ON SOLVING OPTIMIZATION PROBLEMS SUBJECT TO A BUDGET CONSTRAINT WITH ECONOMIES OF SCALE

by

R. J. Hillestad

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The Rand Corporation
Santa Monica, California 90406

SUMMARY

This paper describes a finite procedure for locating a global minimum of a problem which is linear in the objective and constraints except for one nonlinear constraint which is of the "reverse convex" variety. That is, the direction of the inequality is the opposite of that required for a convex constraint. Budget constraints in which the cost functions are subject to economies of scale are typically of this form. An illustrative example of the procedure is provided.

1. INTRODUCTION

This paper describes a finite procedure for solving the problem below, denoted C,

$$\text{(C)} \qquad \min\ cx \quad \text{subject to:} \quad F\begin{cases} Ax \leqq b \\ g(x) \leqq 0 \end{cases}$$

where $C = (C_1, C_2, \ldots, C_n)$; $b = (b_1, b_2, \ldots, b_m)^T$; $A = (a_{ij})$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ (including any nonnegativity constraints) and where g is a differentiable real concave function of $x \in R^n$ and F is assumed to be bounded. Note that concavity of g is the opposite of that required for the nonlinear constraint to produce a convex feasible region. This type of constraint arises frequently in dealing with budget constraints and economies of scale.

It is well known that an optimal solution of a related problem, minimization of a concave function on a bounded polyhedral set, can be obtained by enumeration of vertices. (In fact, with suitable bounds on the objective, the concave objective problem can be put in this form. Let B be a lower bound on $\min_x(f(x) \mid Ax \leqq b)$ where f satisfies the conditions on g above. Then an equivalent bounded problem of form C is, $\min_{x,y}\{y \mid Ax \leqq b, -y \leqq -B, f(x) - y \leqq 0\}$.) Rosen (10) and Meyer (6) show that iterative linearization of problem C will find a stationary point of the Lagrangian when each set of linearized constraints contains an interior. However, such points may be far from the global minimum. Bansal and Jacobsen (1) have investigated a network capacity expansion problem in which expansion is subject to budget limitation and economies of scale. They characterized the local solutions in terms of "basic local solutions" and developed a finite algorithm which depends on the network structure and separability of the single concave cost function to find the global optimum. The branch and bound algorithm of Soland (11) is applicable to this problem when the concave function is separable, but is an infinitely convergent procedure. It is shown in (4) that the more general class of problems permitting multiple reverse convex constraints has a global minimum in at least one of a finite set of solutions of convex subproblems. This paper shows that the optimal solution of C can be obtained by performing a finite number of one dimensional searches. A finite algorithm for obtaining the global minimum is provided and an illustrative example with a tabled summary of iterations is given.

2. LOCAL SOLUTIONS AND EXTREME POINTS

A global solution of C is obviously contained in the set of all local solutions of C. We show now that only a finite number of the local solutions need be considered and as a consequence of the theorems, how to find these solutions.

THEOREM 1. *Let* $\bar{x}$ *be a local solution of* C. *Then* $\bar{x}$ *must solve*

$$\begin{aligned} &\min cx \\ &Ax \leqq b \\ &g(\bar{x}) + \nabla g(\bar{x})(x - \bar{x}) \leqq 0 \end{aligned} \tag{C1}$$

(Note that the linearized feasible region, because of concavity of g, is always contained in the feasible region for C.)

Proof

Problem C satisfies the Arrow-Hurwicz constraint qualification (5) and therefore the Kuhn-Tucker necessary optimality conditions apply at $\bar{x}$. These are that $\bar{\pi}$, $\bar{\gamma}$ exist and $(\bar{\pi}, \bar{\gamma}, \bar{x})$ solves

$$\begin{aligned} C + \pi A + \gamma \nabla g(x) &= 0 \\ \pi(Ax - b) &= 0 \\ \gamma g(x) &= 0 \end{aligned}$$

THEOREM 2. *Each local solution* x *of* C *has a corresponding point* z *such that* $c\bar{x} = cz$ *and* z *is a local extreme point or there exists* y *such that* $cy < c\bar{x}$.

Proof

If $g(\bar{x}) < 0$ for all local solutions then we are done since the result is obviously true for locals constrained by linear constraints.

