VALUES OF LARGE GAMES, II: OCEANIC GAMES

by

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PREFACE

This paper combines, for outside publication purposes, the contents of RM-2649-PR and RM-2650-PR, two early studies on infinite-person games that were written in 1961 as part of the USAF Project RAND program of research in game theory. Except for adding some recent references, no attempt has been made to up-date the presentation.

ABSTRACT

A value theory is developed for voting games in which a sizable fraction of the total vote is controlled by a few major players and the rest is distributed among a continuous infinity of individually insignificant minor players. The latter are referred to collectively as an "ocean," to suggest the total lack of order or cohesion that is assumed.

CONTENTS

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1. The Concept of Oceanic Game

The first note in this series* [1] developed the idea of a sequence of weighted majority games in which certain fixed fractions of the voting strength are held by a few "major" players, while the rest is scattered among a growing number of increasingly insignificant "minor" players. It was shown that the game values to the major players tend to limits that are independent of the particular way in which the minor weights go to zero. An explicit expression for the limits was obtained.

The fact that at least one kind of solution--i.e., the value--is convergent in such a sequence of games suggests that we might profitably deal with the limit of the sequence as a game in its own right. The limit game would of necessity have infinitely many players: a finite discrete set corresponding to the major players, plus a continuum of infinitesimal minor players. We shall refer to the latter as an "ocean," to emphasize the almost total absence of order or cohesion. The voting power of the "oceanic" players will be expressed as a measure, defined on the measurable subsets of the ocean. Any payoffs or value allocations to the oceanic players will be represented similarly in measure-theoretic terms; individual oceanic players will not be considered.

In an oceanic game we would hope to be able to define the major-player values in such a way that continuity is preserved with respect to the values of the finite approximants; i.e., we would expect that

$$\lim_{\ell \to \infty} (\text{val } \Gamma_\ell) = \text{val } (\lim_{\ell \to \infty} \Gamma_\ell). \tag{1.1}$$

That our definition, given below in Section 2, fulfills this hope is confirmed by Theorem 1, proved in Section 3. In the rest of this paper we proceed to reap some of the benefits of dealing directly with the infinte-person game, instead of with a sequence of finite approximants, and derive some interesting properties of the value-solutions. In particular, the cumbersome general formula of [1] is greatly simplified in Section 5 for the important special class of "interior" games--those in which the ocean by itself is a winning coalition. A generalized definition of oceanic game is discussed in Section 6. In the Appendix an interesting special class of oceanic games is analyzed in explicit detail.

2. Definitions

Following the general procedure for defining a game as set forth in reference [1], we take $\mathcal{R}$ to be the Boolean ring generated by the subsets of the finite set $M = \{1, \ldots, m\}$ together with the Lebesgue--measurable subsets of the real unit interval $I = [0, 1]$. Let real numbers $w_1, \ldots, w_m \geqq 0$ be given, and write $w(S)$ for $\Sigma_S w_i$. Define a "voting weight" measure u on $\mathcal{R}$ by

$$(2.1) \qquad u(R) = w(R \cap M) + \alpha\mu(R \cap I), \qquad R \in \mathcal{R},$$

where μ denotes Lebesgue measure and α is a positive constant representing the total weight of the ocean, I. Then the weighted majority game $[c; u]$, with majority quota $c \geqq 0$, is called an <u>oceanic game</u>, and will be denoted by the special symbol

$$(2.2) \qquad [c; w_1, \ldots, w_m; \alpha].$$

We see that a coalition wins if and only if its fraction of the ocean, plus its contingent of major players, "weighs" at least c.

How may we define the value of such a game? The direct formula, applicable to finite-person games,* is not readily extended to the

*Equation (2.2) in [1].

present case. We therefore resort to the "pivotal player" approach, wherein a player's value is taken to be the probability that in a random ordering of all the players he and his predecessors together will have enough votes to win, but his predecessors alone do not. In the finite case this is equivalent to the direct definition. We shall use this approach repeatedly in constructing value theories for infinite-person games.

The notion of a perfectly random shuffling of the continuum of oceanic players, even without the major players, is not an easy thing to formulate precisely.* Fortunately, because of the symmetry of the ocean in the present case, the problem can be sidestepped. We must only be sure to insert the major players into the ocean in a properly random fashion--the ocean having previously been ordered in some fixed way.

Accordingly, let $x_1, \ldots, x_m$ be independent random variables distributed uniformly on the unit interval I. Then for any measurable set A we have

$$\text{Prob}\ \{x_i \in A\} = \mu(A). \tag{2.3}$$

Let P(x) denote the set of major players $i \in M$ such that $x_i < x$. The "predecessors" of a major player i then comprise the finite set $P(x_i)$ together with the oceanic interval $[0, x_i)$. We therefore define the <u>value of the game to player i</u> to be the probability φ_i that

$$w(P(x_i)) + \alpha x_i \leqq c \leqq w(P(x_i)) + w_i + \alpha x_i. \tag{2.4}$$

As for the typical oceanic player, if his position in the ocean is represented by the real number x, then <u>his</u> predecessors will comprise the set $P(x) \cup [0, x)$, and we might arbitrarily call him pivotal if

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*The general question of random orderings of infinite sets will be taken up in a later paper of this series.

$$w(P(x)) + \alpha x = c.$$

This arbitrary definition may be justified by the fact that the <u>total</u> value, Φ, of all the oceanic players combined just complements the values of the major players:

$$\Phi + \varphi(M) = 1. \tag{2.5}$$

We might equally well take (2.5) as the definition of the oceanic value Φ.* Of course the value of any particular oceanic player is zero.

If we examine the cumulative weight function (see Figure 1)

$$f(x) = w(P(x)) + \alpha x,$$

we see that it is monotonic increasing (strictly) with discontinuous jumps at the points $x_1, \ldots, x_m$ and with a constant slope, α, in between. This function is of course a random variable. If its graph happens to intersect the "c" level, then there is an oceanic pivot; if not, then the major player responsible for the jump past that level is the pivot.

A second way to view the situation geometrically is to regard $(x_1, \ldots, x_m)$ as a random point in the m-dimensional unit cube I^m. (See Figure 2.) Let A_i denote the subset of I^m in which the inequalities (2.4) are satisfied. The sets $A_1, \ldots, A_m$ are obviously disjoint, except for some overlapping boundaries of measure zero. The major-player values, $\varphi_1, \ldots, \varphi_m$, are by definition just the m-dimensional volumes of $A_1, \ldots, A_m$, respectively. The oceanic value Φ is the volume of "no man's land" (shaded in the figure).

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*This relationship would fail in the "null game," where $c > w(M) + \alpha$, since there are no winning coalitions and all values are zero. We therefore assume throughout that $0 \leqq c \leqq w(M) + \alpha$.

