AN ABSTRACT SETTING FOR THE
NOTION OF DYNAMIC PROGRAMMING

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SUMMARY

After a brief resume of the needed concepts, the notion of a dynamic programming process is axiomatized and an appropriate version of the "Principle of Optimality" is shown as a consequence.
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1. INTRODUCTION

In this paper one investigates the theory of one-person decision processes alias dynamic programming. In accordance with the writer's personal philosophy relating to applications of mathematics, we give a definition of dynamic programming processes which is obviously too general for use per se, but which may hope to be suitably compromised to a useful level of generality at a later date by restriction of its components. In the present note, we shall define the notion of "optimal policy" but shall not inquire as to existence of such as this clearly requires, in general, additional structure of a nature probably topological. The notion of "good policy" is not even defined since this requires a satisfactory theory of approximation in lattices which is not currently extant. However, a trivial example is given which illustrates the notion of "good policy" in a case where monotoneity and measurability yield satisfactory approximation. As to actual results, we content ourselves for the present with verification of our version of what Dick Bellman calls the "principle of Optimality" [1](1,2).

(1) Numbers in square brackets refer to the list of references concluding the note.

(2) See also the bibliography in [1].
A brief discussion of partial ordering has been included prior to the note proper. This discussion makes the present note reasonably self-contained and available to anyone possessing a rudimentary knowledge of set theory.

2. PREAMBLE ON PARTIAL ORDERING

A binary relation $R$ on a set $S$ (i.e., $R$ is a subset of $S \times S$) is called a partial ordering of $S$ if and only if $R$ has the properties:

1. Reflexivity. For all $a \in S$, $(a, a) \in R$.

2. Antisymmetry. If $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

3. Transitivity. If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

If $R$ is a partial ordering of $S$, one frequently writes something such as $a \preceq b$ or $a \preceq b$ for $(a, b) \in R$ and speaks of the poset (abbreviation for partially ordered set) $(S, \preceq)$ or $(S, \preceq)$, respectively. If $R$ is fixed so that $<$ or $\leq$ is understood, one speaks merely of the poset $S$. The following conventions are understood in any poset $(S, \preceq)$:

4. $a < b$ if and only if $a \leq b$ and $a \neq b$.

5. $a \geq b$ if and only if $b \leq a$.

6. $a > b$ if and only if $b \leq a$ and $a \neq b$.

If $(S, \preceq)$ is a poset and $k \in S$ one writes
(7) $S^k$ for $\{ x \in S | k < x \}$.
(8) $S_k$ for $\{ y \in S | y \leq k \}$.

If $(S, \leq)$ is a poset and $T \subseteq S$ one writes

(9) $T^+$ for $\{ x \in S | y \in T \text{ implies } y \leq x \}$.
(10) $T_+$ for $\{ u \in S | v \in T \text{ implies } u \leq v \}$.

It is obvious that $T \cap T^+$ and $T \cap T_+$ each have at most one element. The element, if any, in $T \cap T^+$ is called the last element of $T$. The element, if any, in $T \cap T_+$ is called the first element of $T$.

If $T^+$ has a first element, the element is called the join of $T$ and written $\bigvee_{x \in T} x$, sup$_T x$, or lub$_T x$. If $T_+$ has a last element, the element is called the meet of $T$ and is written $\bigwedge_{x \in T} x$, inf$_T x$, or glb$_T x$.

If for each element $x$ in a poset $S$ one has $\{ x \}^+ \cup \{ x \}_+ = S$, $S$ is a loset (abbreviation for linearly ordered set).

A poset in which each non-null subset has a first element is a woset (abbreviation for well-ordered set). Clearly, every woset is necessarily a loset.

A poset $(\Delta, \triangleleft)$ is a diset (abbreviation for directed set) if and only if $\{ \alpha, \beta \}^+ \neq \emptyset$ (designates the null set) for each $(\alpha, \beta) \in \Delta \times \Delta$. If $(\Delta, \triangleleft)$ is a diset and $B$ is any set, a mapping $\mathcal{M} : \Delta \rightarrow B$ is called a net in $B$ based on $\Delta$. If $\Gamma \subseteq \Delta$
and \((\Gamma, \prec / \Gamma \times \Gamma)(3)\) is a diset, \(\Gamma\) is called a disubset of \(\Delta\).