Suppose $g(\bar{x}) = 0$. By Theorem 1, $\bar{x}$ must solve the linearized problem with constraint $g(\bar{x}) + \nabla g(\bar{x})(x - \bar{x}) \leqq 0$ and this constraint is tight at $\bar{x}$. For the linearized problem there must exist z^1 such that $cz^1 = c\bar{x}$ and z^1 is an extreme point of this linear problem. By concavity of g and the fact that z^1 is feasible for the linear problem, z^1 must be feasible for C. Suppose that $g(z^1) < 0$. Then, in the linearized problem

$$\begin{aligned} &\min cx \\ &\quad Ax \leqq b \\ &g(z^1) + \nabla g(z^1)(x - z^1) \leqq 0 \end{aligned}$$

the last constraint is not tight at z^1 and since this feasible region is contained by that of C, there must either exist a direction of improvement in C such that there exists y, $cy < c\bar{x}$, or the feasible directions are constrained by another independent linear constraint at z^1 in which case z^1 is a local extreme point of C.

Suppose that $g(\overline{x}) = 0$ and $g(z^1) = 0$. Then it is trivially true that the constraint derived by linearizing g at z^1 is tight at z^1, i.e.,

$$g(z^1) + \nabla g(z^1)(x - z^1) = 0 \quad \text{for } x = z^1. \tag{1}$$

Again it must be shown that either there is a direction of improvement or z^1 is a local extreme point solution of C. If the constraint (1) is linearly independent of the other n - 1 active, linearly independent constraints of $Ax \leqq b$, (an extreme point in R^n is defined by n linearly independent, active linear constraints), then z^1 is an extreme point. Of course this is also true if there are n independent constraints of $Ax \leqq b$ active at z^1, in which case it is either a local extreme point or there is a direction of improvement from z^1. If (1) is not linearly independent, and there are only n - 1 active independent constraints, then the direction or edge defined by the n - 1 independent constraints can be used to find an extreme point or direction of improvement. Let the direction of the edge be d and move such that $c(z^1 + \lambda d) \leqq 0$ for $\lambda > 0$. Since the feasible region is bounded, either $\overline{\lambda}$ such that $c(z^1 + \overline{\lambda} d) < 0$ must be found or a linearly independent constraint of $Ax \leqq b$ becomes tight for some $\overline{\lambda}$ and $z^2 = z^1 + \overline{\lambda} d$ is a local extreme point such that $cz^2 = c\overline{x}$.

The following theorem shows that this class of local solutions is finite and suggests a method of finding each element of the class.

THEOREM 3. The number of local extreme points of C is finite.

Proof

Let $\bar{z}$ be a local extreme point of C. Since by definition $\bar{z}$ is an extreme point of

$$Ax \leqq b$$
$$g(\bar{z}) + \nabla g(\bar{z})(x - \bar{z}) \leqq 0$$

there must be at least n - 1 linearly independent constraints of $Ax \leqq b$ which are active at $\bar{z}$. Now, n - 1 active, linearly independent constraints define a one dimensional surface or edge in R^n. Since there are at most $\binom{m}{n-1}$ edges selectable from $Ax \leqq b$ we need only show that there is a finite number of local extreme points associated with each edge. This is obviously true for those local extreme points such that $g(z) < 0$ since the n^{th} tight constraint must be linear. Consider those local extreme points such that $g(z) = 0$. By concavity of g if there exist more than two points on an edge (of $Ax \leqq b$) such that $g(x) = 0$ then the support of g at all points on a nonzero interval must be colinear with that edge. Linearization of g cannot then serve as the n^{th}

linearly independent equation defining a local extreme point on that edge.

<u>COROLLARY</u>. <u>Any edge of</u> $Ax \leqq b$ <u>can yield at most one local extreme point defined solely by that edge and</u> $g(x) = 0$ <u>if that edge, which lies on the line</u> $\bar{x} \pm \lambda d$, <u>is not such that</u> $cd = 0$.

<u>Proof</u>

The fact that there can be no more than two local extreme points was established in the previous theorem. Suppose there are two. Call these z^1 and z^2 and let $cz^2 < cz^1$. (If $cz^2 = cz^1$ then $d = z^2 - z^1$ and $cd = 0$.) Consider the line defined by $x = z^2 + \lambda(z^2 - z^1)$. It is true that for $\lambda > 0$, $cx < cz^2$. It is also true for $\lambda > 0$ that $g(z^2 +$ [redacted] $\lambda(z^2 - z^1)) \leqq 0$. To see this let $\lambda = \bar{\lambda} > 0$ and $x = z^2 + \bar{\lambda}(z^2 - z^1)$. Hence, $z^2 = x/(1 + \bar{\lambda}) + \bar{\lambda}/(1 + \bar{\lambda})z^1$ and defining $\bar{\gamma} = \bar{\lambda}/(1 + \bar{\lambda})$ we have $z^2 = (1 - \bar{\gamma})x + \bar{\gamma}z^1$ or z^2 expressed as a convex combination of x and z^1. Now $g(z^2) = 0 = g((1 - \bar{\gamma})x + \bar{\gamma}z^1) \geqq (1 - \bar{\gamma})g(x) + \bar{\gamma}g(z^1)$ and therefore,