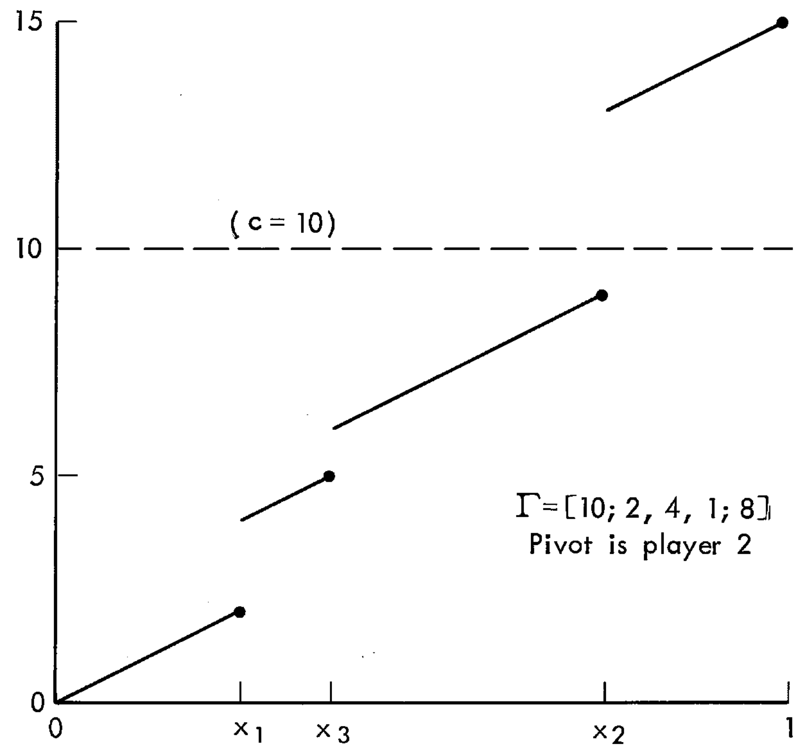

Fig. 1 — Cumulative voting strength after a random ordering

The numerical values given in Figure 2 may be compared with the corresponding numbers for the 12-person approximation game: [8; 4, 1, 1, ..., 1], which are

$$\varphi_1 = .333,\ \varphi_2 = .061,\ \Sigma_3^{12}\ \varphi_i = .606.$$

As the theorem of the next section will show, further refinement of the minor players' weights would produce an even better fit.

3. Continuity of the Value in the Limit

We shall now state and prove the continuity theorem stated symbolically in (1.1).

THEOREM 1. *Let* $\varphi_{i,\ell}$ *denote the value to player* $i \leqq m$ *of the* $(m + n_\ell)$*-person game* $[c_\ell;\ w_{1,\ell},\ \ldots,\ w_{m,\ell},\ a_{1,\ell},\ \ldots,\ a_{n_\ell,\ell}]$, *and let* φ_i *denote the value to major player* i *of the oceanic game* $[c;\ w_1,\ \ldots,\ w_m;\ \alpha]$. *Then*

$$\varphi_i = \lim_{\ell \to \infty} \varphi_{i,\ell}, \tag{3.1}$$

provided that, as $\ell \to \infty$, *we have*

$$c_\ell \to c,\ w_{i,\ell} \to w_i,\ \sum_j a_{j,\ell} \to \alpha,\ \max_j a_{j,\ell} \to 0. \tag{3.2}$$

PROOF. Consider a fixed $i \in M$ and let S be a subset of M not containing i. Let $B_{i,S}$ denote the set of points $(x_1,\ \ldots,\ x_m)$ in I^m such that

$$\begin{cases} x_j < x_i & \text{for all } j \in S, \\ x_j \geqq x_i & \text{for all } j \in M - S. \end{cases} \tag{3.3}$$

Every point of I^m is in exactly one of these sets (given a fixed i),

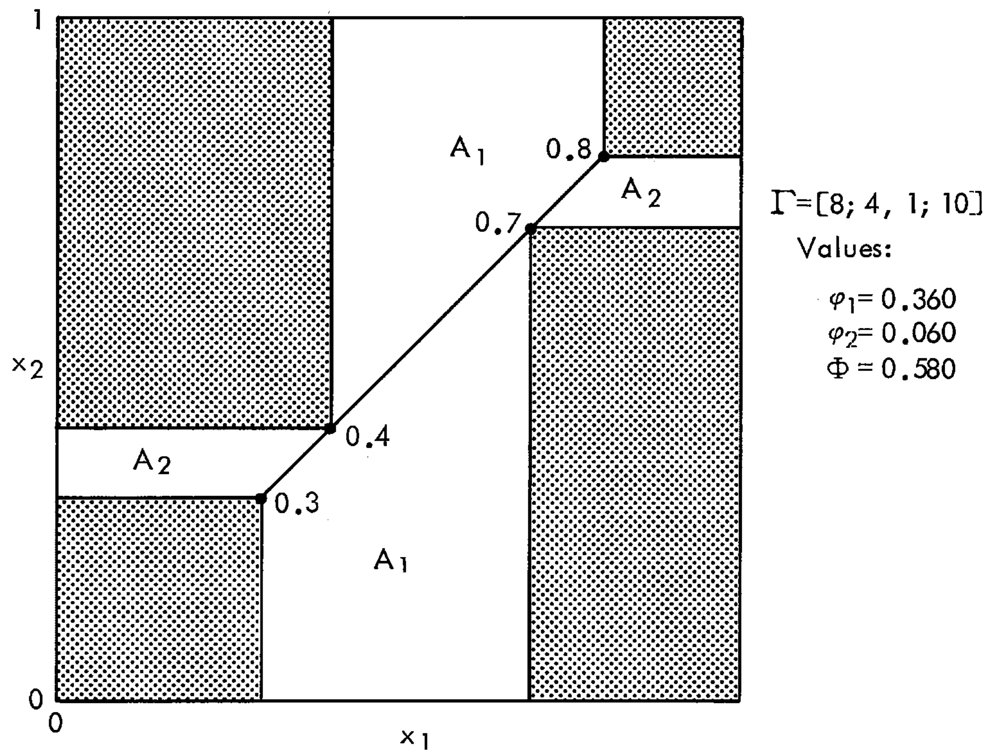

Fig. 2 — Typical partition of I^2

namely the set $B_{i,P(x_i)}$. Recalling our definition of A_i (see Figure 2), we have

$$\varphi_i = \mu^m(A_i) = \sum_{S \subseteq M - \{i\}} \mu^m(A_i \cap B_{i,S}), \tag{3.4}$$

where μ^m denotes m-dimensional Lebesgue measure. Consider a particular set $A_i \cap B_{i,S}$. For a given value of x_i, the other coordinates range independently over one or the other of the intervals $[0, x_i)$ and $[x_i, 1]$, and the (m-1)-dimensional measure of the cross section is $x_i^s(1 - x_i)^{m-s-1}$. Hence we have

$$\mu^m(A_i \cap B_{i,S}) = \int_{t_1}^{t_2} x_i^s(1 - x_i)^{m-s-1} dx_i . \tag{3.5}$$

Here the limits of integration depend on the pivot inequalities (2.4). Taking $P(x_i) = S$ in the latter, and including (via the bracket notation*) the over-all range restriction $0 \leqq x_i \leqq 1$, we find that

$$t_1 = \left\langle \frac{c - w(S \cup \{i\})}{\alpha} \right\rangle, \quad t_2 = \left\langle \frac{c - w(S)}{\alpha} \right\rangle . \tag{3.6}$$

Assembling (3.4), (3.5), and (3.6) gives us precisely the expression derived in [1] for the limit of the finitely defined values $\varphi_{i,\ell}$. This completes the proof of Theorem 1.

4. Variable Quotas. The Oceanic Value. Added Players

Let us now consider the majority quota c of the oceanic game (2.2) as a variable, and write

$$\Gamma(y) = [y; w_1, \ldots, w_m; \alpha]. \tag{4.1}$$

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*We recall that $\langle x \rangle$ denotes the median of 0, x, and 1.

Define the function

(4.2) $$F(y) = \min \{x:\ w(P(x)) + \alpha x \geqq y\},$$

where $0 \leqq y \leqq w(M) + \alpha$. This function is essentially the inverse of the function f(x) illustrated in Figure 1. It gives the location in the ordered ocean of the pivot of the game with quota y. F is continuous, nondecreasing, and piecewise linear, with slope alternating between 0 and $1/\alpha$. (See Figure 3.)

