LOTTERIES AND MENUS: A COMMENT ON UNBOUNDED UTILITIES

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This note, like [16] and [2] before it, busies itself with ramifications of the *St. Petersburg paradox*--that durable and resilient springboard for philosophical speculations about money, risk, utility, and rationality. To some extent we are rebutting [2] and defending [16], but our main purpose here is to present more fully some ideas about unbounded utility that were only briefly mentioned in [16] and that are clearly relevant to the issues considered in both [16] and [2].

In retrospect,* we can hardly claim that [16] undertakes to "resolve the whole paradox," as [2] suggests. But [16] does undertake to show that Nicholas Bernoulli's famous lottery is not very well suited to carry out the one task most commonly assigned to it--that of making a strictly risk-neutral attitude toward money look absurd.** In brief, we argued in [16] first that the apparent absurdity is empirical, rather than logical or mathematical, and second that the Bernoulli lottery, taken literally, involves such unbelievable sums of money that one cannot possibly claim an *empirical* basis for concluding that the gambler's decisions based on expected monetary return are counterintuitive or irrational.*** Finally, by some numerical illustrations we

*Actually, [16] was written several years ago and circulated as a Rand paper; it is now published with only minor revisions.

**Thus, Kim opens with the statement: "The lesson from the St. Petersburg paradox ... has been to reject the assumption of linear utility of income or wealth (and, therefore, the mathematical expectations approach) ..." ([10], p. 148).

***We were then unaware of Fry's essay ([5], pp. 194-199), which makes substantially the same argument.

reminded the reader that if the lottery is truncated, so that only credible sums of money are promised, then decisions based on expected-value calculations are not at all inconsistent with our ordinary intuition and experience as to how "rational" people behave.*

Professor Aumann challenges this argument only obliquely in [2], by raising the prior question of whether unbounded utilities--for money or anything else--can ever be entertained in this context without creating insurmountable difficulties either in logic or in interpretation. This question indeed threatens to foreclose the whole thesis of [16], by denying the very terms of the Bernoulli lottery. In fact, Aumann insists that unboundedness is the "real point" of the St. Petersburg paradox, and that I (and many others) have missed it.

The unboundedness issue was indeed slighted in [16], and this may have led Aumann in [2] to read between the lines and extract what he seemed to think I seemed to think, and call it wrong. There was one remark on the subject in [16], however, in which I claimed that unbounded utility is *not* the real issue in the Bernoulli lottery and that, in any case, it can be confronted more directly with another, much simpler experiment called "Blank Check." In the present note I shall attempt first to justify the first half of this claim--in a way that may provide common ground on which both [16] and [2] can comfortably stand. Then, in a more constructive spirit, I shall attempt to systematize the point about "Blank Check" and show how the whole

*This observation goes all the way back to Gabriel Cramer, writing *before* D. Bernoulli's "St. Petersburg" paper. See also Fellner's discussion ([4], p. 106).

anti-unbounded-utility position can be made independent of gambling behavior and, indeed, of cardinal utility itself.

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For definiteness we recapitulate the Bernoulli lottery, using the "utility" form now since money does not concern us at the moment. Let a rational individual be given, equipped with a real-valued utility function u and facing a countable infinity of mutually exclusive prospects $\{x_n: n = 1, 2, \ldots\}$, the utilities of which have no upper bound. By subsequencing and relabelling we may assume without loss of generality that $u(x_{n+1}) \geqq 2u(x_n) > 0$ for each n. We can then define a lottery--i.e., a random variable, by assigning to each x_n the probability 2^{-n}. This lottery of course has an infinite expected utility, and so the fun begins.

For clarity, we ought to distinguish two rather different "paradoxes" that can arise. As already noted, the first one (I think I am justified in calling it the *classical paradox*) depends on a sharp contrast between what the expected-value theory of rational behavior predicts and what our real-life experience or introspection predicts.*

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*We keep returning to Menger's point about empirical rationality (see [12]) because it is so easily forgotten when the discussion warms up. Yet, a conscientious presentation of the classical paradox will almost always have this essential ingredient buried somewhere in its logic or language, whether or not the author faces up to the implications. Some examples:

> "No *sane man* would ever consider paying the bank one hundred dollars for such a chance." Fry ([5], p. 195).
>
> "... so an *individual should be willing* to pay any finite sum ... A little thought will tell us that *this is not true*; hence the paradox." Brito ([3], p. 123).
>
> "... he 'should' be willing to pay any finite stake. Introspection tells us that *he will not*, and *this 'fact'* will have to be explained." Arrow ([1], p. 407).

