<u>RELATIONS CONNECTING THE DIRAC, HAMILTON-JACOBI, AND GAUGE EQUATIONS</u>

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May 1972

P-4808

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What we want to show in this paper is that classes of exact solutions of the Dirac equation can be constructed out of combinations of gauge functions and action functions. In what follows, standard 4-vector (indicated by a Greek subscript) notation is used, with a pseudo-Euclidean metric. Then the space-time 4-vector is $x_\mu \equiv (x,y,z,ict)$, with c = light velocity, and we can define a generalized 4-momentum $\overline{P}_\mu \equiv P_\mu - \frac{e}{c} A_\mu$, where P_μ is the ordinary 4-momentum ($P_4 = \frac{iE}{c}$, E = energy), e is particle charge, and A_μ is the electromagnetic 4-vector potential ($A_4 = \frac{iV}{e}$, V = electrostatic potential). With m_o = particle rest mass, we have the basic generalized momentum-energy relation:

$$\overline{P}_\mu \overline{P}_\mu \equiv \left(P_\mu - \frac{e}{c} A_\mu\right)\left(P_\mu - \frac{e}{c} A_\mu\right) \equiv -m_o^2 c^2 \ . \tag{1}$$

From Eq. (1) we can go in two directions:

(a) Put $\partial_\mu S \left(\equiv \frac{\partial S}{\partial x_\mu}\right)$ for P_μ in Eq. (1), where S is the action function (a real scalar classically), to get the relativistic Hamilton-Jacobi equation:

$$\left(\partial_\mu S - \frac{e}{c} A_\mu\right)\left(\partial_\mu S - \frac{e}{c} A_\mu\right) + m_o^2 c^2 \equiv 0 \ . \tag{2}$$

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(b) Use the Jordan-Schrödinger rule to get a relativistic quantum-mechanical wave equation. That is, put $\frac{\hbar}{i}\partial_\mu$ for P_μ ($\hbar \equiv$ Planck's constant/2π) and post-multiply by a (scalar) wave function, ψ, to get the Klein-Gordon equation:

$$\left\{\left(\frac{\hbar}{i}\partial_\mu - \frac{e}{c}A_\mu\right)\left(\frac{\hbar}{i}\partial_\mu - \frac{e}{c}A_\mu\right) + m_o^2c^2\right\} \cdot \psi \equiv 0 \ . \qquad (3)$$

As usual, Eqs. (2) and (3) must be gauge invariant, i.e., physical consequences implied by Eqs. (2) and (3) must be unchanged when we replace A_μ by $A_\mu^* \equiv A_\mu + \partial_\mu f$, where f is the gauge function (a scalar). This implies two further conditions:

$$\partial_\mu A_\mu \equiv 0 \ ; \qquad (4a)$$

$$\partial_\mu \partial_\mu f \equiv 0 \ . \qquad (4b)$$

Gauge invariance of Eq. (2) is then ensured by the rule:

(R_1) Replace S by $S^* \equiv S + \frac{e}{c} f$ when A_μ is replaced by A_μ^*.

Further, Eq. (2) is a "classical analogue" for Eq. (3); if we use rule (R_2), then Eq. (3) reduces identically to Eq. (2):

(R_2) Replace ψ by $\psi \equiv \phi \exp\left(\frac{iS}{\hbar}\right)$, with ϕ a scalar, and let $\hbar \to 0$.

Note that in quantum mechanics complex S's are permitted.

For reasons explained fully in texts on quantum mechanics, Eq. (3) is unsatisfactory as a "single-particle" wave equation. Dirac resolved some of the difficulties by "factoring" Eq. (1), in effect, introducing a parameter γ_μ, defined by: (μ,ν run from 1 to 4)

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu \equiv 2\delta_{\mu\nu} \ ; \qquad (5)$$

then,

$$\left[\gamma_\mu\left(P_\mu - \frac{e}{c}A_\mu\right) + im_oc\right]\left[\gamma_\nu\left(P_\nu - \frac{e}{c}A_\nu\right) - im_oc\right] \equiv 0 \; ; \tag{1a}$$

and so:

$$\gamma_\mu\left(P_\mu - \frac{e}{c}A_\mu\right) \pm im_oc \equiv 0 \; . \tag{1b}$$

Then the "factored" form of Eq. (3):

$$\left\{\gamma_\mu\left(\frac{\hbar}{i}\partial_\mu - \frac{e}{c}A_\mu\right) - im_oc\right\} \cdot \psi \equiv 0 \tag{6}$$

is the standard Dirac equation[1] derivable by applying the Jordan-Schrödinger rule to either form of Eq. (1b) [because Eq. (5) defines γ_μ only up to a $\pm$ sign]. Also, we are at liberty to interpret ψ in Eq. (6) as a scalar function of γ_μ and x_μ, i.e., $\psi = \psi(\gamma_\mu, x_\mu)$, provided we also regard γ_μ as a 4-vector [that such an interpretation is possible, and produces the same physical consequences as the usual interpretation which regards ψ as a bi-spinor instead of a scalar, is fully discussed in several references[2]].

