ON INTEGER AND PARTIAL INTEGER LINEAR PROGRAMMING PROBLEMS

George B. Dantzig

June 20, 1958

P-1410

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

10

SUMMARY

The general problem considered is how to solve a linear program in which some variables must have integer values. If the integer condition is ignored the linear programming solution often yields integer solutions and the problem is solved. However if the solution turns out fractional it is necessary to add additional linear constraints and repeat the procedure until an integer solution is achieved. A recent result of R.E. Gomory showed how to add additional constraints if all variables must have integer values. In this paper the partial integer problem is discussed and Gomory's procedure is extended to the case where all but one variable must have integer values.

ON INTEGER AND PARTIAL INTEGER LINEAR PROGRAMMING PROBLEMS

Our objective is to discuss briefly the class of linear constraints discovered by Gomory [1] that may be added to a linear programming problem to cut off fractional extreme points and then extend the method to get conditions for the simplest mixed case of n non-negative variables where <u>one</u> variable may have any non-negative value and the remaining n - 1 variables must have integer values. Under certain conditions this approach may also be used for r non-negative valued variables and n - r non-negative integer variables.*

<u>Pure Integer Case</u>: Suppose that the linear programming problem to be solved partially in integers is in canonical form (1) relative to some set of variables $x_1, x_2, \ldots, x_m$ and that the basic solution is feasible:

$$
(1)\quad \begin{array}{l}
x_1 \qquad\qquad + \bar{a}_{1m+1}x_{m+1} + \cdots + \bar{a}_{1n}x_n = \bar{b}_1 \\
\quad x_2 \qquad + \bar{a}_{2m+1}x_{m+1} + \cdots + \bar{a}_{2n}x_n = \bar{b}_2 \\
\qquad \ddots \\
\qquad\quad x_m + \bar{a}_{mm+1}x_{m+1} + \cdots + \bar{a}_{mn}x_n = \bar{b}_m \,.
\end{array}
$$

If all $\bar{b}_i \geq 0$ are integers then the set of values $(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m, 0, \ldots, 0)$ is an integer solution to the original linear programming problem. If the solution is not in integers then at least one $\bar{b}_i$ is fractional say $\bar{b}_1$. In this situation write $\bar{a}_{1j}, \bar{b}_1$ as integers (positive or negative), N_{1j}, plus

*Based on a discussion with Wheaton Smith and Tom Kawaratani.

a positive proper fractions, f_{1j}, that is to say

(2) $$\bar{a}_{1j} = N_{1j} + f_{1j} \qquad 1 > f_{1j} \geq 0$$
$$\bar{b}_1 = N_{10} + f_{10}$$

Denote by k the linear form

(3) $$f_{1m+1}x_{m+1} + f_{1m+2}x_{m+2} + \cdots + f_{1n}x_n = k$$

It is clear that for any feasible solution to the linear programming problem $k \geq 0$. We are now interested in finding a lower bound for k for any feasible <u>integer</u> solution. To this end subtract the form from the first equation of (1) yielding

(4) $$x_1 + N_{1m+1}\bar{a}_{1m+1} + N_{1m+2} + \cdots + N_{1n}\bar{a}_n - N_{10} = + f_{10} - k$$

where the integral part of $\bar{b}_1$ has been transferred to the left. The left-hand-side for integer solutions is an integer which can never exceed f_{10} because $k \geq 0$, but f_{10} is a proper positive fraction hence $f_{10} - k$ can never exceed 0 or $k \geq f_{10}$; whence from (3)

(5) $$f_{1m+1}x_{m+1} + f_{1m+2}x_{m+2} + \cdots + f_{1n}x_n \geq f_{10}$$

must hold for all integral solutions. This condition does not hold for the basic feasible solution at hand so that adding this linear inequality constraint to the problem actually restricts the convex of solutions of the linear programming problem.