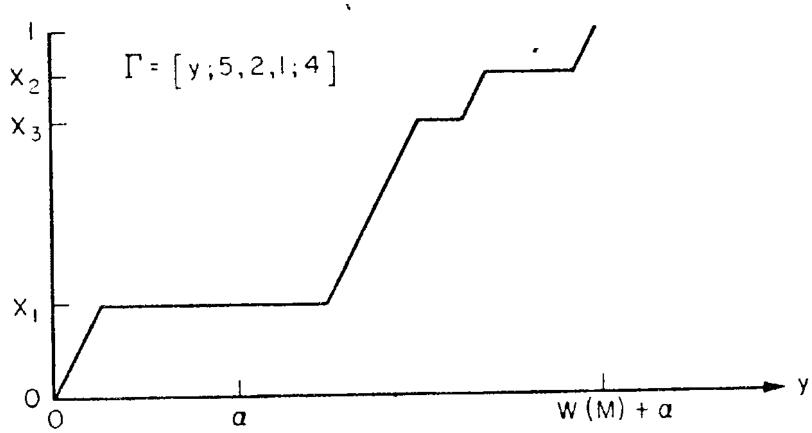

Fig.3—A typical F(y)

Let $\varphi_i(y)$ denote the value of the game $\Gamma(y)$ to the i-th major player, and let $\Phi(y)$ denote the value to the ocean. Applying the remarks that accompany Figure 1, we see that

(4.3) $$\varphi_i(y) = \text{Prob}\ \{F(y) = x_i\}, \qquad i = 1, \ldots, m.$$

Also, since F(y) is differentiable at any particular y with probability 1, we have

(4.4) $$\sum_{i}^{m} \varphi_i(y) = \text{Prob}\ \{F'(y) = 0\}$$

and hence

$$\Phi(y) = \text{Prob }\{F'(y) = 1/\alpha\}. \tag{4.5}$$

Let

$$E\{\cdot\ \cdot\ \cdot\} = \int_{I^m} \cdot\ \cdot\ \cdot\ dx_1 \ldots dx_m$$

denote the "expected value" operator with respect to our basic random variables $x_1, \ldots, x_m$. Then (4.5) can be rewritten

$$\Phi(y) = \alpha E\{F'(y)\}. \tag{4.6}$$

But the operator E commutes with the derivative. Hence the oceanic value is essentially just the slope of the function $E\{F\}$.

THEOREM 2. *The combined value of the oceanic players in* $\Gamma(y)$ *is given by*

$$\Phi(y) = \alpha \frac{d}{dy} E\{F(y)\}. \tag{4.7}$$

We easily verify that $E\{F\}$ is always continuous and has a continuous first derivative. Typical examples of $E\{F\}$ for $m = 1$ are shown in Figure 4. Note that Φ tends to be smaller for central values of y, especially if the major player is very powerful, as in the second case illustrated.

A simple consequence of (4.3) - (4.7) is worth separate mention, because it helps clarify the distinction between voting strengths (as measured by the weights) and "power" (as measured by the values):

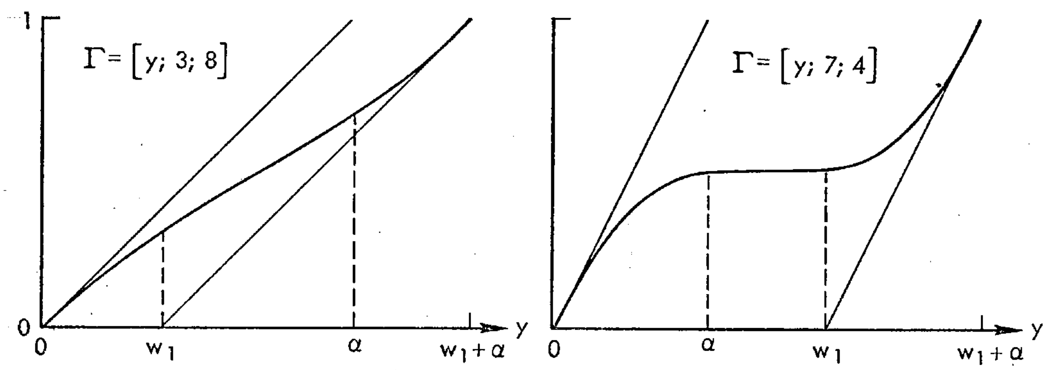

Fig. 4—Some typical $E\{F(y)\}$ for m = 1

THEOREM 3. <u>The value of any major player (or of the ocean) in the game</u> $\Gamma(y)$, <u>averaged over all possible quotas</u> y <u>from</u> 0 <u>to</u> $W = w(M) + \alpha$, <u>is precisely equal to his (its) fraction of the total weight</u>.

By the symmetry of F this theorem remains true if y is averaged only over the interval (W/2, W). This frees us from having to consider "improper" games whose low quotas permit several coalitions to "win" simultaneously.

Note that as y approaches W (the "unanimous" game), the values of all players tend to equalize. This means that the major players become very weak while the ocean becomes all-powerful. To counterbalance this, Theorem 3 implies that for each major player there will be some smaller quota that gives him an exceptionally high value, relative to his voting strength. A detailed study of the case m = 2 (see the Appendix) indicates that the quota y = W/2 (the simple majority case) is usually favorable to the major players in this sense, but not always.

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Now let us consider the oceanic game $\Gamma^+(y)$ obtained by adding a new major player to the game $\Gamma(y)$ of (4.1), thus:

$$(4.8) \qquad \Gamma^+(y) = [y;\ w_1,\ \ldots,\ w_m,\ w_{m+1};\ \alpha].$$

Fix a point $(x_1, \ldots, x_m)$ in I^m. By (4.2), this point defines a particular function $F(y)$ with respect to the game $\Gamma(y)$. Now choose x_{m+1} at random from I^1, and use it to define the corresponding function $F^+(y)$ with respect to $\Gamma^+(y)$. Graphically (see Figure 5), F^+ differs from F only by having a

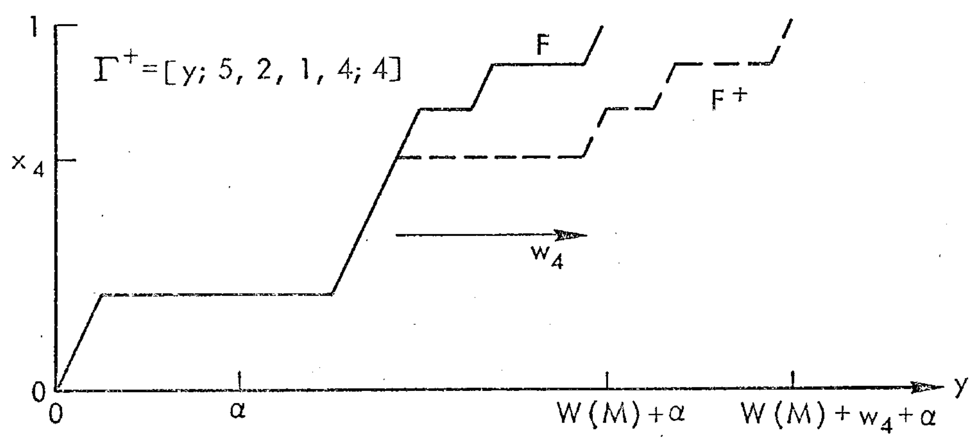

Fig. 5—Effect of added player on F (y)
(compare Fig. 3)

horizontal segment of length w_{m+1} inserted at the x_{m+1} level, with everything to the right moved bodily over to accommodate the insertion. Let $0 \leqq z \leqq w(M) + w_{m+1} + \alpha$. In order that $F^+(z) = x_{m+1}$ (the pivot condition for player $m + 1$ in $\Gamma^+(z)$), we must have

$$(4.9) \qquad F(z - w_{m+1}) \leqq x_{m+1} \leqq F(z).$$

The probability of (4.9) with respect to the single random variable x_{m+1} is simply $F(z) - F(z - w_{m+1})$, provided that both arguments of F in (4.9) are within the domain of definition--i.e., provided that

$$w_{m+1} \leqq z \leqq w(M) + \alpha. \tag{4.10}$$

More generally, it is $F(t_2) - F(t_1)$, with

$$\begin{cases} t_1 = \max(z - w_{m+1}, 0), \\ t_2 = \min(z, w(M) + \alpha). \end{cases} \tag{4.11}$$