From Eq. (1b) we get the analogous linear Hamilton-Jacobi equation by the process noted in (a) above:

$$\gamma_\mu\left(\partial_\mu S - \frac{e}{c}A_\mu\right) - im_oc \equiv 0 \; . \tag{2a}$$

Equations (2) and (2a) have some common solutions for S which are γ_μ-free, as the notation implies.

For a first connecting relation, note that Eq. (4b) can be written as $(\gamma_\mu\partial_\mu)(\gamma_\nu\partial_\nu f) \equiv 0$. Then the equation from such a factorization:

$$\gamma_\nu\partial_\nu f \equiv 0 \tag{7}$$

has the characteristic that any solution of Eq. (7) is also a solution of Eq. (4b) -- the converse is not true -- so that the solutions -- call them f_o -- of Eq. (7) are a subset of the entire set -- call them f -- of the allowable gauge functions f which satisfy Eq. (4b). Suppose we now substitute $\psi \equiv \phi \exp\left(\frac{iB}{\hbar}\right)$ into Eq. (6), where B must be a γ_μ-free function[3] and ϕ must be further specified. The result can be written:

$$\frac{\hbar}{i}\gamma_\mu \partial_\mu \phi + \left[\gamma_\mu \partial_\mu B - \frac{e}{c}\gamma_\mu A_\mu - im_o c\right] \cdot \phi \equiv 0 \ . \qquad (8)$$

Contrary to the case of rule (R_2), Eq. (8) is satisfied identically for every value of $\hbar$ if we choose $B \equiv S$ as a solution of Eqs. (2a) and (2), and ϕ satisfies Eq. (7). Further, ϕ can be written as a product of two functions: f_o, which again satisfies Eq. (7), and $N(\gamma_\mu, \overline{P}_\mu)$, in which the specified explicit argument dependence is on only the γ_μ and the $\overline{P}_\mu$ of Eq. (1). N is a disposable function (see the following paragraphs) allowing ψ to satisfy explicit normalization, boundary, etc., conditions.

The relation $\psi \equiv f_o \cdot N \cdot \exp\left(\frac{iS}{\hbar}\right)$ then connects the wave function ψ, the gauge function f_o, and the action function S in an <u>exact</u> solution of Eq. (6) for nonzero values of $\hbar$ (the argument does not show every exact solution to be of this form).[4]

We could now construct exact solutions of Eq. (6) by combining solutions of Eq. (7) for f_o and Eq. (2a) for S. The simplest standard case of such an exact solution, not requiring detailed computation, is the stationary motion in one dimension ($x_1 = x$) of a particle in a one-dimensional electrostatic potential $A_4 = \frac{iV(x)}{e}$, which we will later specialize further by putting $V(x) = V_c$ = constant; $V_c \equiv 0$ is the free-particle case. Then Eq. (1) becomes (with $P_1 = P_x$, $P_4 = \frac{iE}{c}$, $A_4 = \frac{iV(x)}{e}$, and P_2, P_3, A_1, A_2, A_3 all zero):

$$P_x^2 + \left(\frac{iE}{c} - \frac{iV(x)}{c}\right)^2 \equiv -m_o^2c^2 \ ,$$

so that

$$P_x \equiv \left\{\left(\frac{E}{c} - \frac{V(x)}{c}\right)^2 - m_o^2c^2\right\}^{\frac{1}{2}} \equiv \frac{\partial S}{\partial x} \ ;$$

note that, depending on $V(x)$, P_x can be imaginary. With $P_4 = \frac{iE}{c} = \frac{\partial S}{\partial x_4} = \frac{\partial S}{\partial(ict)}$, the t-dependence of S is just a term $-Et$. Hence the complete action S appearing in $\psi \equiv f_o \cdot N \cdot \exp\left(\frac{iS}{\hbar}\right)$ is given by:

$$S \equiv \int\left\{\left(\frac{E}{c} - \frac{V(x)}{c}\right)^2 - m_o^2c^2\right\}^{\frac{1}{2}} dx - Et \ ,$$

which, further specialized to the case $V(x) = V_c$ = constant becomes simply:

$$S = \left\{\left(\frac{E}{c} - \frac{V_c}{c}\right)^2 - m_o^2c^2\right\}^{\frac{1}{2}} \cdot x - Et \ .$$

In this one-dimensional case the Dirac equation (6) becomes:

$$\frac{\hbar}{i}\gamma_1\frac{\partial\psi}{\partial x} + \frac{\hbar}{i}\gamma_4\frac{\partial\psi}{\partial(ict)} - \gamma_4\frac{iV_c}{c}\psi - im_oc\psi \equiv 0 \ . \tag{6'}$$

