A MIN—MAX SOLUTION OF AN INVENTORY PROBLEM

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ABSTRACT

In this paper we are concerned with the problem of purchasing a quantity of an item in anticipation of a future demand. It is assumed that only the mean and standard deviation of the demand distribution are known. A stock level is found which maximizes the minimum profit for all demand distributions with this mean and standard deviation. This stock level is then compared with the levels obtained by assuming several specific distributions.

I. INTRODUCTION

Most of the authors who have written on the subject of inventory control have made the assumption that future demand is either known precisely, or that it may be described by a definite probability distribution. In any specific problem the selection of a definite probability distribution is made on the basis of a number of factors, such as the sequence of past demands, judgments about trends, etc. For various reasons, however, these factors may be insufficient to estimate the future distribution. As an example, the sample size of the past demands may be quite small, or we may have reason to suspect that the future demand will come from a distribution which differs from that governing past history in an unpredictable way.

It is possible to formulate one aspect of this problem in the language of statistical decision theory. We may assume that the sequence of past demands represents a sample drawn from the same distribution that governs the future demands. A stockage policy is described as a choice of stock level depending on the sequence of past demands. As one possibility, we may examine all stockage policies and choose that one which maximizes the minimum expected profit for all demand distributions. For most inventory models, however, this is too conservative a procedure, and leads to a stockage policy of never stocking. There are other principles that may be employed within this framework, such as the principle of minimax regret. We shall, however, adopt a somewhat different procedure.

In order to discuss the procedure of this paper, we shall find it necessary to make a number of simplifying assumptions about the inventory model to be discussed. We shall assume, for example, that the only pertinent revenues and costs arise the following way. The unit cost of an item is c, regardless of the number purchased, and the unit selling price is r, regardless of the number sold. We shall only consider a one stage model, and assume initially that, subsequent to the demand period, leftover items have no salvage value. In a later section of the paper, we shall introduce a linear salvage function.

This model is very simple to solve if the demand distribution is known precisely. Let us denote the demand distribution by $\Phi(\xi)$. Then if we purchase an amount y, the expected profit is

$$-cy + r\int_0^{\infty} \min\,(y,\xi)\; d\,\Phi(\xi).$$

That value of y which maximizes expected profit is given by the solution of the equation

$$\int_y^{\infty} d\,\Phi(\xi) = \frac{c}{r}\ .$$

What we shall do in this paper is to assume that only the mean (μ) and standard deviation (σ) of the demand distribution are known. These may either be estimated from the past history or determined in other ways. We shall then choose the stockage policy so as to maximize the minimum profit that would occur, considering all distributions with the given mean and standard deviation. In symbols this means that we select the maximizing y in the formula

$$\max_{y} \min_{\substack{\int \xi d\Phi=\mu \\ \int (\xi-\mu)^2 d\Phi=\sigma^2}} \left[r \int_0^{\infty} \min\,(y,\xi)\; d\Phi(\xi) - cy \right].$$

We intend in the next section of this paper to obtain the value of y which maximizes the above expression as a function of μ, σ, and the ratio of cost to price c/r. The appropriate value of y is related in a very simple manner to these parameters, and is given by an expression of the form

$$y = \mu + \sigma f(c/r).$$

One of the objections that can be raised against this approach is that in maximizing the minimum profit we are admitting too many distributions into consideration, and selecting a stock level y that corresponds to a very unreasonable type of distribution. For example, there are a number of specific instances of our problem in which the Poisson distribution seems to describe the future demands fairly accurately. (In this case $\sigma^2 = \mu$.) As a partial answer to this objection we shall compare the stock level obtained above with that obtained by assuming a Poisson, distribution. As we shall show, for a large range of values of c/r, $(.05 < c/r < .95)$ the two stock levels are virtually identical. For $c/r < .05$ the min-max criterion indicates a higher stock level than the Poisson, and for $c/r > .95$ a smaller stock.