Integrating this probability over the other random variables $x_1, \ldots, x_m$ gives us an expression for the value to the $(m + 1)$-st player in the game $\Gamma^+(z)$, as follows:

$$\varphi^+_{m+1}(z) = \underset{I^{m+1}}{\text{Prob}} \{F^+(z) = x_{m+1}\} = \underset{I^m}{E} \{F(t_2) - F(t_1)\}.$$

Theorem 2 now comes into play, enabling us to replace $E\{F\}$ by the integral of Φ. The following useful lemma results:

<u>LEMMA</u>. <u>The added player's value in the game</u> $\Gamma^+(z)$ <u>is related to the oceanic values in the games</u> $\Gamma(y)$ <u>by</u>

$$\varphi^+_{m+1}(z) = \frac{1}{\alpha}\int_{t_1}^{t_2} \Phi(y)\,dy, \tag{4.12}$$

<u>the limits being given by</u> (4.11). <u>If</u> (4.10) <u>holds, this simplifies to</u>

$$\varphi^+_{m+1}(z) = \frac{1}{\alpha}\int_{z - w_{m+1}}^{z} \Phi(y)\,dy. \tag{4.13}$$

This lemma makes it easy to prove the next theorem, which states the not-surprising fact that a very small major player is almost indistinguishable (in value) from a piece of "ocean" of the same size.

The lemma will be used again in the next section.

THEOREM 4. <u>In the general oceanic game, if a major player's weight tends to zero, his value-per-vote ratio</u> φ_i/w_i <u>approaches the value-per-vote ratio</u> Φ/α <u>of the oceanic players</u>.

<u>PROOF</u>. It suffices to establish the principle for the added player in (4.8). Differentiating (4.13) gives us

$$\frac{d\varphi^+_{m+1}(z)}{dw_{m+1}} = \frac{1}{\alpha}\Phi(z - w_{m+1}) \tag{4.14}$$

(subject to (4.10)). Continuity of the values with respect to w_{m+1} makes (4.10) inoperative in the limit as $w_{m+1} \to 0$, and also removes the distinction between Φ and Φ^+ in the limit. The result is now apparent.

5. A Recursion for the Interior Case

The lemma proved in the last section is important because it relates the values of (m + 1)-person oceanic games to the values of m-person oceanic games. It opens the possibility of computing values recursively, from the ground up. (Or, should we say, from the water up?) We could hope to start with games that have no major players--just an ocean--and alternately apply (4.13) and (2.5).* An encouraging sign is the absence of case distinctions that proliferate at each stage of the recursion. So long as we can stick to (4.13) in place of (4.12), we need only deal with a single analytic function at each step. On the other hand, the repeated integrations may prove to be cumbersome, and the repeated application of condition (4.10) threatens to impose ever-narrowing restrictions on the generality of the result. Happily, neither of these difficulties assumes serious proportions.

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*Strangely, no such recursive method for computing values has ever been found for finite games.

Let us first consider the effect of the conditions (4.10). Suppose that we are attempting to derive recursively the value formulas for the m-major-player game

$$(5.1) \qquad [c;\ w_1,\ \ldots,\ w_m;\ \alpha],$$

using (4.13) but not (4.12). It is evident that the y's that appear in the integrals at the first stage of the iteration (m = 0 in the lemma) will collectively cover the entire range from c-w(M) to c. This range will have to be included in the domain of definition of the innermost F function. There being no major players, this domain is simply the interval $[0,\ \alpha]$. Thus we must have, as a necessary condition,

$$(5.2) \qquad 0 \leqq \{c - w(M),\ c\} \leqq \alpha,$$

which we may re-write

$$(5.3) \qquad w(M) \leqq c \leqq \alpha.$$

The reader may recall that (5.3) defines what was called in [1] the interior case. It may be characterized verbally by saying that the ocean is a winning coalition while the set of major players is a losing coalition. Interior games constitute a not inconsiderable fraction of the totality of oceanic games. They are in a sense the furthest removed from the finite (nonoceanic) case, in that the weight of the ocean is bounded below but not above.

We have seen that (5.3) is necessary to the success of the recursion based on (4.13) and (2.5). That it is also sufficient follows rather easily. We need only observe that if $\Gamma^+(z)$ is an interior game, then all games $\Gamma(y)$ called for in the application of (4.13) are interior as well. Hence the single "interior" hypothesis (5.3) efficiently guarantees the satisfaction of all conditions of type (4.10) that will arise later.

When we begin to carry out the iterated integrations, a pleasant surprise greets us. We discover that in an interior oceanic game <u>the values are independent of the majority quota</u>. To see why this is so, note that equation (4.13) has the following property: If $\Phi(y)$ is independent of y, then $\varphi_i^+(z)$ is independent of z, for every i. But the underlined statement is true trivially for $m = 0$. Thus the integration is trivial, and we have the following result.

<u>THEOREM 5</u>. <u>If the games</u>

$$\Gamma = [c; w_1, \ldots, w_m; \alpha]$$

<u>and</u>

$$\Gamma^+ = [d; w_1, \ldots, w_{m+1}; \alpha]$$

<u>are interior, then</u>

$$\varphi_{m+1}^+ = \frac{1}{\alpha} \Phi w_{m+1}. \tag{5.4}$$

<u>COROLLARY</u>. (Compare Theorem 4.) <u>In an interior oceanic game, a major player's value-per-vote ratio is equal to the ocean's value-per-vote ratio with that player eliminated, thus</u>:

$$\frac{\varphi_i[c; w_1, \ldots, w_i, \ldots, w_m; \alpha]}{w_i} = \frac{\Phi[c; w_1, \ldots, 0, \ldots, w_m; \alpha]}{\alpha}. \tag{5.5}$$

Coupling (5.4) and (2.5), we can proceed to calculate the interior-case value formulas for small values of m. An inspection of Table 1 makes it apparent that one obtains a sequence of symmetric, multilinear polynomials in the w_i, of a fairly simple, quasi-homogeneous form. We should be able to write down the general formula as soon as we discover what coefficients to attach to the various products of the weights w_i and their "complements" $\bar{w}_i = \alpha - w_i$.