$$g(x) \leqq \frac{\bar{\gamma}}{(1 - \bar{\gamma})} g(z^1) = 0.$$

Therefore, since

$$c(z^2 + \lambda(z^2 - z^1)) < cz^2 \qquad \text{for } \lambda > 0$$

and

$$g(z^2 + \lambda(z^2 - z^1)) \leqq 0 \qquad \text{for } \lambda > 0$$

there exists a feasible direction of improvement on the edge and z^2 cannot be an extreme point.

These theorems show that a local extreme point corresponding to every possible local solution can be found by locating the linear problem solution ignoring the $g(x) \leqq 0$ constraints (to locate any local extreme point corresponding to $g(x) < 0$) and also searching all edges of $Ax \leqq b$ for intersections with $g(x) = 0$. The edge searches are relatively simple one dimensional searches to which Newton's Method or some other procedure can be applied. The biggest difficulty is the possibly large number of edges which must be searched. Without further information it is necessary to perform $\binom{m}{n-1}$ one dimensional searches. Note however, that only those edges with endpoints x^1, x^2 such that $g(x^1) < 0$, $g(x^2) > 0$ and $c(x^2 - x^1) < 0$ need be searched. And, given that the current best solution is $\bar{x}$ it is only necessary to consider those edges such that $cx^2 < c\bar{x}$.

The following search procedure is suggested for solving problem C. Let S be a set of vertices of $Ax \leqq b$. Initially set $S = \{\emptyset\}$. Let UB be an upper bound on the objective at any stage of the procedure and let $UB = \infty$ initially.

(1) Solve $\min \{cx | Ax \leqq b\}$ and obtain x^1. If $g(x^1) \leqq 0$ stop, x^1 is optimal. Otherwise set $S = \{\emptyset\} \cup \{x^1\}$ and go to step 2.

(2) Select an element from S, say x^i, and go to step 3.
[redacted] If $S = \emptyset$ stop, the current best solution is optimal.

(3) If $cx^i \geqq UB$ remove x^i from S and go to step 2. Otherwise go to step 4.

(4) Let N^i initially be the set of neighboring extreme points of x^i. Obtain an element of N^i, denote it by x^j, and go to step 5. If $N^i = \emptyset$ remove x^i from S and go to step 2.

(5) If $cx^j < cx^i$ or if $x^j \in S$ remove x^j from N^i and go to step 4. If $cx^j - cx^i \geq 0$ and $g(x^j) > 0$ place x^j in S (if $cx^j < UB$), remove x^j from N^i and go to step 4. (It is necessary to avoid allowing an extreme point being placed back into S after it has been previously removed from S. This could happen here if $cx^j = cx^i$.) Otherwise we have that $g(x^i) > 0$, $g(x^j) \leqq 0$, and $c(x^j - x^i) \geqq 0$ and the edge must be searched. Find $\bar{\lambda}$ which solves,

$$\bar{\lambda} = \min \lambda$$
$$g(\lambda x^j + (1 - \lambda)x^i) = 0$$
$$\lambda \in [0, 1]$$

and let $x^{\lambda} = \bar{\lambda}x^{j} + (1 - \bar{\lambda})x^{i}$. If $cx^{\lambda} < UB$ let $UB = cx^{\lambda}$ and store x^{λ} as the current best solution. In either case, remove x^{j} from N^{i} and go to step 4.

This procedure searches the edges of $Ax \leqq b$ in an upward direction with respect to the objective. By bounding and partitioning the extreme points defining those edges it generally allows termination prior to complete enumeration. When the set S is empty the edges from all extreme points such that $g(x) > 0$ have been considered. When all points x^{k} in S are such that $cx^{k} \geqq UB$ there is no value in searching additional edges. Note that this algorithm uses the fact that it is possible, when considering any neighbor x^{j} of x^{i}, to eliminate from consideration those neighbors such that $c(x^{j} - x^{i}) < 0$ since x^{j} must be found on at least one other path from x^{k} such that $c(x^{j} - x^{k}) \geqq 0$ from the original linear program solution. If this wasn't true then x^{j} would be a strong local solution of the linear problem involving only the original linear constraints, implying that x^{1} was not optimal.