Clearly, if \(\Gamma\) is a disubset of \(\Delta\) and \(\gamma \in \Gamma\), then \(\Gamma\gamma\) and \(\Gamma\gamma\) are also disubsets of \(\Delta\). If \(\Gamma\) is a disubset of \(\Delta\), \(\mathfrak{U}\) is a net based on \(\Gamma\), and \(\gamma \in \Gamma\), the nets \(\mathfrak{U}/\Gamma\gamma\) and \(\mathfrak{U}/\Gamma\gamma\) are denoted by \(\mathfrak{U}\gamma\) and \(\mathfrak{U}\gamma\), respectively. One sees by induction that if \(\Gamma\) is a finite subset of a non-null diset, then \(\Gamma^+ \neq \emptyset\).

A non-null poset \((\wedge, \prec)\) is a lattice if and only if
\[
\bigvee_{\gamma \in \{\alpha, \beta\}} \gamma \quad \text{and} \quad \bigwedge_{\gamma \in \{\alpha, \beta\}} \gamma
\]
exist for each \((\alpha, \beta)\) in \(\wedge \times \wedge\).

One usually writes \(\alpha \lor \beta\) and \(\alpha \land \beta\) for \(\bigvee_{\gamma \in \{\alpha, \beta\}} \gamma\) and \(\bigwedge_{\gamma \in \{\alpha, \beta\}} \gamma\), respectively. One sees by induction that if \(\Gamma\) is a non-null finite subset of a lattice, then \(\bigvee_{\gamma \in \Gamma} \gamma\) and \(\bigwedge_{\gamma \in \Gamma} \gamma\) exist. For any lattice \(\wedge\), the mappings \(\bigvee: \wedge \times \wedge \rightarrow \wedge\)
and \(\bigwedge: \bigwedge \times \wedge \rightarrow \wedge\) may be thought of as algebraic operations. They have the properties:

(11) **Commutativity.** For all \((\alpha, \beta)\) in \(\wedge \times \wedge\),
\[
\alpha \lor \beta = \beta \lor \alpha \quad \text{and} \quad \alpha \land \beta = \beta \land \alpha.
\]

(12) **Associativity.** For all \((\alpha, \beta, \gamma)\) in \(\wedge \times \wedge \times \wedge\),
\[
\alpha \lor (\beta \lor \gamma) = (\alpha \lor \beta) \lor \gamma \quad \text{and} \quad \alpha \land (\beta \land \gamma) = (\alpha \land \beta) \land \gamma.
\]

(13) **Idempotency.** For all \(\alpha\) in \(\wedge\), \(\alpha \lor \alpha = \alpha \land \alpha = \alpha\).

(14) **Alternation.** \(\alpha \lor \beta = \alpha\) if and only if \(\alpha \land \beta = \beta\).

Conversely, if \((\wedge, \lor, \wedge)\) is a double groupoid (i.e., \(\lor\) and \(\wedge\) are binary operations under which \(\wedge\) is closed) and if (11), (12), (13), and (14) subsist (*'), and if one defines
\[
\bigwedge_{\Gamma \times \Gamma} \text{indicates the relation} \prec \text{restricted to the subset} \Gamma \times \Gamma \text{of} \Delta \times \Delta.
\]

(*) As a postulate list this is somewhat redundant. Cf. [2] and [4].
\( \alpha \leq \beta \) if and only if \( \alpha \wedge \beta = \alpha \), then \( (\wedge, \leq) \) is a lattice in which join and meet are the original operations \( \vee \) and \( \wedge \), respectively. Thus, a lattice may be regarded either from the viewpoint of poset theory as a poset with extrema for finite subsets or from the viewpoint of universal algebra as a double semilattice with alternation (\(^5\)).

A lattice \( \wedge \) is called upper conditionally complete if and only if \( \Gamma \leq \wedge \) and \( \Box \uplus \Gamma^+ \uplus \wedge \) imply the existence of \( \bigvee_{\gamma \in \Gamma} \gamma \); that is, every non-null subset which is bounded above has a least upper bound. Lower conditional completeness is dually defined and the conjunction of the two properties is referred to as merely conditional completeness.

3. DYNAMIC PROGRAMMING PROCESSES

Definition. A dynamic programming process, which may be indicated as \( \text{DPP}((\Delta, \preceq), B, \mathcal{Y}, A, t, (\wedge, \leq), \vee) \), is a mathematical system consisting of the objects listed as (1) to (7) below subject to the postulates (8) to (12) below.

