
In the second section a three-stage problem is also discussed and solved for a restricted case.

Lemma 1.

The production sequence on either machine can be made the same as that of the other machine without loss of time.

Proof.

On the time scales for each machine place the A's and B's in any position subject to the rules. If the orders are different, the elements out of order will be placed something like the following.



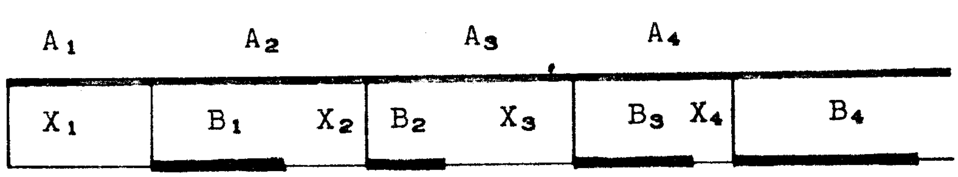
Then without loss of time we can make the ordering of stage $\begin{Bmatrix}1\\2\end{Bmatrix}$ the same as the ordering of stage $\begin{Bmatrix}2\\1\end{Bmatrix}$ by successive interchanges, starting from the $\begin{Bmatrix}\text{left}\\\text{right}\end{Bmatrix}$ of consecutive pairs of those items which are out of order.

Next, since the orders are now the same, we may start each item as soon as possible to minimize the total time. Thus there are no delay times on the first stage.

Notation.

Let X_i be the inactive period of time for the second machine immediately before the i-th item comes onto the second machine.

If, for example, we consider the sequence S = 1, 2, 3, ···, n, we have the following time scales for each machine:



We have

$$X_1 = A_1$$

$$X_2 = \max\,(A_1 + A_2 - B_1 - X_1,\ 0)$$

$$X_1 + X_2 = \max\,(A_1 + A_2 - B_1,\ A_1)$$

$$X_3 = \max\,(\sum_1^3 A_i - \sum_1^2 B_i - \sum_1^2 X_i,\ 0)$$

$$\sum_1^3 X_i = \max\,(\sum_1^3 A_i - \sum_1^2 B_i,\ \sum_1^2 X_i)$$

$$= \max\,(\sum_1^3 A_i - \sum_1^2 B_i,\ \sum_1^2 A_i - B_1,\ A_1)\ .$$

In general,

$$\sum_1^n X_i = \max_{1 \leq u \leq n} K_u$$

where

$$K_u = \sum_{i=1}^{u} A_i - \sum_{i=1}^{u-1} B_i\ .$$

Let

$$F(S) = \max_{1 \leq u \leq n} K_u\ .$$

We want a sequence S^* such that $F(S^*) \leq F(S_o)$ for any S_o.

Solution of problem.

Consider S' the sequence formed by interchanging the j-th and j+1-st items in S. Then

$$F(S') = \max_{1 \leq u \leq n} K_u'$$

where

$$K'_u = \sum_{i=1}^{u} A'_i - \sum_{i=1}^{u} B'_i \; ; \quad A'_i = A_i \; ; \quad B'_i = B_i \qquad \text{for } i \neq j, \; j+1 \; ,$$

and

$$A'_j = A_{j+1}, \; B'_j = B_{j+1}, \; A'_{j+1} = A_j, \; B'_{j+1} = B_j \; .$$

Then

$$K'_u = K_u \quad \text{if} \quad u \neq j, \; j+1 \; .$$

Thus $F(S') = F(S)$ unless possibly if $\max (K_j, K_{j+1}) \neq \max (K'_j, K'_{j+1})$.

<u>Theorem 1</u>. An optimal ordering is given by the following rule.

Item (j) precedes item (j+1) if

$$\text{(I)} \qquad \max (K_j, K_{j+1}) < \max (K'_j, K'_{j+1}) \; .$$

If there is equality, either ordering is optimal provided it is consistent with all the definite preferences (see case 4 in Lemma 2).

By subtracting $\sum_1^{j+1} A_i - \sum_1^{j-1} B_i$ from each term in (I) it becomes

$$\max (-B_j, -A_{j+1}) < \max (-B_{j+1}, -A_j)$$

or

$$\text{(II)} \qquad \min (A_j, B_{j+1}) < \min (A_{j+1}, B_j) \; .$$

This ordering is transitive (proof follows), thus leading to a sequence S^*, unique except for some indifferent elements.