With the above S in the wave function ψ and with the stated characteristics of f_o, N, Eq. (6') gives:

$$\left[\gamma_1\{\}^{\frac{1}{2}} + \gamma_4\left(\frac{iE}{c} - \frac{iV_c}{c}\right) - im_oc\right] \cdot (f_o \cdot N) \equiv 0 \ , \tag{1b'}$$

which indeed vanishes by virtue of Eq. (1b) for every value of $(f_o \cdot N)$, the amplitude of the wave function ψ. To further specify this amplitude, we note that because of Eqs. (1) and (1a), Eq. (1b') is always satisfied very generally by putting

$$(f_o \cdot N) \equiv \left[\gamma_1\{\}^{\frac{1}{2}} + \gamma_4\left(\frac{iE}{c} - \frac{iV_c}{c}\right) + im_oc\right] \cdot F \ ,$$

where the proportionality factor F is still essentially disposable but satisfies Eq. (7); F must post-multiply because it can still contain γ_μ.

At this level of specificity, we can now treat, in the following way, the simplest case of the Klein paradox as the magnitude of V_c varies, where we suppose, for a wave initially propagating to the right from $-\infty$ at energy E, $V_c \equiv 0$ in the left half of the x space $(x < 0)$, and V_c = constant > 0 in the right half $(x > 0)$, with an infinitely steep jump in the potential at $x = 0$. Then ψ_{left} can be considered as an incoming wave $(\ell \to r)$ propagating to the right and a reflected wave $(\ell \to \ell)$ propagating to the left, while ψ_{right} is a wave $(r \to r)$ propagating only to the right. The respective amplitudes from the above relationship are then:

$$(f_o \cdot N)_{\ell\to r} = \left[\gamma_1 P_{x_\ell} + \gamma_4 \frac{iE}{c} + im_oc\right] \cdot F_{\ell\to r} \ ,$$

$$(f_o \cdot N)_{\ell\to\ell} = \left[-\gamma_1 P_{x_\ell} + \gamma_4 \frac{iE}{c} + im_oc\right] \cdot F_{\ell\to\ell} \ ,$$

$$(f_o \cdot N)_{r\to r} = \left[\gamma_1 P_{x_r} + \gamma_4\left(\frac{iE}{c} - \frac{iV_c}{c}\right) + im_oc\right] \cdot F_{r\to r} \ :$$

The boundary condition at $x = 0$ is:

$$(f_o \cdot N)_{\ell\to r} + (f_o \cdot N)_{\ell\to\ell} = (f_o \cdot N)_{r\to r} \;.$$

This condition, along with details of the γ_μ algebra, gives the ratios of these amplitudes in a standard way as V_c varies,[5] and permits the Klein paradox to be analytically discussed.

Finally, coupling together the classical rule (R_1) and rule (R_2) shows immediately how ψ has to be transformed when we make a gauge transformation in Eqs. (3) and (6); the rule is of course: (note f is assumed γ_μ-free here)

(R_3) Replace ψ by $\psi^* \equiv \psi \cdot \exp\left(\frac{ie}{\hbar c} f\right)$ if we replace A_μ by A_μ^*.

There are naturally deeper reasons why such intimate connections exist between relativistic wave mechanics and "classical" relativistic mechanics.

FOOTNOTES, REFERENCES

(1) P.A.M. Dirac, *Proc. Roy. Soc.* A 117, 610 (1928). Naturally, as a "single particle" theory, Eq. (6) gives a good description only for such phenomena in which effects of creation or annihilation of real or virtual particles are not very important.

(2) Arnold Sommerfeld, *Atombau und Spektrallinien,* II Band (Chapter IV), F. Vieweg & Sohn, Braunschweig, 1960; R. Feynman, *Quantum Electrodynamics,* Lectures 9 through 14, W. A. Benjamin, Inc., New York, 1962.

(3) If B is not γ_μ-free we cannot assume we get a cancellable exponential factor from the substitution.

(4) An equation of the form of Eq. (8) provides a basis for a general series solution of the Dirac equation. W. Pauli, *Helv. Phys. Acta,* Band 5, 1932.

(5) We can assume each F to be written in the form $F_{\ell\to r} = C_{\ell\to r} \cdot (1+\gamma_4)$, etc., for example, with $C_{\ell\to r}$, etc. an ordinary numerical constant and where the factor $(1+\gamma_4)$ occurs in each F [such F's clearly are just the simplest kind of solutions of Eq. (7), namely constants]. Then, because $\gamma_4(1+\gamma_4) \equiv 1 + \gamma_4$, the boundary condition gives a relation of the form: [Eq. (1)] $\cdot\ \gamma_1(1+\gamma_4)$ + [Eq. (2)] $\cdot\ (1+\gamma_4) \equiv 0$. The brackets [Eq. (1)], [Eq. (2)] are then separately equated to zero to get the desired relations in the C's.

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