II. THE MATHEMATICAL SOLUTION OF THE PROBLEM

We are interested in determining the value of y which maximizes the following expression:

$$P(y) = \min_{\substack{\int \xi d\Phi = \mu \\ \int (\xi-\mu)^2 d\Phi = \sigma^2}} \left[r\int_0^\infty \min(y,\xi)d\Phi(\xi) - cy \right]$$

The computation of the function $P(y)$ is based upon the fact that a minimizing distribution may be chosen with all of its weight concentrated at two points. The proof that only two point distributions need be examined depends on the following lemma.

Lemma: Let y, μ, and σ be fixed. Then there exists a quadratic function $Q(\xi) = \alpha + \beta\xi + \gamma\xi^2$ such that $Q(\xi) \leq \min(y,\xi)$ for $\xi \geq 0$ with equality holding at only two points a and b. Moreover there exists a two point distribution situated at a and b, with mean μ and standard deviation σ.

Let us assume for the moment that this lemma is correct, and let us designate the particular two point distribution described in the lemma by $F(\xi)$. Then it is easy to see that $F(\xi)$ actually minimizes $\int_0^\infty \min(y,\xi)\, d\Phi(\xi)$ for all distributions $\Phi(\xi)$ with mean μ and standard deviation σ. For let $\Phi(\xi)$ be such a distribution. Then

$$\int_0^\infty \min(y,\xi)\, d\Phi(\xi)$$

$$= \int_0^\infty [\min(y,\xi) - Q(\xi)]\, d\Phi(\xi) + \int_0^\infty Q(\xi)\, d\Phi(\xi)$$

$$\geq \int_0^\infty Q(\xi)\, d\Phi(\xi) = \alpha + \beta\mu + \gamma(\mu^2 + \sigma^2)$$

But

$$\int_0^\infty \min(y,\xi)\, dF(\xi) = \int_0^\infty Q(\xi)\, dF(\xi)$$

$$= \alpha + \beta\mu + \gamma(\mu^2 + \sigma^2),$$

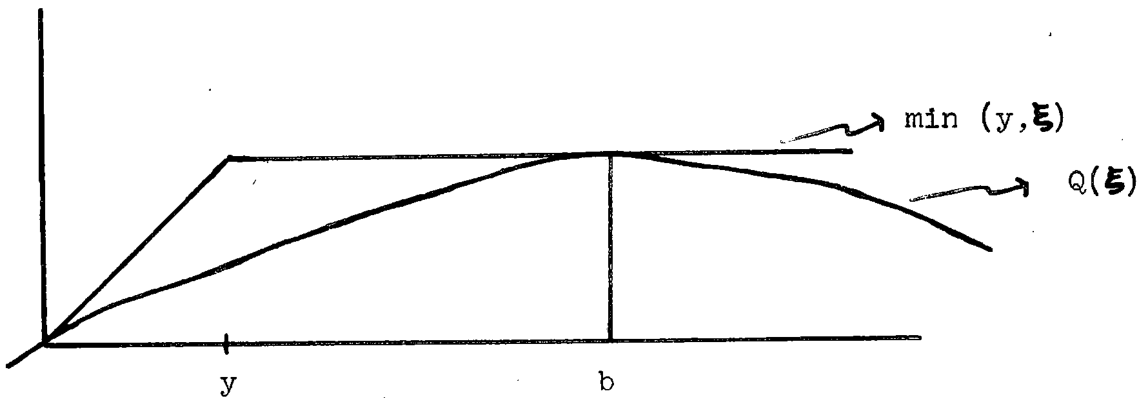
since $Q(\xi)$ and $\min(y,\xi)$ are equal on those points where F has all of its weight. Therefore subject to verification of the above lemma, we have demonstrated that a minimizing two-point distribution may be selected.

The proof of the lemma breaks down into two cases.

Case 1. $\quad y \leq \dfrac{\mu^2 + \sigma^2}{2\mu}$.

