A SOLUTION CONTAINING 
AN ARBITRARY CLOSED COMPONENT 

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An n-person game solution is constructed around an arbitrary closed set, showing that there is practically no limit to the possible complexity of solutions.
A SOLUTION CONTAINING AN ARBITRARY CLOSED COMPONENT

A class of n-person game solutions is described in this note which consist of an arbitrary closed subset, J, of a certain \((n - 3)\)-dimensional region of the fundamental simplex, together with one other closed set, \(K - H\), depending on but disjoint from J. The arbitrariness of J illustrates the great variety of solution sets that are possible in the von Neumann-Morgenstern theory, and provides a ready source of examples, pleasant and pathological, for the testing of conjectures about the topological properties of solutions in general.

Let \(N = \{1, \ldots, n\}\) be the set of players, \(n \geq 4\). We shall consider the simple game with winning coalitions \(N - \{1\}\), \(N - \{2\}\), \(N - \{3\}\), and \(N\).* In the 0-1 normalization, the imputation space \(A\) is the simplex of nonnegative \(n\)-tuples \(\alpha = <\alpha_1, \ldots, \alpha_n>\) with sum 1. Let \(\text{dom}_k \beta\) denote the set of \(\alpha \in A\) with \(\alpha_i < \beta_i\), all \(i \neq k\). Then the set of imputations dominated by \(\beta\) is

\[
\text{dom} \beta = \text{dom}_1 \beta \cup \text{dom}_2 \beta \cup \text{dom}_3 \beta,
\]

and the defining properties of a solution \(V\) can be expressed

\[
V \cap \text{dom} V = \emptyset \quad \text{and} \quad V \cup \text{dom} V = A,
\]

where, of course, \(\text{dom} V\) denotes \(\sup_{\beta \in V} \text{dom} \beta\).

*It can be described as a direct product of the simple majority game on \(\{1, 2, 3\}\) and the pure bargaining (or "unanimous") game on \(\{1, \ldots, n\}\).
We shall now define the sets $J$, $K$, and $H$ referred to above. Let $U$ be the simplex of nonnegative $(n-3)$-tuples $u = (u_4, \ldots, u_n)$ with sum $\leq 1$. Let $u_0$ denote the difference $1 - \sum_{j=4}^{n} u_j$, and let $U_0$ be the face of $U$ on which $u_0 = 0$. Let $C$ be an arbitrary, closed set in $U - U_0$, and let $J$ be the image of $C$ under the one-one map:

$$(u_4, \ldots, u_n) \longleftrightarrow <0, \frac{u_0}{2}, \frac{u_0}{2}, u_4, \ldots, u_n>$$

of $U$ into $A$. This is the "arbitrary" part of the solution.

Let $\rho$ be the distance function in $U$:

$$\rho(u, w) = \sum_{j=4}^{n} |u_j - w_j|$$

and let $\rho(u, W) = \inf_{w \in W} \rho(u, w)$ for sets $W$ in $U$. Note that

$$\rho(u, W) \leq \rho(u, v) + \rho(v, W),$$

and that $\rho(u, U_0) = u_0$ for all $u \in U$.

The set $K$ can now be defined; it consists of all imputations of the form:

$$<\frac{u_0 - \rho(u, \bar{C})}{2}, x, y, u_4, \ldots, u_n>, \ u \in U,$$

where $\bar{C} = C \cup U_0$. Observe that the first component is always nonnegative, and that the sum of the second and third is determined by $u$; they run between 0 and $u_0 + \lceil \rho(u, \bar{C}) \rceil/2$, sweeping out a line segment for each fixed $u$ in $U - U_0$. Thus $K$ is a continuous hypersurface of dimension $n-2$, separating
A into two parts. It is disjoint from J since any imputation in J ∩ K would have u ∈ C and u₀ - ρ(u, C) = 0, implying both u₀ ≠ 0 and u₀ = 0.

Finally, H is defined as K ∩ dom J, the part of K dominated by J. Since α₁ = 0 in J, we have H = K ∩ dom₁ J. We shall show that J ∪ (K - H) is a solution.

Our construction is illustrated in Figure 1 for n = 4. J consists of two dots and a dash, so spaced as to make the "holes" H clearly visible. An inspection of the horizontal cross-sections shows a relationship between our solutions and the familiar symmetric and discriminatory solutions of the 3-person simple majority game. (The dotted lines are medians.)