1. \( (\Delta, \preceq) \) is a diset called "stage space."
2. \( B \) is a set called "decision space."
3. \( \mathcal{Y} \) is a set of nets in \( B \) each of which is based

(\(^5\) Cf. [3]. A groupoid which is associative is called a semigroup. A commutative semigroup in which every element is idempotent is called a semilattice.)
on some disubset of $\Delta$. $\mathcal{H}$ is called "policy space."

(4) $A$ is a set called "phase space" whose elements are called "states."

(5) $t$ is a mapping $t: A \times \mathcal{H} \rightarrow A$. $t$ is called the "transition operator."

(6) $(\land, \leq)$ is an upper conditionally complete lattice called "value space."

(7) $\nu$ is a mapping $\nu: A \rightarrow \land$. $\nu$ is called the "payoff."

(8) If $\eta \in \mathcal{H}$ and $\eta$ is based on $\gamma$ and $\gamma \in \Gamma$, then $\eta \gamma \in \mathcal{H}$ and $\eta \gamma \in \mathcal{H}$.

(9) If $\Gamma$ is a disubset of $\Delta$ and if $\gamma \in \Gamma$ and if $\eta_1 \in \mathcal{H}$, $\eta_2 \in \mathcal{H}$, $\eta_1$ is based on $\gamma \gamma$ and $\eta_2$ is based on $\gamma \gamma$, then there is $\eta \in \mathcal{H}$ based on $\gamma$ with $\eta \gamma = \eta_1$ and $\eta \gamma = \eta_2$.

(10) If $\Box \in \mathcal{H}$, then, for all $a \in A$, $(a, \Box)t = a$.

(11) If $\eta \in \mathcal{H}$ is based on $\gamma$ and $\gamma \in \Gamma$, then, for all $a \in A$, $(a, \eta)t = ((a, \eta \gamma)t, \eta \gamma)t$.

(12) For any $a \in A$ and any disubset $\Gamma$ of $\Delta$,

\[
\{(a, \eta)t \mid \eta \in \mathcal{H} \text{ and } \eta \text{ based on } \Gamma \}^+ \cup \Box.
\]

Conventions.

(13) $\mathcal{H}(\Gamma) = \{ \eta \in \mathcal{H} \mid \eta \text{ is based on } \Gamma \}$.

(14) One writes $a \eta$ for $(a, \eta)t$.

(15) If $\eta \in \mathcal{H}$, $\Gamma(\eta)$ denotes the base of $\eta$. 
(16) If \( \mathcal{X} (\Gamma) \uparrow \Box \), a mapping \( \nu_\Gamma : A \to \Lambda \) is defined as follows: \( \nu_\Gamma = \bigvee_{\eta \in \mathcal{X} (\Gamma)} \eta \nu \). This indicates that the join exists by Postulate (12). The mapping \( \nu_\Gamma \) is called the upper value over \( \Gamma \) of the DPP.

**Remark.** If \( \mathcal{X} (\Gamma) \uparrow \Box \) and \( \gamma \in \Gamma \), then \( \mathcal{X} (\Gamma \gamma) \uparrow \Box \) and \( \mathcal{X} (\Gamma \gamma) \uparrow \Box \), by Postulate (8).

**Definition.** A policy \( \eta \in \mathcal{X} (\Gamma) \) is called an optimal \( \Gamma \) policy in \( D \subset A \) if \( \eta \nu / D = \nu_\Gamma / D \). An optimal \( \Gamma \) policy in \( A \) is called merely an optimal \( \Gamma \) policy.

4. **PRINCIPLE OF OPTIMALITY**

We now contemplate a fixed DPP and assume \( \mathcal{X} (\Gamma) \uparrow \Box \) for the \( \Gamma \) under consideration.

**Lemma 1.** If \( \gamma \in \Gamma \) and \( \eta \in \mathcal{X} (\Gamma \gamma) \), a \( \eta \nu_\gamma \leq \nu_\Gamma \).

**Proof.** \( \eta \nu_\gamma = \bigvee_{\eta' \in \mathcal{X} (\Gamma \gamma)} \eta' \nu \). By Postulate (9), there is \( \eta'' \in \mathcal{X} (\Gamma) \) with \( \eta'' / \Gamma \gamma = \eta \) and \( \eta'' / \Gamma \delta = \eta' \). By Postulate (11), a \( \eta \eta'' = a \eta'' \) so a \( \eta \eta'' \nu = a \eta'' \nu \leq \bigvee_{\eta''' \in \mathcal{X} (\Gamma)} \eta''' \nu = a \nu_\Gamma \).