In this case, let

$$Q(\xi) = \frac{2y}{\left(\mu + \frac{\sigma^2}{\mu}\right)}\, \xi - \frac{y}{\left(\mu + \frac{\sigma^2}{\mu}\right)^2}\, \xi^2.$$



It may be verified that $Q(\xi)$ is tangent to the curve $\min(y,\xi)$ at $\xi = \mu + \frac{\sigma^2}{\mu}$, and therefore lies below that part of the curve parallel to the ξ-axis. In order to have $Q(\xi) < \xi$ for $0 < \xi < y$, we must have

$$\frac{2y}{(\mu + \frac{\sigma^2}{\mu})} - \frac{y\xi}{(\mu + \frac{\sigma^2}{\mu})^2} < 1$$

and this follows from the defining condition of case 1. The two point distribution referred to in the lemma has mass $\frac{\sigma^2}{\mu^2 + \sigma^2}$ at 0 and $\frac{\mu^2}{\mu^2 + \sigma^2}$ at $\xi = \mu + \frac{\sigma^2}{\mu}$.

Case 2. $y > \frac{\mu^2 + \sigma^2}{2\mu}$.

In this case the quadratic $Q(\xi)$ does not pass through the origin, but is tangent to the curve $\min(y,\xi)$ at two points, one less than y, and the other greater. Let us, first of all, remark that if a two point distribution puts mass p at a and $1-p$ at b, with $a < b$, then

$$b = \mu + \frac{\sigma^2}{\mu - a} \quad \text{and} \quad p = \frac{\sigma^2}{(\mu-a)^2 + \sigma^2}.$$

Therefore all two-point distributions with a given mean and standard deviation, form a one parameter family, indexed by the parameter a (with $0 \leq a < \mu$).

For any value of $0 \leq a < \min(y,\mu)$, let us construct a quadratic $Q_a(\xi)$ which is tangent to $\min(y,\xi)$ at $\xi = a$, and parallel to the ξ axis at $\xi = b$. Let $h(a)$ represent the height of $Q_a(\xi)$ above the curve $\min(y,\xi)$ at $\xi = b$, i.e.

$$h(a) = Q_a(b) - \min(y,b).$$

The lemma will be demonstrated if we can show that a value of a exists with $h(a) = 0$. In order to demonstrate this, let us examine the function h for several values of a. For $a = 0$, a simple calculation shows us that

$$Q_0(\xi) = \xi - \frac{1}{2b}\xi^2,$$

and therefore

$$h(0) < 0 \text{ if } y > \frac{b}{2} = \frac{\mu^2 + \sigma^2}{2\mu}.$$

Since this latter condition is the defining condition of case 2, it follows that $h(0) < 0$.

If $y < \mu$ then a glance at the geometry of the situation shows us that $h(y) > 0$. On the other hand if $y \geq \mu$, then $\lim_{a \to \mu} Q_a(\xi) = \xi$, and $b \to \infty$ so that $\lim_{a \to \mu} h(a) = \infty$. We have therefore shown that there is a value of a between 0 and $\min(y,\mu)$ with $h(a) = 0$. The corresponding quadratic satisfys the condition of the lemma, and the distribution associated with this quadratic is the one defined by the corresponding value of a.

We now turn to a computation of the function $P(y)$. The method that we have used to demonstrate the above lemma shows us that if $y \leq \frac{\mu^2 + \sigma^2}{2\mu}$, then a minimizing distribution is the two point distribution with mass $\frac{\sigma^2}{\mu^2 + \sigma^2}$ at 0 and $\frac{\mu^2}{\mu^2 + \sigma^2}$ at $\mu + \frac{\sigma^2}{\mu}$. Therefore for $y \leq \frac{\mu^2 + \sigma^2}{2\mu}$,

$$P(y) = r \frac{\mu^2 y}{\mu^2 + \sigma^2} - cy.$$

For $y > \frac{\mu^2 + \sigma^2}{2\mu}$, we may use the following procedure to determine $P(y)$. Evaluate the expression

$$r \int_0^{\infty} \min (y, \xi)\, d\Phi - cy$$

for each fixed y as a function of a, and choose the value of a which minimizes this function. The answer is

$$P(y) = r \left\{ \frac{\mu + y}{2} - 1/2 \sqrt{(y - \mu)^2 + \sigma^2} \right\} - cy.$$

The next question that arises is the determination of that value of y which maximizes the minimum expected profit $P(y)$. We notice first of all that the function $P(y)$ is concave, and has a continuous derivative at all points. We shall find it convenient to consider two cases separately.