The proof that J ∪ (K - H) is a solution is most conveniently presented as a series of "corrections" to the near-solution K. Thus, putting V₁ = K, we shall show

(I) \( V₁ ∩ \text{dom } V₁ = ∅ \)

(II) \( V₁ ∪ \text{dom } V₁ = A - J. \)

Putting \( V₂ = V₁ ∪ J \), we shall find

(III) \( V₂ ∩ \text{dom } V₂ = H \)

(IV) \( V₂ ∪ \text{dom } V₂ = A. \)

Finally, \( V₃ = V₂ - H \) will give us the desired result

(V) \( V₃ ∩ \text{dom } V₃ = ∅ \)

(VI) \( V₃ ∪ \text{dom } V₃ = A. \)
since \((K \cup J) - H = J \cup (K - H)\).

Proof of I: \(K \cap \text{dom } K = \emptyset\).

Suppose \(\alpha \in \text{dom } \beta\), with \(\alpha, \beta \in K\):

\[
\alpha = \left< \frac{u_0 - \rho(u, \overline{c})}{2}, x, y, u_1, \ldots, u_n \right>,
\]

\[
\beta = \left< \frac{w_0 - \rho(w, \overline{c})}{2}, x', y', w_1, \ldots, w_n \right>.
\]

Then \(u < w\) (in all components), and \(\rho(u, w) = u_0 - w_0\). If \(\alpha \in \text{dom}_1 \beta\) then \(x + y < x' + y'\), or

\[
\frac{w_0 + \rho(w, \overline{c})}{2} > \frac{u_0 + \rho(u, \overline{c})}{2},
\]

giving the contradiction \(\rho(w, \overline{c}) > \rho(u, w) + \rho(u, \overline{c})\). But if \(\alpha \in \text{dom}_2 \beta\) or \(\alpha \in \text{dom}_3 \beta\), we have

\[
\frac{w_0 - \rho(w, \overline{c})}{2} > \frac{u_0 - \rho(u, \overline{c})}{2},
\]

giving the contradiction \(\rho(u, \overline{c}) > \rho(u, w) + \rho(w, \overline{c})\).

Proof of II: \(K \cup \text{dom } K = A - J\).

Given \(u \in U\), let \(A(u)\) be the crosssection of \(A\) consisting of points of the form \(\left< \alpha_1, \alpha_2, \alpha_3, u_1, \ldots, u_n \right>\). Define \(K(u) = K \cap A(u)\); it is a single point if \(u \in U_0\), otherwise it is a line segment parallel to the \(\alpha_1 = 0\) face of \(A(u)\), not more than halfway from that face to the opposite vertex (see Figure 2). Define \(L(u) = A(u) \cap \overline{\text{dom } K(u)}\), where "\overline{\text{dom}}" is taken
with respect to the two-person sets \( \{1,2\} \), \( \{1,3\} \), and \( \{2,3\} \).

(The "\( \text{dom} \)" pattern in \( A(u) \) is the same as the domination pattern of the 3-person simple majority game.) It is clear that \( K(u) \) and \( L(u) \) are disjoint and exhaust \( A(u) \), except in the extreme case \( K(u) = QR \) (see the figure), when the point \( P \) is not covered. But the extreme case occurs only when \( p(u, \bar{C}) = 0 \) and \( u \notin U_0 \) — that is, precisely when \( u \in C \) — and we see that the uncovered point is an element of \( J \). To sum up, if \( \alpha \) is an arbitrary point of \( A \), then it is either in \( J \), in \( K \), or in \( L(u) \) for some \( u \notin U_0 \). If the latter, we have in fact \( \alpha \in \text{dom} K \), since we can pick \( u' > u \) and have \( \alpha \in \text{dom} K(u') \) by making the differences \( u'_j - u_j \) sufficiently small, by the uniform continuity of \( p \). Thus \( J \cup K \cup \text{dom} K = A \), and half of
our result, namely: $K \cup \text{dom } K \supseteq A - J$, has been established.

To prove the reverse inequality: $K \cup \text{dom } K \subseteq A - J$, it suffices to show that $J$ and $\text{dom } K$ are disjoint. Suppose therefore that $\alpha \in \text{dom } \beta$ where $\alpha \in J \cap A(u)$, $\beta \in K \cap A(w)$ for some $u, w \in U$. If $\alpha \in \text{dom}_1 \beta$, then we would have $\alpha_1 > \beta_1$, impossible since $\alpha_1 = 0$. If $\alpha \in \text{dom}_2 \beta$, then $\alpha_3 < \beta_3$ and $u < w$, by definition. But $\alpha_3 = u_0/2$ and $\beta_3 \leq \lfloor w_0 + \rho(w, \bar{c}) \rfloor/2$, hence $\rho(w, \bar{c}) > u_0 - w_0 = \rho(w, u)$. This is also impossible, since $u \in C$. The case $\alpha \in \text{dom}_3 \beta$ is similar. This completes the proof.