Table 1

VALUES OF INTERIOR OCEANIC GAMES

m	Typical major value φ_1	Oceanic value Φ
0	-	1
1	$\frac{w_1}{\alpha}$	$\frac{\bar{w}_1}{\alpha}$
2	$\frac{w_1\bar{w}_2}{\alpha^2}$	$\frac{1}{\alpha^2}\left(\bar{w}_1\bar{w}_2 + w_1w_2\right)$
3	$\frac{w_1}{\alpha^3}\left(\bar{w}_2\bar{w}_3+w_2w_3\right)$	$\frac{1}{\alpha^3}\left(\bar{w}_1\bar{w}_2\bar{w}_3 + w_1w_2\bar{w}_3 + w_1\bar{w}_2w_3 + \bar{w}_1w_2w_3 - 2w_1w_2w_3\right)$

Accordingly, let $\pi(S)$ be defined, for subsets S of M, as the product $u_1u_2\ldots u_m$, where u_i is w_i/α for $i \in S$ and $\bar{w}_i/\alpha$ for $i \in M-S$. Let $\pi_i(S)$ be defined similarly for subsets S of $M - \{i\}$. The sought-for coefficients may be defined by the recurrence

(5.6) $$a_n = 1 - na_{n-1},$$

with $a_0 = 1$. So the following sequence of numbers is generated:

(5.7) $$a_0 = 1,\ a_1 = 0,\ a_2 = 1,\ a_3 = -2,\ a_4 = 9,\ a_5 = -44,\ \ldots .$$

We see that the a_i are alternately positive and nonpositive. Moreover, we have

(5.8) $$a_n = n!\left[\frac{1}{n!} - \frac{1}{(n-1)!} + \dots + (-1)^n\right].$$

From this we see that $|a_n|$ is the nearest integer to $n!/e$, for all $n > 0$.*

THEOREM 6. In the interior oceanic game

$$[c;\ w_1,\ \dots,\ w_m;\ \alpha], \qquad w(M) \leqq c \leqq \alpha,$$

the value to the i-th major player is

(5.9) $$\varphi_i = \frac{w_i}{\alpha} \sum_{S \subseteq M-\{i\}} a_s \pi_i(S),$$

and the combined value of the oceanic players is

(5.10) $$\Phi = \sum_{S \subseteq M} a_s \pi(S),$$

where s denotes the number of elements in S.

(In other words, the coefficient of any term in Table 1 is simply a_s/α^m, where s is the number of unbarred w_i factors. The fact that a_1 is zero gives a deceptive simplicity to the formulas for small m.)

PROOF. Table 1 suffices to start the induction. The inductive step involves checking to see that the formulas given satisfy both (5.4) and (2.5). The former is immediate. As for the latter, we have

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*Another incidental fact: $|a_n|$ is the number of "total disarrangements" of n objects (i.e., permutations without fixed points).

$$\sum_{i \in M} \varphi_i = \sum_{i \in M} \frac{w_i}{\alpha} \sum_{S \subseteq M-\{i\}} a_s \pi_i(S)$$

$$= \sum_{i \in M} \sum_{\substack{S \subseteq M \\ S \ni i}} a_{s-1} \pi(S).$$

Each $S \subseteq M$ appears in the double sum exactly s times, once for each of its elements i. Hence we continue

$$\sum_{i \in M} \varphi_i = \sum_{S \subseteq M} s a_{s-1} \pi(S)$$

$$= \sum_{S \subseteq M} (1 - a_s) \pi(S) \qquad \text{(using (5.6))}$$

$$= \sum_{S \subseteq M} \pi(S) - \sum_{S \subseteq M} a_s \pi(S)$$

$$= 1 - \Phi,$$

as was to be shown.

6. Generalized Oceanic Games *

It is natural to try to generalize the definitions of Section 2 to a game in which the voting strength of the ocean is distributed inhomogeneously. Equation (2.1), defining the oceanic game, would be replaced by

(6.1) $$u(R) = w(R \cap M) + \nu(R \cap I), \qquad \text{all } R \in \mathcal{R},$$

where ν is some more or less arbitrary "voting measure" on I.

It is evident that any atoms that may occur in ν will be essentially indistinguishable from major players, and vice versa.**

*For more recent work in this area, see [8] and [9].

**A major player can be inserted into the ocean at any point having zero ν-measure, by adding the obvious step-function to ν.

We could therefore dispense entirely with major players in our formulation, if we wish, and simply define a generalized oceanic game as an arbitrary weighted majority game [c; u] (see [1]) in which the Boolean ring $\mathcal{R}$ of player sets consists of the measurable subsets of the unit interval I.*

If there are only a finite number of atoms in ν, however, then it is more convenient to reverse the above procedure and pull the atoms out as explicit major players. This enables us to restrict our attention to non-atomic measures ν in (6.1). A non-atomic measure on I can always be represented as the limit of a sequence of finite step-functions, with the weight of the largest step going to zero. The inhomogeneous oceanic game can therefore be represented as the limit of a sequence of finite games, fulfilling the hypotheses of the fundamental limit theorem [1]. While we have not yet defined the value for the inhomogeneous case, we can nevertheless conclude that there is a unique extension of the finite value definition that is continuous (in the sense of (1.1) or Theorem 1), at least in so far as the major players are concerned. For the present discussion we shall take this continuous extension as the definition of value, since we are not yet equipped to derive the value from a random-ordering principle and then prove it continuous.

Under this definition, the first point of interest is the observation that the values to the major players in an inhomogeneous game are the same as in the corresponding homogeneous game. One merely sets $\alpha = \nu(I)$. The irregularities in voting strength in the ocean have no effect on the major players.

We can also determine without much difficulty the distribution of value within an inhomogeneous ocean. Not surprisingly, it turns out to be directly proportional to the measure ν:

$$\text{(6.2)} \qquad \varphi(S) = \frac{\nu(S)}{\nu(I)}\,\Phi, \qquad \text{all measurable } S \subseteq I.$$

---

*One might prefer to withhold the adjective "oceanic" unless the support of μ is a continuum, or has the power of the continuum.

To prove this, consider an S that has a rational fraction of the total measure, say $\nu(S)/\nu(I) = r/s$, with r and s integers. Since ν is continuous, S can be partitioned into subsets $A_1, \ldots, A_r$, and $I - S$ into subsets $B_1, \ldots, B_{s-r}$, all of which have ν-measure exactly equal to $\nu(I)/s$. In the finite, (m+s)-person game obtained by considering these subsets as individual players, the values of S and I are clearly in the desired ratio r/s, by symmetry. But the partition just described can be refined indefinitely, preserving symmetry. Therefore we obtain (6.2) in the limit. The extension to irrational-fractional subsets S is immediate.

We note that a game with two or more homogeneous oceans is a special case of the inhomogeneous theory we have just presented. Our results show that multiple oceans can be "pooled" without affecting the major players' values.

If there are an infinite number of atoms in the measure ν of (6.1), then we are confronted with what amounts to a denumerable infinity of major players, with or without an accompanying ocean. Consideration of this case must be deferred until we have developed an approach to the theory of countable-person games.

APPENDIX: A CORPORATION WITH TWO LARGE STOCKHOLDERS*

In this appendix we shall examine the case of two major players more fully, as its value theory exhibits a surprising wealth of subtle detail despite the apparently uncomplicated setting. In the discussion, we shall generally interpret the <u>value</u> as an index of <u>power</u>; this usage is in effect a definition of the latter term. We trust that the qualitative properties of the value as they are revealed in this example will serve to justify the choice of terminology.

Consider a corporation with common stock held by two large interests and a great number of very small interests. Assume that control of the corporation hinges entirely on a simple majority vote of the stock, and forget about proxies, the board of directors, marketability of shares, the wishes of management, and other such mundane matters. A formal model of the resulting, stripped-down political structure is then provided by the oceanic game

$$[1/2;\ w_1,\ w_2;\ \alpha], \tag{A.1}$$

where always $\alpha \equiv 1 - w_1 - w_2$.

The two-dimensional domain of the parameters makes a right triangle in the $(w_1,\ w_2)$-plane and can conveniently be split into four cases, as shown in Figure 6. Region I, with its large ocean, represents the theoretically important "interior" case. In Region II the ocean is smaller, but still holds the "balance of power" between the two major players. In the other regions, a single player has dictatorial power and the trivial games that result are interesting only as limiting cases.