Case 1. $\frac{c}{r} \left(1 + \frac{\sigma^2}{\mu^2}\right) < 1.$

In this case, the formula given above shows that $P(y)$ is strictly increasing in the interval $(0, \frac{\mu^2 + \sigma^2}{2\mu})$. Since $P(y)$

is negative for every large y, we see that the maximum will be attained for $y > \frac{\mu^2 + \sigma^2}{2\mu}$. We have

$$P(y) = r\left\{\frac{\mu + y}{2} - 1/2 \sqrt{(y - \mu)^2 + \sigma^2}\right\} - cy$$

and

$$P'(y) = r\left\{1/2 - 1/2 \left(\frac{y - \mu}{\sqrt{(y - \mu)^2 + \sigma^2}}\right)\right\} - c.$$

If we equate $P'(y)$ to zero we obtain

$$y = \mu + \sigma f(c/r)$$

where

$$f(a) = 1/2\left(\frac{1 - 2a}{\sqrt{a(1 - a)}}\right) \text{ for } 0 < a < 1.$$

We shall discuss the meaning of this stock level in the next section.

Case 2. $\frac{c}{r}\left(1 + \frac{\sigma^2}{\mu^2}\right) > 1.$

In this case $P(y)$ is initially decreasing. It is a concave function and therefore always regative, except for $y = 0$. In this case the optimal policy is to buy none of the stock. The intuitive idea here is that the ratio of standard deviation to mean is so high, that the transaction is a very risky one, and unless the ratio of cost to price is low, it is best not to enter into it at all.

The results of this section may be summarized as follows:

THE STOCK LEVEL WHICH MAXIMIZES THE MINIMUM EXPECTED PROFIT FOR ALL DEMAND DISTRIBUTIONS WITH MEAN μ AND STANDARD

DEVIATION σ IS GIVEN BY

$$y = \begin{bmatrix} 0 \; ; & \text{FOR } \frac{c}{r}\left(1 + \frac{\sigma^2}{\mu^2}\right) > 1 \\ \mu + \sigma f\left(\frac{c}{r}\right); & \text{FOR } \frac{c}{r}\left(1 + \frac{\sigma^2}{\mu^2}\right) < 1 \end{bmatrix}$$

WHERE

$$f(a) = 1/2\left(\frac{1 - 2a}{\sqrt{a(1 - a)}}\right).$$

It is easy to verify that if this policy is used, the minimum expected profit is the larger of

$$(r - c)\mu - \sigma\sqrt{c(r - c)} \quad \text{and} \quad 0,$$

and that this amount is guaranteed against all distributions with mean μ and standard deviation σ. The term $(r - c)\mu$ represents the profit if the demand is known to be precisely μ, and the term $\sigma\sqrt{c(r - c)}$ may be looked upon as the loss attributable to an unprecise knowledge of the demand. The point at which zero units are stocked is precisely the point at which the loss term becomes greater than the profit term.

III. SOME OBSERVATIONS ON THE SOLUTION

The solution given in the previous section breaks into two parts: if $c\left(1 + \frac{\sigma^2}{\mu^2}\right) > r$, we stock none of the item; if $c\left(1 + \frac{\sigma^2}{\mu^2}\right) < r$, we stock an amount $y = \mu + \sigma f(c/r)$, where $f(a) = 1/2\left(\frac{1 - 2a}{\sqrt{a(1 - a)}}\right).$ This section will be devoted to a discussion of some of the properties of this solution, and a comparison with the solution obtained by assuming a Poisson distribution.

We notice that the function $f(a)$ is strictly decreasing from $+\infty$ at 0 to $-\infty$ at 1. It is positive between 0 and $1/2$, negative between $1/2$ and 1, and equal to zero at $1/2$. It is easy to verify that the relationship $f(a) = -f(1-a)$ is satisfied.

These properties of $f(a)$ imply the following general conclusions about the stockage level y:

1. If $c/r < 1/2$, you stock more than the mean demand, and the higher the standard deviation, the more you stock (unless $\frac{c}{r}\left(1 + \frac{\sigma^2}{\mu^2}\right) > 1$).