The second paradox, however, is not empirical at all but is imbedded in the logic of the theory of cardinal utility under risk. As this theory is usually formulated,* *a utility function* is given (or derived) that maps a space of mutually exclusive *prospects* into a space of *utilities*, the latter represented as real numbers. The prospect space is assumed to be closed under the operation of forming lotteries, with compound or multistage lotteries being reduced according to the usual rules for combining probabilities. Moreover, the utility function is assumed (or is derived) to be *risk neutral*, so that the utility of any lottery is equal to the expected value of the utility of its outcome. Other assumptions or conditions may also be imposed on the utility function, but one does *not* usually require its range to be bounded, *a priori*. On the other hand, the prospect space is usually *not* required to be closed under probability mixes over infinite sets. Indeed, such infinite lotteries would at once force the range to be bounded, since otherwise there would be a "Bernoulli" lottery, as above, whose utility is not a real number.** Since it is difficult on intuitive or even on practical grounds to rule out infinite lotteries, it seems advisable

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> "Some introspection, however, should convince anyone that *no player would pay very much* ... to play this game. If I charged $1 billion ... *I would surely have no takers*." Nicholson ([13], p. 147).
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> "...intuitively obvious that no *reasonable person* would prefer ... this game ... to the outright receipt of any appreciable amount of money. *Search your mind, reader, and I am sure you will agree!*" Fellner ([4], p. 102).

(Italics added.) See also the authors we quoted in [16].

*For example, see Marschak [11] or Herstein and Milnor [7].

**The extended real line $R \cup \{\infty, -\infty\}$ offers no escape, as the indispensible continuity or "Archimedean" axiom would have to be sacrificed.

to rule out unbounded cardinal utilities. Logically we cannot have both.*

One may well ask whether this purely logical argument, resting directly on the "utility" form of Bernoulli's lottery, might not be so devastating that it preempts the classical, empirical paradox. Does it not effectively deny the possibility of the proposed lottery, with its presumed unbounded payoffs, and so leave us nothing empirical to talk about? The answer is no, not really. There is a simple recourse: finitize the lottery by imposing a "house limit" or a stop rule, while taking care to set the limit so very, very high that the expected value of the lottery, though finite, is outrageously large. Then all the usual classical arguments can still be applied.** Technical, mathematical boundedness is restored by this device, but the force of the empirical paradox, such as it is, remains unabated.

It therefore appears that to dispose of the classical paradox requires not only that utilities be bounded (which is a mathematical statement), but that they somehow be "reasonably bounded" or "not unbelievably large" (which is an empirical statement).*** If Aumann's

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*See e.g., Hirshleifer ([8], p. 228), Isbell ([9], p. 360), or Savage ([15], p. 95); the latter, however, seems more concerned with "reasonable boundedness" (see below) than with strict mathematical boundedness.

**In [16] we could modify Step (II) to read "... to toss coins with him 10^{100} times or until he fails to win ...," and Step (III) to read "... that his utility for this game is very large, and in particular that it exceeds $1000."

***Turning from the *existence* of a bound to its *magnitude* raises the intriguing side question: large compared to what? Both the zero and the unit of cardinal utility are arbitrary, so how can we express the limits of "reasonableness" in an intrinsic or invariant way?

For one answer, we could take the utility of some fixed prospect (the status quo?) plus a suitably large multiple of the typical or "average" utility-difference met with in everyday decisionmaking. Another final paragraph in [2] is taken in this sense of "reasonable boundedness" (as I believe it must), rather than strict mathematical boundedness, then I quite agree with his remarks as far as they go. My further point in [16], however, is that unreasonably large amounts of *money* must also be disallowed if our everyday experience is to be brought to bear in deciding whether someone's rational behavior is consistent with a linear utility for wealth or income.*