Then $F(S^*) \leq F(S_o)$ for any sequence S_o since S^* can be obtained from S_o by successive interchanges of consecutive items, according to the rule (II), and each interchange will give a value of F smaller than or the same as before.

Lemma 2.

Rule (II) is transitive.

Suppose min $(A_1, B_2) \leq$ min (A_2, B_1) and min $(A_2, B_3) \leq$ min (A_3, B_2). Then min $(A_1, B_3) \leq$ min (A_3, B_1) except possibly when item 2 is indifferent to both 1 and 3.

Proof.

Case 1. $A_1 \leq B_2, A_2, B_1$ and $A_2 \leq B_3, A_3, B_2$.

Then $A_1 \leq A_2 \leq A_3$ and $A_1 \leq B_1$ so that $A_1 \leq$ min (A_3, B_1).

Case 2. $B_2 \leq A_1, A_2, B_1$ and $B_3 \leq A_2, A_3, B_2$.

Then $B_3 \leq B_2 \leq B_1$ and $B_3 \leq A_3$ so that $B_3 \leq$ min (A_3, B_1).

Case 3. $A_1 \leq B_2, A_2, B_1$ and $B_3 \leq A_2, A_3, B_2$.

Then $A_1 \leq B_1$ and $B_3 \leq A_3$ so that min $(A_1, B_3) \leq$ min (A_3, B_1).

Case 4. $B_2 \leq A_1, A_2, B_1$ and $A_2 \leq B_3, A_3, B_2$.

Then $A_2 = B_2$ and we have item 2 indifferent to item 1 and item 3. In this case 1 may or may not precede 3 but there is no contradiction to transitivity as long as we order item 1 and item 3 first, then put item 2 anywhere.

Note.

Using (II), there is an extremely simple, practical way of ordering the items in n steps.

Working rule.

1) List the A's and B's in two vertical columns.

i	A_i	B_i
1	A_1	B_1
2	.	.
3	.	.
4	.	.
n	A_n	B_n

2) Scan all the time periods for the shortest one.

3) If it is for the first machine (i.e., an A_i), place the corresponding item first.

4) If it is for the second machine (i.e., a B_i), place the corresponding item last.

5) Cross off both times for that item.

6) Repeat the steps on the reduced set of 2n − 2 time intervals, etc. Thus we work from both ends toward the middle.

7) In case of ties, for the sake of simplicity order the item with the smallest subscript first. In case of a tie between A_i and B_i, order the item according to the A.

To illustrate the method, the following somewhat extreme example is worked out. Consider

i	A_i	B_i
1	4	5
2	4	1
3	30	4
4	6	30
5	2	3

The rule gives an optimal sequence (5, 1, 4, 3, 2). The total delay time for this sequence is 4 units, and the total elapsed time is 47 units. If one reversed the order of the items, the total time would be 78 units, the worst value possible.

§2. Three-stage production schedule.

For three different machines or stages (at most one item at a time on each machine), the problem loses some of the nice structure of the two-stage case. However, the problem is formulated, and for the special case where min $A_i \geq$ max B_i the complete solution is found analogously to the two-stage problem.

Lemma 3.

An optimal ordering can be reached if we assume the same ordering of the n items for each machine.

By Lemma 2 the orders on the first and third machines can be made the same as that of the second. Thus the lemma is proved. Note these arguments are not sufficient for a four-stage problem.

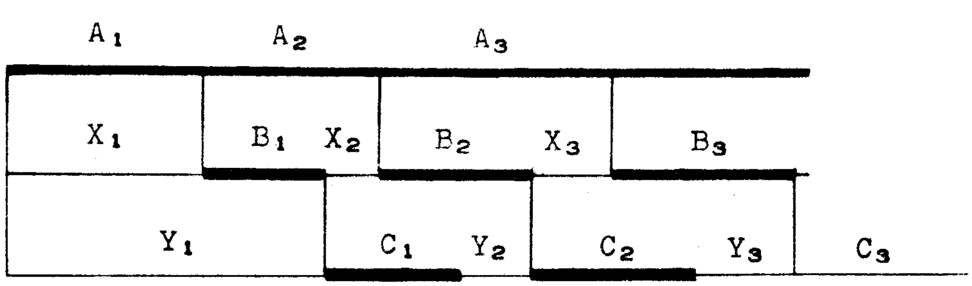
Notation.

