A SOLUTION CONTAINING AN ARBITRARY CLOSED COMPONENT

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SUMMARY

An n-person game solution is constructed around an arbitrary closed set, showing that there is practically no limit to the possible complexity of solutions.

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A class of n-person game solutions is described in this note which consist of an arbitrary closed subset, J, of a certain (n - 3)-dimensional region of the fundamental simplex, together with one other closed set, K - H, depending on but disjoint from J. The arbitrariness of J illustrates the great variety of solution sets that are possible in the von Neumann-Morgenstern theory, and provides a ready source of examples, pleasant and pathological, for the testing of conjectures about the topological properties of solutions in general.

Let $N = \{1, \ldots, n\}$ be the set of players, $n \geq 4$. We shall consider the simple game with winning coalitions $N - \{1\}$, $N - \{2\}$, $N - \{3\}$, and N.* In the 0-1 normalization, the imputation space A is the simplex of nonnegative n-tuples $\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle$ with sum 1. Let $\mathrm{dom}_k \beta$ denote the set of $\alpha \in A$ with $\alpha_i < \beta_i$, all $i \neq k$. Then the set of imputations dominated by β is

$$\mathrm{dom}\, \beta = \mathrm{dom}_1 \beta \cup \mathrm{dom}_2 \beta \cup \mathrm{dom}_3 \beta,$$

and the defining properties of a solution V can be expressed

$$V \cap \mathrm{dom}\, V = \emptyset \qquad \text{and} \qquad V \cup \mathrm{dom}\, V = A,$$

where, of course, dom V denotes $\sup_{\beta \in V} \mathrm{dom}\, \beta$.

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*It can be described as a direct product of the simple majority game on $\{1,2,3\}$ and the pure bargaining (or "unanimous") game on $\{4, \ldots, n\}$.

We shall now define the sets J, K, and H referred to above. Let U be the simplex of nonnegative (n-3)-tuples $u = (u_4, \ldots, u_n)$ with sum ≤ 1. Let u_0 denote the difference $1 - \Sigma_4^n u_j$, and let U_0 be the face of U on which $u_0 = 0$. Let C be an arbitrary, closed set in $U - U_0$, and let J be the image of C under the one-one map:

$$(u_4, \ldots, u_n) \longleftrightarrow \quad < 0, \frac{u_0}{2}, \frac{u_0}{2}, u_4, \ldots, u_n >$$

of U into A. This is the "arbitrary" part of the solution.

Let ρ be the distance function in U:

$$\rho(u, w) = \Sigma_4^n \ |u_j - w_j|$$

and let $\rho(u, W) = \inf_{w \in W} \rho(u, w)$ for sets W in U. Note that

$$\rho(u, W) \leq \rho(u, v) + \rho(v, W),$$

and that $\rho(u, U_0) = u_0$ for all $u \in U$.

The set K can now be defined; it consists of all imputations of the form:

$$< \frac{u_0 - \rho(u, \overline{C})}{2}, \ x, \ y, \ u_4, \ldots, u_n > , \quad u \in U,$$

where $\overline{C} = C \cup U_0$. Observe that the first component is always nonnegative, and that the sum of the second and third is determined by u; they run between 0 and $[u_0 + \rho(u, \overline{C})]/2$, sweeping out a line segment for each fixed u in $U - U_0$. Thus K is a continuous hypersurface of dimension $n - 2$, separating

A into two parts. It is disjoint from J since any imputation in $J \cap K$ would have $u \in C$ and $u_0 - \rho(u, \bar{C}) = 0$, implying both $u_0 \neq 0$ and $u_0 = 0$.

Finally, H is defined as $K \cap \text{dom } J$, the part of K dominated by J. Since $\alpha_1 = 0$ in J, we have $H = K \cap \text{dom}_1 J$. We shall show that $J \cup (K - H)$ is a solution.

Our construction is illustrated in Figure 1 for $n = 4$. J consists of two dots and a dash, so spaced as to make the "holes" H clearly visible. An inspection of the horizontal cross-sections shows a relationship between our solutions and the familiar symmetric and discriminatory solutions of the 3-person simple majority game. (The dotted lines are medians.)