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In order to motivate the rest of this note, consider the following situation. We start with a wealthy sponsor and a rational subject (as in the "money" form of the Bernoulli lottery). The sponsor writes out a check to the subject and signs it, but leaves the amount of money blank. In return for a specified fee or other commitment, he hands over the check and the subject is now free to fill in any amount he pleases and go to the bank to cash the check. (The First Rational Bank of St. Petersburg, perhaps?) We can now ask the obvious, if faintly

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method might be to follow Aumann in [2] and use as a measuring rod the difference between "a long, happy and useful life" and "death"; we would certainly agree that 10^{100} such units is unreasonably large. Yet another method would be to borrow a leaf from psychometry and postulate not only a "least noticeable difference," but also a "greatest conceivable difference" in utility; the ratio between the two would be independent of scale factors and zero points and might be regarded as an observable personal or socio-cultural constant.

**In principle, however (as Reinhard Selten has pointed out to us), there should be no reason to exclude empirically unbelievable prospects from at least hypothetical consideration.

"I can't believe *that*," said Alice.

"Can't you?" the Queen said, in a pitying tone. "Try again: draw a long breath, and shut your eyes."

Alice laughed, "There's no use trying," she said: "one *can't* believe impossible things."

"I daresay you haven't had much practice," said the Queen. "When I was your age, I always did it for half-an-hour a day. Why, sometimes I've believed as many as six impossible things before breakfast!"

ridiculous question: What is the blank check worth? What is a reasonable fee, or other consideration?

Serious questions of credibility arise, of course, similar to those discussed in [16]; in this case they are even more starkly exposed. But leaving them aside, we can hardly escape the conclusion that if the utility of money is not assumed to be bounded, then the utility of the blank check is infinite. In other words, this "game," like Bernoulli's, is a way of manufacturing a prospect of infinite utility out of a set of prospects of finite, but unbounded, utility. The intriguing feature of this model, however, is that we can now argue against unbounded *ordinal* utilities, as no cardinal ideas are involved. Choice has replaced chance.

Let us sketch some highlights of a possible formal approach. Let A denote a domain of mutually exclusive alternatives or prospects, and let $\mathcal{A}$ be a suitably rich family of subsets of A (say, the set of all finite or countable subsets, or any σ-field that includes the singletons). Let "Sel" be a mapping from $\mathcal{A} - \{\phi\}$ to A, with Sel(S) interpreted as a list, or "menu," offering the selection S. More precisely, Sel(S) will represent the *prospect of having to select an outcome from the set* S.* The axioms for a "selection space" (A, $\mathcal{A}$, Sel) might then include the following:

A1 $$\mathrm{Sel}(\{x\}) = x, \qquad \text{all } x \in A$$

*"Sel" should not be confused with a "choice function." Note that generally $\mathrm{Sel}(S) \notin S$.

(single-choice menus), and

A2 $\quad \mathrm{Sel}(\{\mathrm{Sel}(S): S \in \mathcal{S}\}) = \mathrm{Sel}(\bigcup_{S \in \mathcal{S}} S), \quad$ all $\mathcal{S} \subset \mathcal{A}$

(reduction of compound menus); in the latter we might wish to restrict the sets $\mathcal{S}$ to be countable or even finite collections of subsets of A. Finally, if u is a utility function from A to the reals R, it might be required to satisfy

A3 $$u(\mathrm{Sel}(S)) = \sup_{x \in S} u(x), \quad \text{all } S \in \mathcal{A}.$$

This expresses a kind of neutrality: there is neither pain nor pleasure in having to make a choice.

The parallel with cardinal utility theory is striking. To make the analogy, let the domain of prospects now be a measurable space (M, $\mathcal{C}$), and let $\underset{\sim}{M}$ be a suitably rich (and suitably convex) family of probability measures on M, including all the "sure things" δ_x, $x \in M$. Let "Lot" be a mapping from $\underset{\sim}{M}$ to M, and interpret Lot(μ) for $\mu \in \underset{\sim}{M}$ as the *prospect of having to enter the lottery defined by the probability measure* μ. The axioms for a "mixture space" (M, $\mathcal{C}$, $\underset{\sim}{M}$, Lot) might then include the following:

B1 $$\mathrm{Lot}(\delta_x) = x, \quad \text{all } x \in M$$

(one-outcome lotteries), and

B2 $$\mathrm{Lot}\Big(\int_{\underset{\sim}{M}} \mathrm{Lot}(\mu)d\alpha(\mu)\Big) = \mathrm{Lot}\Big(\int_{\underset{\sim}{M}} \mu d\alpha(\mu)\Big)$$