3. AN EXAMPLE

Consider the following problem of form C.

$$\begin{aligned}
\min\ & -3x_1 - 4x_2 - x_3 - 7x_4 \\
\text{s.t.}\ & 8x_1 + 3x_2 + 4x_3 + x_4 \leqq 7 \\
& 2x_1 + 6x_2 + x_3 + 5x_4 \leqq 3 \\
& x_1 + 4x_2 + 5x_3 + 2x_4 \leqq 8
\end{aligned}$$

$$g(x) = 40(x_1-x_1^2) + 160(x_2-x_2^2) + 10(x_3-x_3^2) + 160(x_4-x_4^2) - 10 \leqq 0$$

$$x_1,\ x_2,\ x_3,\ x_4 \geqq 0$$

The optimal LP solution, ignoring the reverse convex constraint, is $\bar{x} = (.84, 0, 0, .26)$ with objective $c\bar{x} = -4.3684$. At this solution $g(\bar{x}) = 26$ so a feasible solution must be found. The sequence of iterations of the procedure of the previous section is given in Table 1.

Table 1

Extreme Points in S	Extreme Points Being Considered (x^i)	Neighbor (x^j)	$g(x^j)$	cx^i	Status of x^j or Intersection if Edge Search Required	$cx^j - cx^i$
1	1		26.	-4.37		
1	1	2	28.4		Add to S	.168
1,2	1	3	25.8		Add to S	1.06
1,2,3	1	4	17.81		Add to S	.737
1,2,3,4	1	5	-5.63		Intersection (x = .87, 0, 0, .036) cx = 02.86 UB = -2.86	1.74
2,3,4	2		28.4	-4.2		
2,3,4	2	1	26.		$cx^j - cx^i < 0$	-.168
2,3,4	2	6	30.		$cx^j > UB$	2.2
2,3,4	2	7	16.8		Add to S	.59
2,3,4,7	2	8	-10.		Intersection (x = 0, 0, 0, .067) cx = -.4689	4.2
3,4,7	3		25.8	-3.3		
3,4,7	3	1	26.		$cx^j - cx^i < 0$	-1.06
3,4,7	3	6	30.		$cx^j > UB$	1.31
3,4,7	3	9	16.8		$cx^j > UB$	.69
3,4,7	3	5	-5.63		Intersection (x = .86, .034, 0, 0) cx = -2.72	.685
4,7	4		17.81	-3.63		
4,7	4	1	26.		$cx^j - cx^i < 0$	-.737
4,7	4	7	16.8		Already in S	.0229
4,7	4	9	16.85		$cx^j > UB$	1.02
4,7	4	10	-16.1		Intersection (x = .09, 0, 1.53, .1) cx = -2.55	1.79
7	7		16.8	-3.61		
7	7	2	28.4		$cx^j - cx^i < 0$	-.59
7	7	4	17.8		$cx^j - cx^i < 0$	-.0229
7	7	11	16.2		$cx^j > UB$	1.14
7	7	12	-19.6		Intersection (x = 0, 0, 1.55, .13) cx = -2.49	2.0
∅						

The extreme points corresponding to numbers 1-12 are:

	x_1	x_2	x_3	x_4	cx	g(x)
1	.842	0.	0.	.263	-4.37	26.
2	0.	0.	0.	.6	-4.2	28.4
3	.786	.238	0.	0.	-3.37	29.8
4	.105	0.	1.47	0.26	-3.63	17.81
5	.875	0.	0.	0.	-2.63	-5.63
6	0.	.5	0.	0.	-2.0	30.
7	0.	0.	1.47	.304	-3.61	16.8
8	0.	0.	0.	0.	0	-10.
9	.09	.238	1.39	0.	-2.61	16.8
10	.08	0.	1.58	0.	-1.84	-16.1
11	0.	.27	1.38	0.	-2.47	16.2
12	0.	0.	1.6	0.	-1.61	-19.6

At termination when $S = \emptyset$, the smallest intersection already found is the optimal solution. Hence, $c\bar{x} = -2.86$ with $\bar{x} = (.87, 0, 0, .036)$.

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