The proof that $J \cup (K - H)$ is a solution is most conveniently presented as a series of "corrections" to the near-solution K. Thus, putting $V_1 = K$, we shall show

(I) $\quad V_1 \cap \text{dom } V_1 = \emptyset$

(II) $\quad V_1 \cup \text{dom } V_1 = A - J.$

Putting $V_2 = V_1 \cup J$, we shall find

(III) $\quad V_2 \cap \text{dom } V_2 = H$

(IV) $\quad V_2 \cup \text{dom } V_2 = A.$

Finally, $V_3 = V_2 - H$ will give us the desired result

(V) $\quad V_3 \cap \text{dom } V_3 = \emptyset$

(VI) $\quad V_3 \cup \text{dom } V_3 = A,$

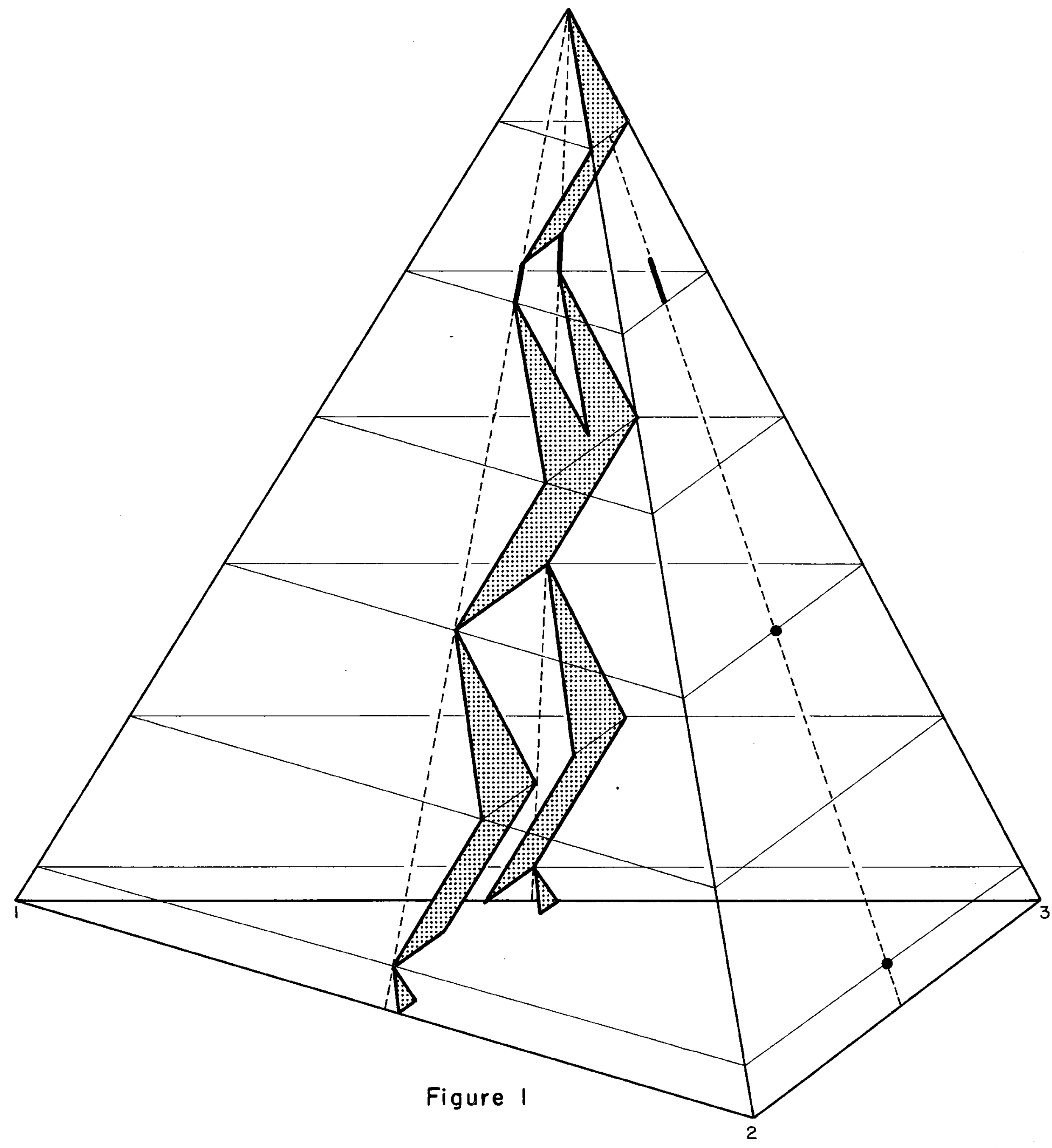

Figure 1

since $(K \cup J) - H = J \cup (K - H)$.

Proof of I: $K \cap \mathrm{dom}\, K = \emptyset$.