The power index of the first major player is given by:

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*This appendix is a condensed version of [3].

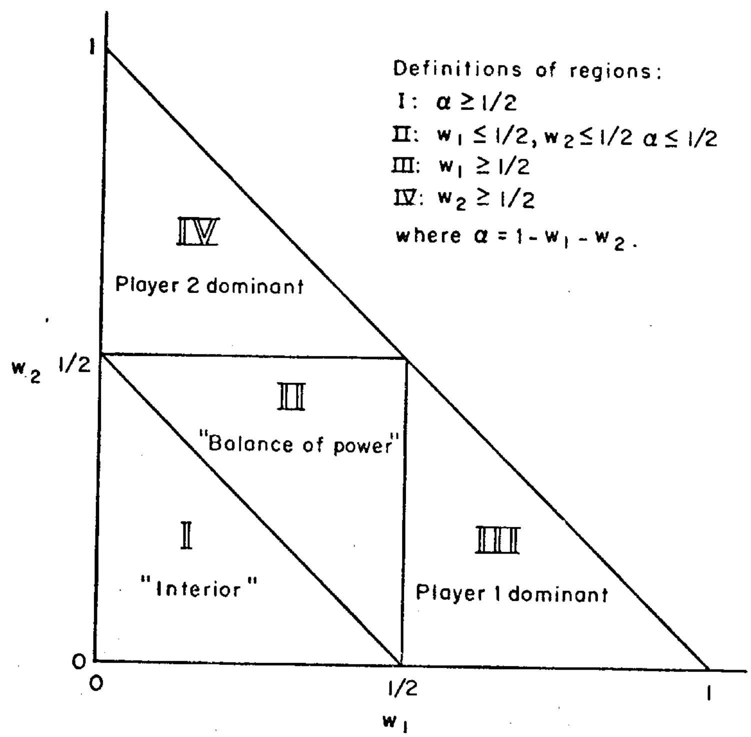

Fig. 6 — Range of parameters in the oceanic game $[1/2; w_1; w_2; \alpha]$

$$\varphi_1 = \begin{cases} \dfrac{w_1}{\alpha} - \dfrac{w_1 w_2}{\alpha^2} & \text{in Region I} \\[2ex] \dfrac{(1 - 2w_2)^2}{4\alpha^2} & \text{in Region II} \\[2ex] 1 & \text{in Region III} \\ 0 & \text{in Region IV.} \end{cases}$$

The second player's index φ_2 is similar, and the total power of the ocean is given by

$$\Phi = 1 - \varphi_1 - \varphi_2.$$

We shall be interested also in the relative powers per share of stock, or *power ratios*, which are given by

$$R_1 = \varphi_1/w_1, \qquad R_2 = \varphi_2/w_2, \qquad R_{oc} = \Phi/\alpha.$$

Figure 7 displays φ_1 and R_1 as functions of the game parameters; the corresponding figure for the second player may be obtained by reflecting in the line $w_1 = w_2$. Figure 8 gives similar information for the oceanic players. For example, if the first player holds 10 percent of the stock and the second player 40 percent, then their power indices are 0.04 and 0.64 respectively, and their power ratios are 0.4 and 1.6 respectively. The oceanic players have a combined power of 0.32 and a power ratio of 0.64.

Some comments on these graphs are in order. The functions plotted are continuous everywhere except at the critical point $w_1 = w_2 = 1/2$, and their first derivatives are continuous in the interiors of the regions and across the boundary between Regions I

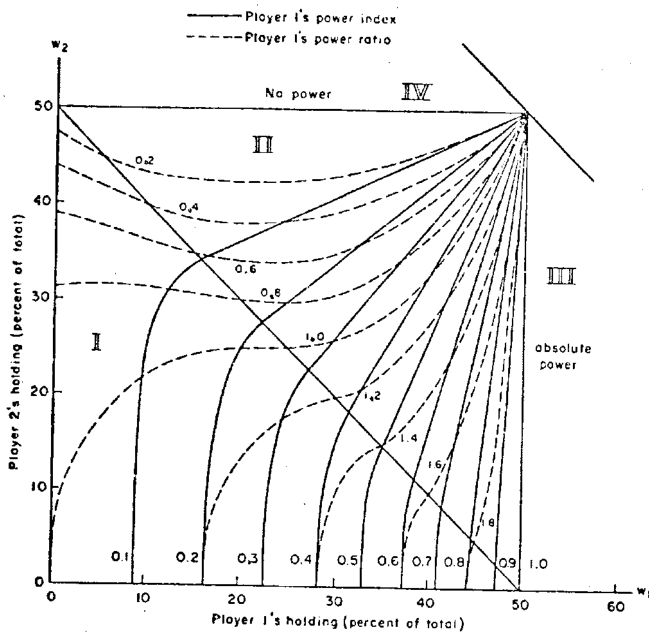

Fig. 7——Power of player 1

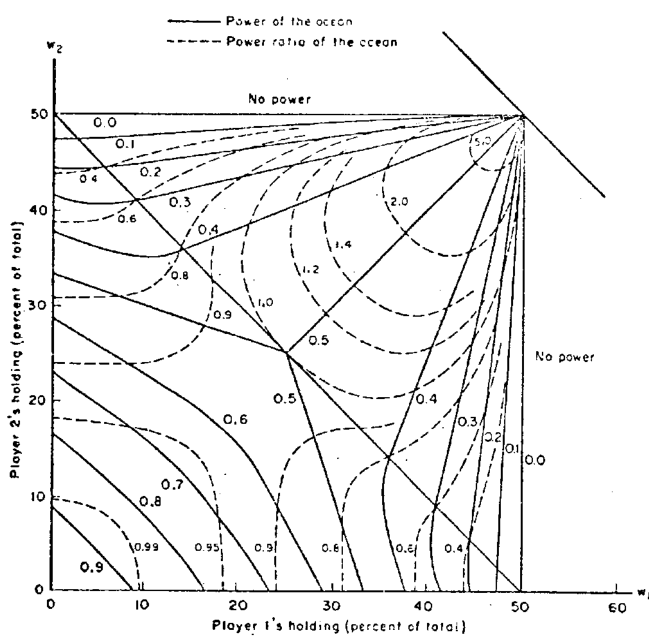

Fig. 8——Combined power of the oceanic players

and II. The presence of oceanic players seems to wipe out the irregular "fine structure" that is usually found in finite games with many small players.

When one of the major holdings is small, the power of the other major player is not sensitive to it, as we can see from the near-vertical slope of the φ_1 and R_1 contours as $w_2 \to 0$ in Figure 7. The same is true for the ocean's power ratio, as we would expect from Theorem 4, but not for its total power, since α depends on w_2. If we let w_2 go to zero, then player 2 vanishes into the ocean without a ripple, providing a perfectly continuous transition to the one-major-player case.

We see that moderately powerful major players, as in the center of Region I, do somewhat better than the minor players on a power-per-share basis. (The rankings of power ratios are shown in Figure 9.) As we cross into Region II, however, the ebbing tide of oceanic players becomes increasingly significant. The "balance of power" effect is strikingly illustrated as the two major holdings simultaneously approach 50 percent ownership, as in a fiercely contested proxy battle. The ocean actually holds on to half the <u>total</u> power as long as the big players are evenly matched, and the ocean's power ratio increases without limit near the critical point $w_1 = w_2 = 1/2$.

Table 2 and Figure 10 present in detail the calculated fluctuations in the power distribution that would accompany a hypothetical growth of w_1 from zero to more than 50 percent of the stock, with w_2 held constant at 30 percent. Perhaps the most interesting thing about this exercise is the fact that there <u>is</u> so much detail, despite the extreme simplicity of the game model itself. Note that the cross section chosen cuts through six of the eight regions of Figure 9.