2. If $c/r > 1/2$, you stock less than the mean and the higher the standard deviation, the less you stock.

3. If $c/r = 1/2$, you stock the mean, independently of the value of the standard deviation, unless $1/2\left(1 + \frac{\sigma^2}{\mu^2}\right) > 1$, or $\sigma > \mu$, in which case you stock zero.

As we maintained previously, there are a number of practical instances of this problem, in which it is reasonable to assume that the demand function is given by a distribution close to a Poisson distribution. We shall present in the following tables a comparison of the stock levels obtained by assuming a Poisson distribution (the technique is the one outlined in the introduction), and by using our stock formula with $\sigma = \sqrt{\mu}$.

$\mu = 100; \quad \sigma = 10$

$\frac{c}{r}$	Min–Max Level	Poisson Level
.98	63.7	80+
.95	79.4	84+
.9	86.7	87+
.8	92.5	92+
.6	97.8	98–
.5	100.0	100+
.3	104.5	105+
.1	113.3	113+
.05	120.6	117+
.02	134.3	121+

$\mu = 36; \quad \sigma = 6$

$\frac{c}{r}$	Min–Max Level	Poisson Level
.98	0	24+
.95	23.6	26+
.9	28.0	29–
.8	31.5	31+
.6	34.7	35–
.5	36.0	36+
.3	38.7	40–
.1	44.0	44+
.05	48.4	47–
.02	56.6	49+

$\mu = 4; \quad \sigma = 2$

$\frac{c}{r}$	Min–Max Level	Poisson Level
.98	0	1+
.95	0	2–
.9	0	2+
.75	2.84	3+
.5	4	4+
.2	5.5	6+
.1	6.64	7+
.05	8.12	8+
.02	10.86	9+
.01	13.84	10–

$\mu = .25; \quad \sigma = .5$

$\frac{c}{r}$	Min–Max Level	Poisson Level
.98	0	0+
.9	0	0+
.6	0	0+
.4	0	1–
.3	0	1–
.2	.62	1–
.1	.91	1+
.02	1.96	2–
.01	2.71	2+
.005	3.75	3–

These charts indicate that the min-max solution stocks less than the Poisson for a high ratio of cost to price, but stocks more than the Poisson for a low ratio. For intermediary values of this ratio (say, $.05 < c/r < .95$) the two stockage policies agree quite well.

For large values of the mean it is possible to approximate the Poisson distribution by a Gaussian distribution with mean μ and standard deviation $\sqrt{\mu}$. The appropriate stock level would then be given by $y = \mu + \sqrt{\mu}\ g\left(\frac{c}{r}\right)$, where $g(a)$ is defined by

$$a = \frac{1}{\sqrt{2\pi}} \int_{g(a)}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy.$$

A comparison between $g(a)$ and $f(a)$ is given by Figure 1. It may be seen that for $.1 \leq a \leq .9$, the two curves differ by no more than $.2$ so that the two stockage policies differ by no more than $.2\sqrt{\mu}$.

IV. THE RESULT WITH SALVAGE VALUE

We may be interested in modifying our original assumption that the excess of stock over demand has no salvage value. For example, we may treat by the same methods as above, the case in which the excess of stock over demand may be sold for a reduced price r', with $r' < c < r$. The stock levels then become

$$y = \begin{bmatrix} 0 \; ; & \text{for } \left(\frac{c - r'}{r - r'}\right)\left(1 + \frac{\sigma^2}{\mu^2}\right) > 1 \\ \\ \mu + \sigma f\left(\frac{c - r'}{r - r'}\right) \; ; & \text{for } \left(\frac{c - r'}{r - r'}\right)\left(1 + \frac{\sigma^2}{\mu^2}\right) < 1 \end{bmatrix} .$$

This formula reflects the intuitively plausible observation that the automatic disposal of surplus items at a price r' may be described by a translation of both the cost and price by an amount r'.