(reduction of compound lotteries); in the latter, α is a probability measure on $\underset{\sim}{M}$, perhaps restricted to having a countable or even finite carrier.* Finally, of course we have risk neutrality:

B3 $$u(\mathrm{Lot}(\mu)) = \int_M u(x)d\mu(x),$$

which says there is no pleasure or pain in gambling.**

Pursuing the analogy, we recall that a mixture space M is often just the set of all probability measures on a basic set P of "sure" or "pure" outcomes, with the natural definition:

$$\mathrm{Lot}(\mu) = \int_{x \in M} x d\mu(x), \qquad \text{all } \mu \in \underset{\sim}{M}.$$

If P is finite then M takes the form of a finite-dimensional simplex.*** Similarly, a selection space A can be just the set of all nonempty subsets of a basic set Q of no-option or "compulsory" outcomes, with

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*The probability measure $\int_{\underset{\sim}{M}} \mu d\alpha(\mu) \in \underset{\sim}{M}$ is defined by:

$$\Big[\int_{\underset{\sim}{M}} \mu d\alpha(\mu)\Big](S) = \int_{\underset{\sim}{M}} \mu(S)d\alpha(\mu), \qquad \text{all } S \in \mathcal{C}.$$

**Another key condition, of course, is the "axiom of closure": Lot $(\mu) \in M$; this, like its analogue Sel(S) $\in$ A in the other theory, is buried in our definitions.

***Marschak [11].

the natural definition:

$$\text{Sel}(S) = \bigcup_{x \in Q} x, \qquad \text{all } S \in \mathcal{A}.$$

If Q is finite then A takes the form of a finite Boolean algebra, omitting the zero.*

As already indicated, the Bernoulli game tells us that if we have a mixture space with a sufficiently rich class $\underset{\sim}{M}$ of admissible "mixtures," then any real-valued utility function satisfying B3 must be bounded. On the other hand, the "Blank Check" game tells us that if we have a selection space with a sufficiently rich class $\mathcal{A}$ of admissible "menus," then any real-valued utility function satisfying A3 must be bounded, at least from above. Moreover, the supremum of the range u(A) is actually attained, something that we cannot conclude from the Bernoulli lottery.**

To complete our argument, we maintain that *menu-forming* is an intrinsically more fundamental process than *lottery-forming*, and that it would be far harder to deny A3 than B3 in most application. Indeed, users of utility theory -- economists in particular -- are

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*The general question of when a selection space is representable as a *subset* of a Boolean algebra has a certain formal resemblance to the question of when a mixture space is representable as a *subset* of a linear vector space (see Hausner [5]), with closure under Boolean addition in the former corresponding to convexity in the latter.

**Certain other properties of u(A) can be inferred from A3; they are not, however, as strong as the convexity of the range u(M) that we get from B3.

often content to omit risk and probability from their models. On occasion, they will even allow their entrepreneurs to form markets for almost everything under the sun *except* man-made lotteries. This exception is of course tied to a desire to avoid taking the step from ordinal to cardinal utilities.* On the other hand, users of utility theory can seldom get very far if they eschew models involving choice situations, or refuse to assign utilities to such situations. For example, it is commonplace to assume that the utility to a consumer of a certain income is the supremum of his utilities for the bundles in the budget set defined by that income; this is just our A3. Moreover, the case for allowing infinite menus, once finite menus are admitted, is at least as strong as the case for taking the step from finite to infinite lotteries. So boundedness seems forced on us by "Blank Check" as soon as we adopt the real numbers as an ordinal utility scale, and long before gambling and cardinality enter the picture.

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In sum, the thrust of these remarks has been towards showing that the unbounded utility issue is independent of the considerations surrounding the St. Petersburg paradox. On the one hand, the Bernoulli game can be modified to be (mathematically) bounded without losing whatever power it might have to make risk-linear money look absurd. On the other hand, the logical challenge that the Bernoulli game raises

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*Radner [14]; Shubik [17].

against unbounded utilities (as opposed to "unreasonably large" but finite utilities) can be mounted equally well in a more basic setting, not involving lotteries at all.

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