The following comments are keyed to Table 2:

a. Player 2 has a commanding position.

b. Player 1's power ratio is slightly larger than the ocean's in this initial phase.

c. Almost all of player 1's gain so far has been at the expense of the ocean.

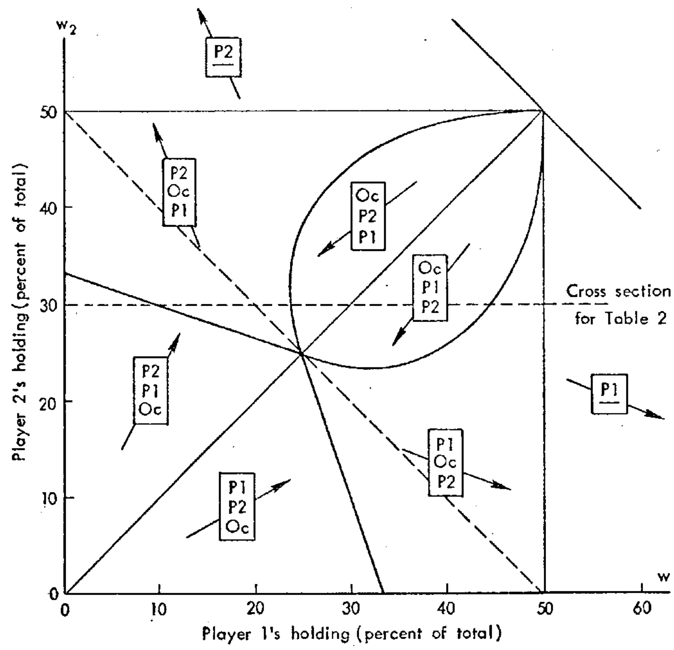

Fig. 9—Ranking of players by power ratio

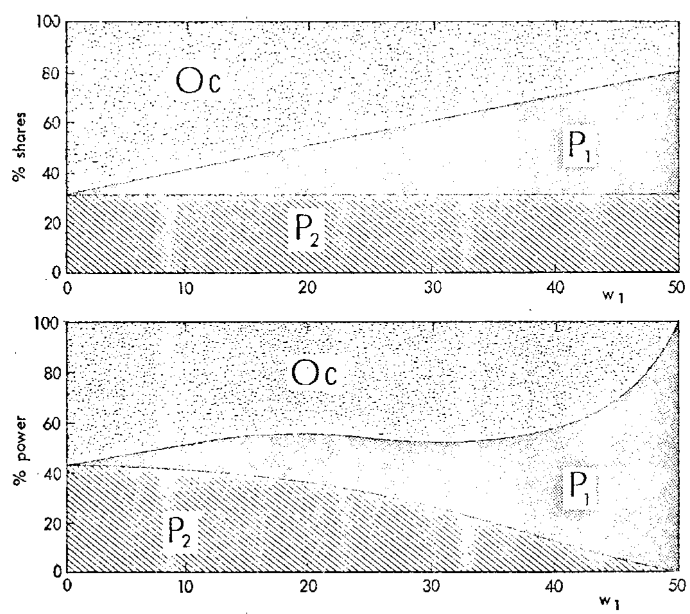

Fig. 10—Shareholding and power as a function of w_1, with w_2 = 30.

d. The ocean's power ratio surpasses that of player 1, which now starts to drop.

e. The ocean's total power starts to rise, despite its dwindling size. Player 2 starts to slip significantly and Player 1's power ratio is lower now than ever before.

f. The game enters Region II. Note that Player 2's shares are still "worth" half again as much as Player 1's.

g. The ocean for the first time has a power ratio less than 1.

h. This is the turning point for Player 1's power ratio.

i. The symmetric point. The oceanic power index reaches a local maximum.

j. Player 1's power ratio at last breaks out of the narrow interval within which it has been oscillating.

k. Player 2 has gone into a sharp decline; he has less than half the power of Player 1 even though their share percentages are not too different.

l. The oceanic players' power ratio reaches its maximum, even as their total power begins to drop rapidly.

m. Player 1 sweeps toward absolute control.

n. Player 2's block of shares is "worth" much less than the smaller quantity of scattered oceanic shares.

o. The game enters Region III.

* * * * * * *

The reader should be cautioned against attaching too much significance to this little story of the path to corporate power through share acquisition. The value of a game can be justified as a measure of power, we believe, but only on a static, _ceteris paribus_ basis. Once a game acquires a context, or a dynamic history, then the delicate interplay of symmetry and ignorance on which the validity of the power index depends is lost.

This warning notwithstanding, we would like to speculate for a few paragraphs on letting the power index--or power ratio--influence the way that holdings change over time. Of course, in practice any such influence would have to compete with many other tendencies, incentives, and operational constraints. But we are in no mood to enlarge our pleasantly uncluttered formal model to include even the most important factors (e.g., money, time) that realism would surely

TABLE 2

Shareholdings			Power indices			Power ratios			Notes
w_1	w_2	α	φ_1	φ_2	Φ	R_1	R_2	R_{oc}	*
0	30	70	.000	.429	.571	(.816)	1.429	.816	a
1	30	69	.008	.428	.563	.819	1.428	.816	b
2	30	68	.016	.428	.555	.822	1.427	.817	
3	30	67	.025	.428	.548	.824	1.426	.817	
4	30	66	.033	.427	.540	.826	1.423	.818	
5	30	65	.041	.426	.533	.828	1.420	.819	
6	30	64	.050	.425	.525	.830	1.416	.821	
7	30	63	.058	.423	.519	.831	1.411	.823	c
8	30	62	.067	.421	.512	.832	1.405	.826	
9	30	61	.075	.419	.506	.833	1.397	.829	
10	30	60	.083	.417	.500	.833	1.389	.833	d
11	30	59	.092	.414	.494	.833	1.379	.838	
12	30	58	.100	.410	.490	.832	1.367	.845	
13	30	57	.108	.406	.486	.831	1.354	.852	
14	30	56	.116	.402	.483	.829	1.339	.862	
15	30	55	.124	.397	.479	.826	1.322	.871	
16	30	54	.132	.391	.477	.823	1.303	.884	
17	30	53	.139	.384	.476	.819	1.282	.899	
18	30	52	.146	.377	.476	.814	1.257	.916	e
19	30	51	.153	.369	.478	.807	1.230	.936	
20	30	50	.160	.360	.480	.800	1.200	.960	f
21	30	49	.167	.350	.483	.793	1.168	.986	
22	30	48	.174	.340	.486	.789	1.134	1.013	g
23	30	47	.181	.330	.489	.787	1.100	1.040	
24	30	46	.189	.319	.492	.787	1.065	1.068	h
25	30	45	.198	.309	.494	.790	1.029	1.098	
26	30	44	.207	.298	.496	.795	.992	1.127	
27	30	43	.216	.286	.498	.801	.954	1.157	
28	30	42	.227	.274	.499	.810	.915	1.188	
29	30	41	.238	.262	.500	.821	.874	1.219	
30	30	40	.250	.250	.500	.833	.833	1.250	i
31	30	39	.263	.237	.500	.848	.791	1.281	
32	30	38	.277	.224	.499	.866	.748	1.312	j
33	30	37	.292	.211	.497	.886	.704	1.342	
34	30	36	.309	.198	.494	.908	.658	1.372	
35	30	35	.327	.184	.490	.933	.612	1.399	
36	30	34	.346	.170	.484	.961	.565	1.425	k
37	30	33	.367	.155	.478	.993	.517	1.447	
38	30	32	.391	.141	.469	1.028	.469	1.465	
39	30	31	.416	.126	.458	1.067	.420	1.477	
40	30	30	.444	.111	.444	1.111	.370	1.481	l
41	30	29	.476	.096	.428	1.160	.321	1.476	
42	30	28	.510	.082	.408	1.215	.272	1.458	
43	30	27	.549	.067	.384	1.276	.224	1.423	
44	30	26	.592	.053	.355	1.345	.178	1.365	
45	30	25	.640	.040	.320	1.422	.133	1.280	m
46	30	24	.694	.028	.278	1.510	.093	1.158	
47	30	23	.756	.017	.227	1.609	.057	.987	
48	30	22	.826	.008	.165	1.722	.028	.751	
49	30	21	.907	.002	.091	1.851	.008	.432	n
50	30	20	1.000	.000	.000	2.000	.000	.000	o
51	30	19	1.000	.000	.000	1.961	.000	.000	

*See text.

demand, so we shall make only a few elementary, essentially qualitative observations that may help to round out our exploration of the role of the power index in oceanic voting games.