Suppose $\alpha \in \mathrm{dom}\, \beta$, with $\alpha, \beta \in K$:

$$\alpha = \left\langle \frac{u_0 - \rho(u, \bar{c})}{2}, x, y, u_4, \ldots, u_n \right\rangle,$$

$$\beta = \left\langle \frac{w_0 - \rho(w, \bar{c})}{2}, x', y', w_4, \ldots, w_n \right\rangle.$$

Then $u < w$ (in all components), and $\rho(u, w) = u_0 - w_0$. If $\alpha \in \mathrm{dom}_1 \beta$ then $x + y < x' + y'$, or

$$\frac{w_0 + \rho(w, \bar{c})}{2} > \frac{u_0 + \rho(u, \bar{c})}{2},$$

giving the contradiction $\rho(w, \bar{c}) > \rho(u, w) + \rho(u, \bar{c})$. But if $\alpha \in \mathrm{dom}_2 \beta$ or $\alpha \in \mathrm{dom}_3 \beta$, we have

$$\frac{w_0 - \rho(w, \bar{c})}{2} > \frac{u_0 - \rho(u, \bar{c})}{2}$$

giving the contradiction $\rho(u, \bar{c}) > \rho(u, w) + \rho(w, \bar{c})$.

Proof of II: $K \cup \mathrm{dom}\, K = A - J$.

Given $u \in U$, let $A(u)$ be the crosssection of A consisting of points of the form $\langle \alpha_1, \alpha_2, \alpha_3, u_4, \ldots, u_n \rangle$. Define $K(u) = K \cap A(u)$; it is a single point if $u \in U_0$, otherwise it is a line segment parallel to the $\alpha_1 = 0$ face of $A(u)$, not more than halfway from that face to the opposite vertex (see Figure 2). Define $L(u) = A(u) \cap \overline{\mathrm{dom}}\, K(u)$, where "$\overline{\mathrm{dom}}$" is taken

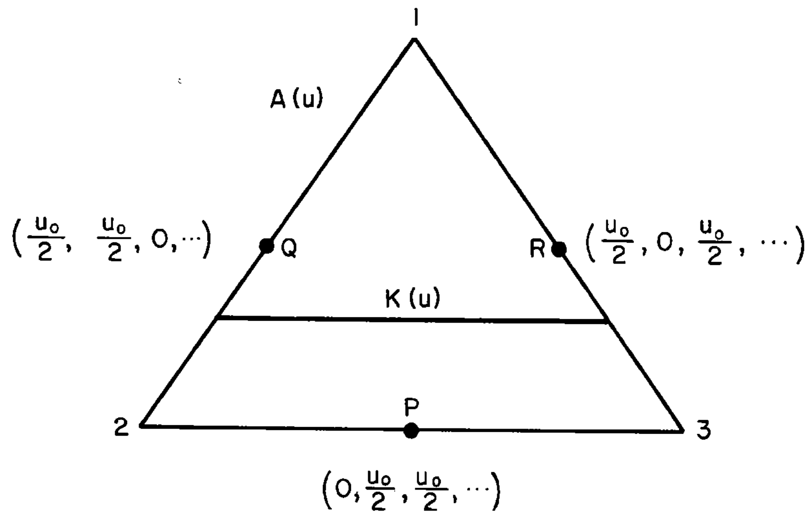

Figure 2

with respect to the two-person sets $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$. (The "$\overline{\text{dom}}$" pattern in A(u) is the same as the domination pattern of the 3-person simple majority game.) It is clear that K(u) and L(u) are disjoint and exhaust A(u), except in the extreme case $K(u) = \overline{QR}$ (see the figure), when the point P is not covered. But the extreme case occurs only when $\rho(u, \bar{C}) = 0$ and $u \notin U_0$ — that is, precisely when $u \in C$ — and we see that the uncovered point is an element of J. To sum up, if α is an arbitrary point of A, then it is either in J, in K, or in L(u) for some $u \notin U_0$. If the latter, we have in fact $\alpha \in$ dom K, since we can pick $u' > u$ and have $\alpha \in$ dom K(u') by making the differences $u'_j - u_j$ sufficiently small, by the uniform continuity of ρ. Thus $J \cup K \cup$ dom K = A, and half of our result, namely: $K \cup \text{dom } K \supseteq A - J$, has been established.