**Proof of III:** $(K \cup J) \cap \text{dom } (K \cup J) = H.$

Expanding the left member of III, we have

$$(K \cap \text{dom } K) \cup (K \cap \text{dom } J) \cup (J \cap \text{dom } K) \cup (J \cap \text{dom } J).$$

The first term is empty by I; the second is $H$ by definition; the third is empty by II. It remains to show that the fourth is empty. But this is trivial, since $\alpha \in \text{dom } \beta$ implies $\alpha_1 \neq \beta_1$, whereas $\alpha_1 = 0$ in $J$.

**Proof of IV:** $(K \cup J) \cup \text{dom } (K \cup J) = A.$

Immediate, by II. (This is the trivial step of the first "correction.")

**Proof of V:** $[\overline{J \cup (K - H)}] \cap \text{dom } [J \cup (K - H)] = \emptyset.$

Immediate, by III. (This is the trivial step of the second "correction.")
Proof of VI: \[ j \cup (K - H) \] \cup \text{dom} \[ j \cup (K - H) \] = A.

We intend to prove that both \( H \) and \( \text{dom} H \) are contained in \( \text{dom} [ j \cup (K - H) ] \). Then the left member of VI can be written:

\[ [ j \cup (K - H) ] \cup H \cup \text{dom} [ j \cup (K - H) ] \cup \text{dom} H, \]

which is equal to \( A \), by IV. We shall show in particular that \( H \) and \( \text{dom}_1 H \) are contained in \( \text{dom} J \), and that \( \text{dom}_2 H \) and \( \text{dom}_3 H \) are contained in \( \text{dom} (K - H) \). First, by definition, we have \( H \subseteq \text{dom} J \); also:

\[ \text{dom}_1 H \subseteq \text{dom}_1 (\text{dom} J) = \text{dom}_1 (\text{dom}_1 J) \subseteq \text{dom}_1 J \subseteq \text{dom} J, \]

using the transitivity of "dom". (The last two inclusions are actually equalities.) This leaves \( \text{dom}_2 H \) and \( \text{dom}_3 H \).

Suppose that \( \gamma \in \text{dom}_2 \beta, \beta \in H \). Define \( \beta' \) by

\[
\begin{cases}
\beta'_1 = \beta_1, & \beta'_2 = 0, & \beta'_3 = \beta_2 + \beta_3, \\
\beta'_j = \beta_j, & j = 4, \ldots, n.
\end{cases}
\]

Then \( \beta' \), like \( \beta \), is in \( K \), and clearly \( \alpha \in \text{dom}_2 \beta' \). However, \( \beta' \) is not in \( H \). Indeed, if it were, there would be \( \gamma \in J \) with \( \beta' \in \text{dom} \gamma = \text{dom}_1 \gamma \) and we would have:

\[
\begin{align*}
\beta'_1 &= \beta'_1 + \beta'_2 = 1 - \sum_3^n \beta'_j > 1 - \sum_3^n \gamma_j = \gamma_1 + \gamma_2 \\
&= \gamma_2 = \gamma_3 > \beta'_3 = \beta'_2 + \beta'_3;
\end{align*}
\]

which is an impossible inequality for \( \beta' \in K \). Hence
\( \text{dom}_2 H \subseteq \text{dom}_2 (K - H) \). Similarly \( \text{dom}_3 H \subseteq \text{dom}_3 (K - H) \). This completes the proof.

**Remarks.**

(1) It is easy to see that \( K - H \) is a connected set, by considering imputations like the \( \beta' \) of the last proof.

(2) Our particular games are not constant-sum, but their constant-sum extensions \((n + 1)\) players\) will possess discriminatory solutions that are essentially the same as those we have constructed. There is no reason to believe that solutions with arbitrary closed components cannot be found in a much wider class of games — perhaps in almost all games.

(3) It would be interesting to know of more general situations in which the method of successive corrections:

\[
V_{n+1} = (V_n - X) \cup Y
\]

where \( V_n \cap \text{dom } V_n = X, \ V_n \cup \text{dom } V_n = A - Y, \) can be shown to lead to a solution.

(4) Two interesting questions regarding cardinality are not answered by our present example, since \( K - H \) is always uncountable, namely: Do countably infinite solutions ever occur? Is there an upper bound (say \( 2^n \)) to the number of imputations in a finite solution?