**Lemma 2.** \( \bigvee_{\eta \in \mathcal{X} (\Gamma \gamma)} \eta \nu_\gamma \leq \nu_\Gamma \).

**Proof.** By Lemma 1, a \( \nu_\Gamma \in \{ a \eta \nu_\gamma / \eta \in \mathcal{X} (\Gamma \gamma) \}^+ \).
Lemma 3. \( \supseteq \frac{\gamma}{\Gamma} \leq \bigvee_{\eta \in \mathcal{Y}(\Gamma)} a \eta \gamma \Gamma \).

Proof. Let \( \eta \in \mathcal{Y}(\Gamma) \). By Postulate (8) and Postulate (11), \( a \eta \gamma = a \eta \gamma \eta \gamma \leq a \eta \gamma \gamma \gamma \leq \bigvee_{m \in \mathcal{Y}(\Gamma)} a m \gamma \gamma \gamma \).

Thus, \( \bigvee_{m \in \mathcal{Y}(\Gamma)} a m \gamma \gamma \gamma \in \{ a \eta \gamma \eta \in \mathcal{Y}(\Gamma) \}^+ \).

Theorem. If \( \mathcal{Y}(\Gamma) \uparrow \Omega \) and if \( \gamma \in \Gamma \), then

\[ a \gamma = \bigvee_{\eta \in \mathcal{Y}(\Gamma)} a \eta \gamma \Gamma \gamma . \]

Proof. Proposition is conjunction of Lemmas 2 and 3.

Corollary. If \( \mathcal{M} \) is an optimal \( \Gamma \) policy and if \( \gamma \in \Gamma \), then \( \gamma \gamma \gamma \) is an optimal \( \Gamma \) policy in \( A \mathcal{M} \).

Proof. Let \( c \in A \mathcal{M} \gamma \). Then, for some \( a \in A \), \( c = a \mathcal{M} \gamma \).

Thus, applying the Theorem, \( c \mathcal{M} \gamma \gamma = a \mathcal{M} \gamma \mathcal{M} \gamma \gamma = a \mathcal{M} \gamma = a \gamma = \bigvee_{\eta \in \mathcal{Y}(\Gamma)} a \eta \gamma \Gamma \gamma \gamma \leq a \mathcal{M} \gamma \Gamma \gamma = c \gamma \Gamma \gamma . \)

5. A TRIVIAL EXAMPLE

The following should be borne in mind concerning the example:

(1) The example is trivial.

(2) The purpose of the example is to illustrate the components of a DPP, to show that optimal policies rarely
exist, and to exhibit the sort of thing one might mean by the
term "good policy" in appropriate circumstances where value
space possesses a pseudonorm or norm and there is monotone
convergence in this pseudonorm toward upper values.

To obtain the example we define components as follows:
(3) \((\Delta, \prec)\) is the set of positive integers in their
usual ordering.
(4) \(B\) is the interval \([0,1]\).
(5) \(\mathcal{N}\) is the set of all nets in \(B\) whose bases are
either intervals \([n, n+k]\) or rays \([n, \infty)\) or integers. The
null-net based on the null-set is also included in \(\mathcal{N}\).
(6) \(A\) is the interval \([0,1]\).
(7) If \(a \in A\) and \(\eta\) is based on \([n, n+k]\), let a \(\eta\) be
\(ax_1x_2...x_{k+1}\), where \(x_1\) is the value of \(\eta\) at \(n+1\) and
juxtaposition indicates ordinary multiplication. If \(\eta\) is
based on \([n, \infty)\), let a \(\eta\) be \(\lim_{k \to \infty} ax_1x_2...x_k\), where the \(x_i\)
have meaning as before.
(8) Let \((\wedge, \leq)\) be the lattice of Borel sets \(^{(6)}\) in
\([0,1]\) ordered by inclusion.
(9) Let a \(\vee = [0,a]\).

Then, the null-net is an optimal \(\square\) policy. For \(\Gamma\) any
interval or ray, a \(\bigvee_{\Gamma} = [0,a]\). There are in this case no
optimal \(\Gamma\) policies but there are "good \(\Gamma\) policies" in the

\(^{(6)}\) We use Borel set in the sense of Hausdorff rather than
Halmos.
sense that for \( \varepsilon > 0 \) there is \( \eta_\varepsilon \in \mathcal{N}(\Gamma) \) with measure
\(( \nu_\Gamma \oplus \eta_\varepsilon \nu ) < \varepsilon \). (7)

\(^{(7)}\) \( \oplus \) indicates symmetric difference.
6. REFERENCES


The RAND Corporation