To prove the reverse inequality: $K \cup \text{dom } K \subseteq A - J$, it suffices to show that J and $\text{dom } K$ are disjoint. Suppose therefore that $\alpha \in \text{dom}\beta$ where $\alpha \in J \cap A(u)$, $\beta \in K \cap A(w)$ for some u, $w \in U$. If $\alpha \in \text{dom}_1\beta$, then we would have $\alpha_1 > \beta_1$, impossible since $\alpha_1 = 0$. If $\alpha \in \text{dom}_2\beta$, then $\alpha_3 < \beta_3$ and $u < w$, by definition. But $\alpha_3 = u_0/2$ and $\beta_3 \leq [w_0 + \rho(w, \overline{C})]/2$, hence $\rho(w, \overline{C}) > u_0 - w_0 = \rho(w, u)$. This is also impossible, since $u \in C$. The case $\alpha \in \text{dom}_3\beta$ is similar. This completes the proof.

<u>Proof of III: $(K \cup J) \cap \text{dom } (K \cup J) = H$.</u>

Expanding the left member of III, we have

$$(K \cap \text{dom } K) \cup (K \cap \text{dom } J) \cup (J \cap \text{dom } K) \cup (J \cap \text{dom } J).$$

The first term is empty by I; the second is H by definition; the third is empty by II. It remains to show that the fourth is empty. But this is trivial, since $\alpha \in \text{dom}\beta$ implies $\alpha_1 \neq \beta_1$, whereas $\alpha_1 \equiv 0$ in J.

<u>Proof of IV: $(K \cup J) \cup \text{dom } (K \cup J) = A$.</u>

Immediate, by II. (This is the trivial step of the first "correction.")

<u>Proof of V: $[J \cup (K - H)] \cap \text{dom } [J \cup (K - H)] = \emptyset$.</u>

Immediate, by III. (This is the trivial step of the second "correction.")

Proof of VI: $[J \cup (K - H)] \cup \text{dom}[J \cup (K - H)] = A.$

We intend to prove that both H and dom H are contained in dom $[J \cup (K - H)]$. Then the left member of VI can be written:

$$[J \cup (K - H)] \cup H \cup \text{dom}\ [J \cup (K - H)] \cup \text{dom}\ H,$$

which is equal to A, by IV. We shall show in particular that H and $\text{dom}_1 H$ are contained in dom J, and that $\text{dom}_2 H$ and $\text{dom}_3 H$ are contained in dom (K – H). First, by definition, we have $H \subseteq \text{dom}\ J$; also:

$$\text{dom}_1 H \subseteq \text{dom}_1(\text{dom}\ J) = \text{dom}_1(\text{dom}_1 J) \subseteq \text{dom}_1 J \subseteq \text{dom}\ J,$$

using the transistivity of "dom_k". (The last two inclusions are actually equalities.) This leaves $\text{dom}_2 H$ and $\text{dom}_3 H$. Suppose that $\gamma \in \text{dom}_2 \beta$, $\beta \in H$. Define β' by

$$\begin{cases} \beta_1' = \beta_1, \quad \beta_2' = 0, \quad \beta_3' = \beta_2 + \beta_3, \\ \beta_j' = \beta_j, \quad j = 4, \ldots, n. \end{cases}$$

Then β', like β, is in K, and clearly $\alpha \in \text{dom}_2\ \beta'$. However, β' is <u>not</u> in H. Indeed, if it were, there would be $\gamma \in J$ with $\beta' \in \text{dom}\ \gamma = \text{dom}_1 \gamma$ and we would have:

$$\beta_1' = \beta_1' + \beta_2' = 1 - \Sigma_3^n \beta_j' > 1 - \Sigma_3^n \gamma_j = \gamma_1 + \gamma_2$$

$$= \gamma_2 = \gamma_3 > \beta_3' = \beta_2' + \beta_3';$$

which is an impossibile inequality for $\beta' \in K$. Hence

$dom_2 H \subseteq dom_2(K - H)$. Similarly $dom_3 H \subseteq dom_3(K - H)$. This completes the proof.

Remarks.

(1) It is easy to see that $K - H$ is a connected set, by considering imputations like the β' of the last proof.

(2) Our particular games are not constant-sum, but their constant-sum extensions ($n + 1$ players) will possess discriminatory solutions that are essentially the same as those we have constructed. There is no reason to believe that solutions with arbitrary closed components cannot be found in a much wider class of games — perhaps in almost all games.

(3) It would be interesting to know of more general situations in which the method of successive corrections:

$$V_{n+1} = (V_n - X) \cup Y$$

where $V_n \cap \text{dom } V_n = X$, $V_n \cup \text{dom } V_n = A - Y$, can be shown to lead to a solution.

(4) Two interesting questions regarding cardinality are not answered by our present example, since $K - H$ is always uncountable, namely: Do countably infinite solutions ever occur? Is there an upper bound (say 2^n) to the number of imputations in a finite solution?