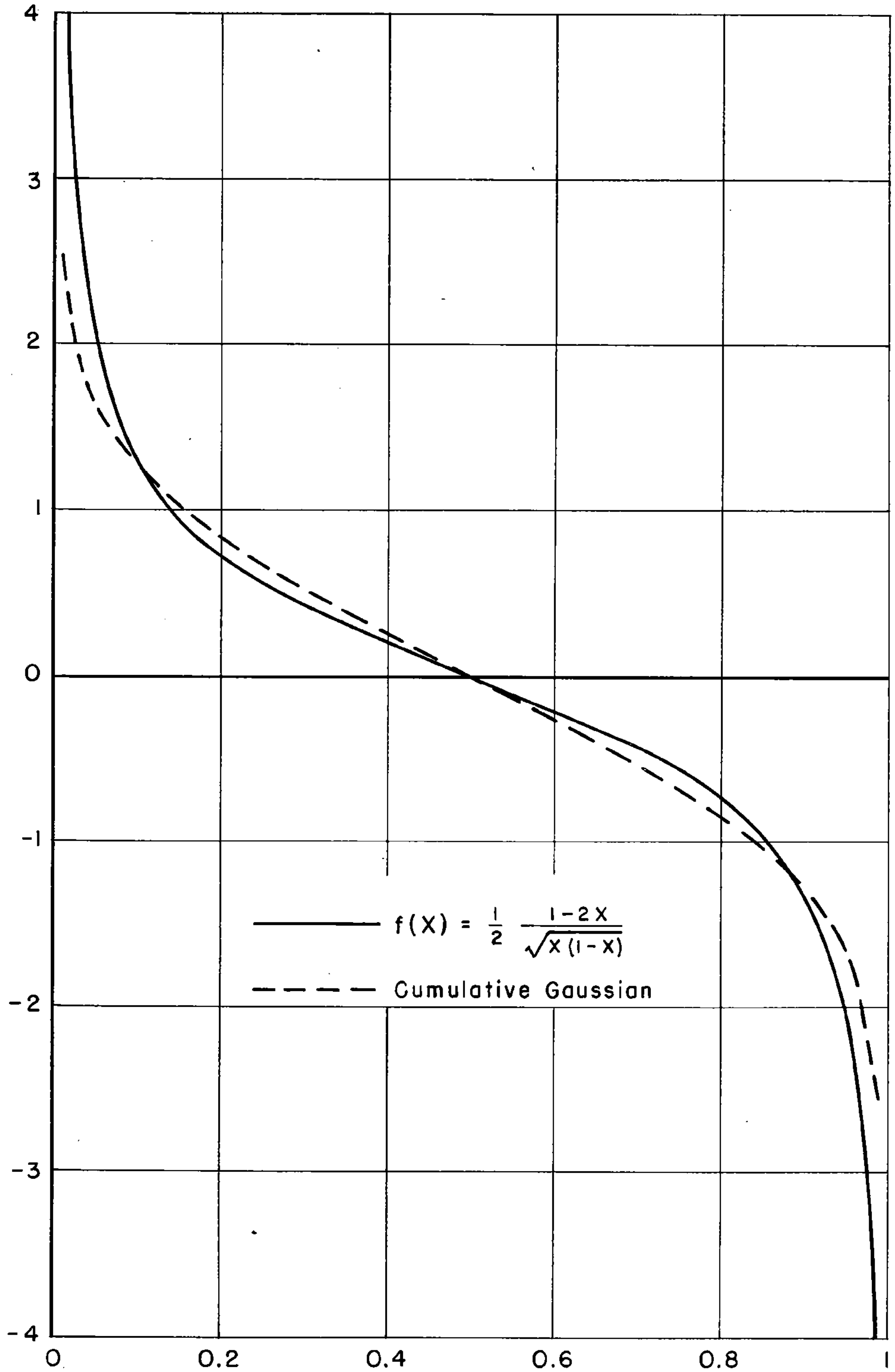

4
3
2
1
0
-1
-2
-3
-4
0
0.2
0.4
0.6
0.8
1
$f(X) = \frac{1}{2} \frac{1-2X}{\sqrt{X(1-X)}}$
Cumulative Gaussian

Fig. I