Let A_i, B_i, X_i be defined as in the two-stage problem.

Let C_i = setup time plus work time for the i-th item on the third machine.

Let Y_i = the delay interval on the third machine immediately preceding the entry of the i-th item onto the third machine.

Consider the time scales for each machine.



We have

$$Y_1 = X_1 + B_1 = A_1 + B_1$$

$$\vdots$$

$$Y_n = \max\left(\sum^{n} B_i + \sum^{n} X_i - \sum^{n-1} C_i - \sum^{n-1} Y_i,\ 0\right)$$

so that

$$\sum^{n} Y_i = \max\left(\sum^{n} B_i - \sum^{n-1} C_i + \sum^{n} X_i,\ \sum^{n-1} Y_i\right)$$

$$= \max\left(\sum^{n} B_i - \sum^{n-1} C_i + \sum^{n} X_i,\right.$$

$$\left.\sum^{n-1} B_i - \sum^{n-2} C_i + \sum^{n-1} X_i,\ \cdots,\ B_1 + X_1\right) .$$

Let

$$H_v = \sum^{v} B_i - \sum^{v-1} C_i, \quad v = 1, 2, \cdots, n, \text{ and}$$

$$K_u = \sum^{u} A_i - \sum^{u-1} B_i, \quad u = 1, 2, \cdots, n, \text{ as before.}$$

Then

$$\sum^{n} Y_i = \max_{1 \le u \le v \le n} (H_v + \max K_u) = \max_{1 \le u \le v \le n} (H_v + K_u) .$$

As before, we interchange the j-th and j+1-st items. Then the H's and K's are unchanged except possibly those with subscripts j and j+1.

Now we compare

$$\max\left(H_{j+1} + K_u,\ 1 \le u \le j + 1;\ H_j + K_u,\ 1 \le u \le j\right)$$

with

$$\max\,(H'_{j+1} + K'_u,\ 1 \le u \le j+1;\ H'_j + K'_u,\ 1 \le u \le j)\ .$$

Notice these terms no longer involve just the subscripts j and j+1, and thus the decision is not independent of what precedes the interchanged elements.

§3. Special case where $\min A_i \ge \max B_j$.

Here $\max_{u \le v} K_u = K_v$, so that we now compare fewer terms. Our rule now states that the j-th item precedes the j+1-st item if

$$\text{(III)} \quad \max\,(H_{j+1} + K_{j+1},\ H_j + K_j) < \max\,(H'_{j+1} + K'_{j+1},\ H'_j + K'_j)\ .$$

In case of equality, we make the ordering of indifferent items consistent with the ordering given by the definite inequalities.

Then by subtracting

$$\sum^{j+1} A_i - \sum^{j-1} B_i + \sum^{j+1} B_i - \sum^{j-1} C_i$$

from both sides of (III), it becomes

$$\max\,(-B_j - C_j,\ -B_{j+1} - A_{j+1}) < \max\,(-B_{j+1} - C_{j+1},\ -B_j - A_j)$$

or

$$\text{(IV)} \quad \min\,(A_j + B_j,\ C_{j+1} + B_{j+1}) < \min\,(A_{j+1} + B_{j+1},\ C_j + B_j)\ .$$

Lemma 4.

Rule (IV) is transitive.

Proof is the same as for Lemma 2.

By the same arguments as before, we can reach an optimal sequence by successive interchanges of adjacent elements in any sequence following this rule. Thus we have

Theorem 2.

If $\min A_i \geq \max B_i$, $1 \leq i \leq n$, then an optimal three-stage production schedule is given by the following rule. Item i precedes item j if

$$\min (A_i + B_i, C_j + B_j) < \min (A_j + B_j, C_i + B_i) .$$

If equality holds, the two items are indifferent and either is permissible provided we order these items in a manner consistent with the orders given by the definite inequalities.

REFERENCES

1. Melvin E. Salveson, "On a Qualitative Method in Production Planning and Scheduling," Econometrica, Vol. 20, No. 9, October, 1952, pp. 571–574.
2. G. B. Dantzig, "Linear Model of Production," RAND Publication P-375A, February, 1953.