The new condition that Gomory adds to the linear programming problem is the equation

$$f_{1m+1}x_{m+1} + f_{2m+2}x_{m+2} + \cdots + f_{1n}x_n - f_{10} = x_{n+1} \tag{5.1}$$

where $x_{n+1} \geq 0$ is a new variable. It is clear that $x_{n+1} = k - f_{10}$ and by (4) that x_{n+1} is an integer for any integer solutions of the original problem. Thus the augmented problem has one more equation and one more variable which must also be solved in integers. Gomory states that he can prove that the repeated addition of such conditions can lead to an integral solution in a finite number of steps. (5) is not the only linear inequality constraint that can be generated out of the first equation of (1). Indeed multiplying by any integer t yields

$$t\, x_1 + \sum_{j=m+1}^{n} t\, \bar{a}_{1j}x_j = t\, \bar{b}_1 \tag{6}$$

If now $t\, \bar{a}_{1j}$ and $t\, \bar{b}_1$ are split into an integer part, N^t_{1j}, (positive or negative) and a positive proper fractional f^t_{1j} so that

$$(7) \qquad \begin{aligned} t\, \bar{a}_{1j} &= N^t_{1j} + f^t_{1j} \qquad\qquad 1 > f^t_{1j} \geq 0 \\ t\, \bar{b}_1 &= N^t_{0j} + f^t_{0j} \end{aligned}$$

then analogous to (5) one generates the <u>infinite</u> class of constraints

(8) $$f^t_{1,m+1}x_{m+1} + f^t_{1m+2}x_{m+2} + \cdots + f^t_{1n}x_n \geq f^t_{10}$$

for $t = 1,2,\ldots$.

However if the $\bar{a}_1$ and $\bar{b}_1$ are rational then the class of inequalities is really finite,—for let D be their least common denominator, then it is not difficult to see that $t\ \bar{a}_{1j}$ and $t\ \bar{b}_1$ will have the same fractional parts for $t = k$ and $t = D + k$.

In particular setting $t = D - 1$ yields a relation complimentary to (5)

(9) $$\bar{f}_{1m+1}x_{m+1} + \bar{f}_{1m+2}x_{m+2} + \cdots + \bar{f}_{1n}x_n \geq \bar{f}_{10}$$

where

(10) $$\bar{f}_{1j} = \begin{cases} 1 - f_{1j} & \text{for } 1 > f_{1j} > 0 \\ 0 & \text{for } \quad f_{1j} = 0 \ . \end{cases}$$

This same relation holds for non-rational $\bar{a}_{1j}$, $\bar{b}_1$. This may be verified by a suitable limit process using rational approximations but may also be inferred directly by setting

(11) $$\begin{aligned} \bar{a}_{1j} &= N_{1j} - \bar{f}_{1j} \qquad\qquad 1 > \bar{f}_{1j} \geq 0 \\ \bar{b}_1 &= N_{10} - \bar{f}_{10} \end{aligned}$$

where N_{1j} is an integer positive or negative and $\bar{f}_{1j}$ a proper non-negative fraction. In this case the form analogous to (3) is <u>added to</u> (instead of subtracted from) the first equation of (1) to generate the analogue to (4), [to prove that the ~~value of the form must exceed $\bar{f}_{10}$.~~]

Adding expressions (5) and (9) yields the relation

(12) $$x_{m+1} + x_{m+2} + \cdots + x_n \geq 1$$

if all fractional parts are strictly positive. However if $f_{1r} = 0$ then the term x_r is dropped so that

(13) $$x_{m+1} + \cdots + x_{r-1} + x_{r+1} + \cdots + x_n \geq 1$$

In general all terms x_j corresponding to $f_{1j} = 0$ are omitted.

It is interesting to note that the expressions (12) can easily be derived directly by noting that the linear form on the left is always non-negative, and for integer x_j must be an integer. If the sum vanished then all $x_j = 0$ for non-basic variables and this would imply the unique fractional basic feasible solution. Hence for integer solutions at least one non-basic $x_j > 0$, which means for non-negative integer, non-basic x_j's that their sum must exceed unity. This relation (not due to Gomory) can be found in [2]. It is clear that condition (12) while simpler than is also weaker than those proposed by Gomory because two Gomory type conditions (5) and (9) imply (12).