One dynamic hypothesis might be that shares tend to migrate to positions of increased power ratio. The resulting "drift" would then correspond roughly to the arrows shown in Figure 9. We see at once, however, that this hypothesis would also imply motion within Regions III and IV, regions which intuitively ought to be very stable. Powerless shares would gravitate into the portfolio of the winning player despite his lack of any interest in acquiring them.

The basic flaw in this hypothesis is the assumption that shares, rather than people, are attracted by the prospect of increased power. To put the motivation where it belongs, we ought to assume that the marginal power to the player governs the migration of shares. If a share happens to represent a smaller increment of power to its owner than to some other player, it may be supposed that a transfer could be arranged at a mutually agreeable price.

To illustrate the drift that this principle would produce, we have plotted in Figure 11 the vector field $\vec{u} = (u_1, u_2)$ defined by*

$$u_i = \frac{d\varphi_i}{dw_i} - \frac{\Phi}{\alpha}, \qquad i = 1, 2.$$

(This involves the further assumption that shares move only into and out of the ocean--i.e., the "open market"--and do not move between the two major players directly.) There are several noteworthy features of this vector field. One is that the larger of the two major holdings always grows more rapidly than the smaller; indeed, the latter sometimes even decreases. The ocean never increases, however, even when its power ratio is very large. But in the "balance of power" situation ($w_1 \approx w_2$ in Region II), the ocean holds its own, since one of the big players will be selling shares almost as fast as the other one buys them.

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*In drawing Figure 11 we have distorted the lengths for better legibility. The field actually shown has the form $c\vec{u}/\sqrt{|\vec{u}|}$.

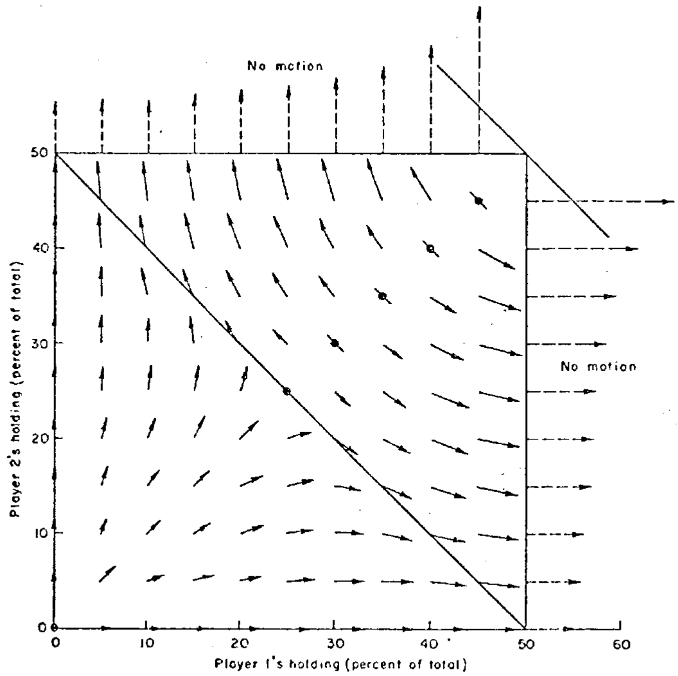

Fig. 11—Migration of shareholdings induced by differences in marginal power per share

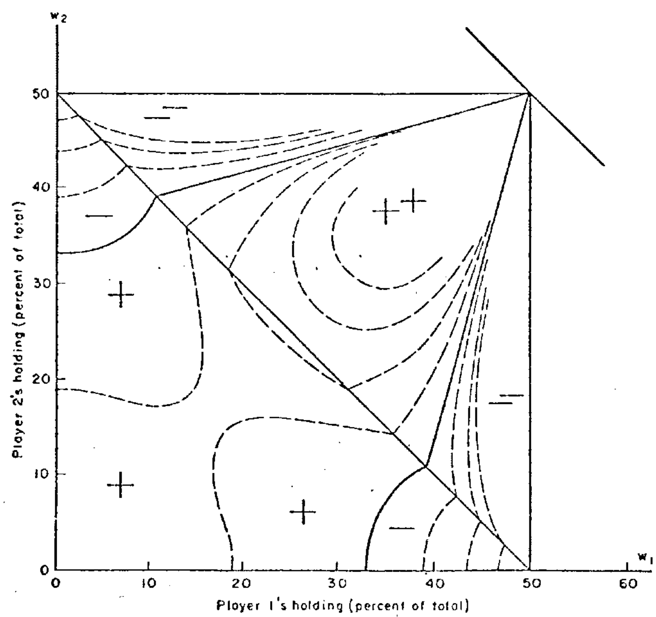

Fig. 12—Prospects for the emergence of a third major player (+ = favorable, — = unfavorable)

Another interesting point is the greater stability of Region I as compared with Region II, if we use the magnitude of $\vec{u}$ as a measure of stability. But the transition between Regions I and II is continuous. At the boundaries of Regions III and IV, on the other hand, there is a sharp discontinuity: $|\vec{u}|$ drops abruptly to 0 when one player achieves absolute control.

Another kind of dynamic evolution that might be investigated is the entry of a third major interest, via a process of "crystallization" out of the ocean. Theorem 4 indicates that there would be no first-order incentive for this to occur,* since for w_3 near 0 both the power ratio φ_3/w_3 and the marginal power per share $d\varphi_3/dw_3$ are approximately equal to the oceanic ratio Φ/α. In other words, the situation in regard to the birth of a new major player is in equilibrium.

But whether the equilibrium is stable or unstable depends on the second-order terms: Specifically, one must consider the difference

$$\Delta = \varphi_3\ [1/2;\ w_1,\ w_2,\ \varepsilon;\ \alpha - \varepsilon] - \frac{\varepsilon}{\alpha}\,\Phi[1/2;\ w_1,\ w_2,\ 0;\ \alpha].$$

In conformity with Theorem 4, this is of the order of magnitude of ε^2. Figure 12 shows how the sign of Δ depends on the game parameters. A "+" means that conditions favor the entrance of a third major player; a "−" means the opposite. The dashed curves are contours of constant Δ. We see again that the "balance of power" region is less stable than the "interior" region.

The foregoing may seem incomplete without a discussion of power-induced splits and mergers among the major interests. We may remark that either of these maneuvers can be profitable under some circumstances. They have little to do with the "oceanic" character of our present example, however, since they can just as well be studied in the context of ordinary finite games.

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*The case considered here is slightly different, in that the new player's shares are not exogenous, but are subtracted from the existing ocean.

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VALUES OF LARGE GAMES, II: OCEANIC GAMES

Milnor & Shapley