Partial Integer Case:

Let us suppose in (1) that all x_j except the non-basic variable x_{m+1} must be in integers. Then the constraint to be added to the system analogous to (5) or (9) is

(14) $$\bar{a}_{1m+1}x_{m+1} + f_{m+2}x_{m+2} + \cdots + f_n x_n \geq f_0 \quad \text{if} \quad \bar{a}_{1m+1} \geq 0$$

or

$$-\bar{a}_{1m+1}x_{m+1} + \bar{f}_{m+2}x_{m+2} + \cdots + \bar{f}_n x_n \geq \bar{f}_0 \quad \text{if} \quad \bar{a}_{1m+1} \leq 0 .$$

The proof is similar to those given earlier and will therefore not be repeated.

If several x_j say $x_{m+1}, x_{m+2}, \cdots, x_p$ are not required to be integers and it is known that the form

(15) $$\bar{a}_{1m+1}x_{m+1} + \cdots + \bar{a}_{1p}x_p$$

is non-negative [as for example if it happens that all $\bar{a}_{1j} \geq 0$ for $j = m + 1, \cdots, p$] then, analogous to (5), the new constraint is

(16) $$\bar{a}_{1m+1}x_{m+1} + \cdots + \bar{a}_{1p}x_p + f_{p+1}x_{p+1} + \cdots + f_n x_n \geq f_0$$

where it is assumed that

(17) $$\bar{a}_{1m+1}x_{m+1} + \cdots + \bar{a}_{1p}x_p \geq 0 .$$

More generally analogous to (8) one can generate the class of new constraints for $t = 1,2,\ldots$

(18) $$\bar{a}_{1m+1}x_{m+1} + \cdots + \bar{a}_{1p}x_p + f^t_{p+1}x_{p+1} + \cdots + f^t_n x_n \geq f^t_0$$

assuming (17) to be true or if instead of (17),

(19) $$\bar{a}_{1m+1}x_{m+1} + \cdots + \bar{a}_{1p}x_p \leq 0$$

then

$$(20) \quad -\bar{a}_{1m+1}x_{m+1} \cdots -\bar{a}_{1p}\, x_p + \bar{f}^t_{p+1}x_{p+1} + \cdots + \bar{f}^t_n x_n \geq \bar{f}^t_0$$

where

$$\bar{f}_j = \begin{cases} 1 - f_j & \text{for } 1 > f_j > 0 \\ 0 & \text{for } \quad f_j = 0 \,. \end{cases}$$

However, only the case of one variable which need not be an integer can be considered as completely solved by this approach, since conditions (17) and (19) need not hold in general for $p > 1$.

REMARK 1: It is possible to use a combination of devices when there is more than one variable which need not be in integers. If the only equation in the canonical form in which they appear is the one corresponding to x_1 whose value is fractional, then one could try to solve <u>two alternative linear programming problems</u>, one in which (18) is used as the added constraint, the other where (20) is used instead. As long as this situation does not reoccur in the new problems this may be a useful approach.

REMARK 2: On the other hand if x_{m+1} and x_{m+2} say are non-basic variables which may take on fractional values and these occur with <u>rational</u> non-vanishing coefficients in the equations associated with integer variables, say x_1 and x_2 in the canonical form, then it is possible to form from them a single equation in which say x_{m+2} is eliminated and x_1 and x_2 have integer coefficients say

$$N_1X_1 + N_2X_2 + \alpha_{m+1}x_{m+1} + 0\, x_{m+2} + \cdots + \alpha_n x_n = N_1\bar{b}_1 + N_2\bar{b}_2$$

where N_i are positive or negative integers. If the right-hand-side is fractional then an added constraint analogous to (14) may be added. This approach of course may be generalized to several variables.

REMARK 3: It is not known whether an admissible solution can be obtained for the partial integer case by adding a finite number of constraints of the type indicated.

REFERENCES

1. Gomory, Ralph E., "Essentials of an Algorithm for Integer Solutions to Linear Programs, " communicated to the Bull. Amer. Math. Society in letter from Princeton dated April 23, 1958.

2. Dantzig, G.B., Solving Linear Programs in Integers, The RAND Corporation, Paper P-1359, May 5, 1958.