BIRTH-DEATH PROCESSES AND TANDEM QUEUES *

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P-863

June 22, 1956

*To be submitted for publication in the Annals of Mathematical Statistics.

(6)

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We shall point out a simple property of stationary birth and death processes, which implies that for certain queues, the output process is closely related to the input process. We apply this to a situation where customers proceed to a second queue after having been processed at a first queue. Theorem 2 was first essentially proved by P. J. Burke [6], by a different technique.

Let $n(t)$ be the population of a stationary birth and death process at time t. Let the matrix of transition probabilities $P = (P_{ij}(h))$ from state i to state j in time h be

$$(1) \qquad \begin{cases} P_{n,n+1}(h)=\lambda_n h+o(h), & \lambda_n > 0,\ n=0,1,\ldots, \\ P_{n,n-1}(h)=\mu_n h+o(h), & \mu_n > 0,\ n=1,2,\ldots,\ \ \mu_0=0, \\ P_{nn}(h)=1-\lambda_n h-\mu_n h+o(h). & \end{cases}$$

We assume that $p_k=\Pr\left\{n(t)=k\right\} > 0$, and use p to denote the vector $(p_0,p_1,\ldots)$. Then p satisfies

$$(2) \qquad pL=0$$

with

$$L=\lim_{h\to 0} h^{-1}(P-E).$$

The time reversed process $r(t)=n(-t)$ is, of course, also a Markov process. The following result is quite possibly known (Heuristic forms of the necessary argument date to Ehrenfest [4]).

THEOREM 1. <u>$r(t)$ is a birth and death process whose transition matrix</u> <u>$Q=(Q_{ij}(h))$ is identical with</u> P.

Proof. It is sufficient to show that $P_{ij}(h) = Q_{ij}(h) + o(h)$; that is,

$$Q_{ij} = p_j P_{ji}/p_i = P_{ij} + o(h).$$

Since this relation is obviously true for $|i-j| \neq 1$, it need only be shown that

$$p_j L_{ji} = p_i L_{ij}, \qquad |i-j|=1.$$

This, in turn, follows by induction on the columns of (2), using the fact that the row sums of L vanish.

COROLLARY. <u>If</u> $\lambda_n = \lambda$, $(n=0,1,\ldots)$, <u>then the death times form a Poisson process of density</u> λ.

Outline of proof of corollary. Since λ_n is constant the birth times of $n(t)$ are Poisson with density λ. The stochastic process $r(t)=n(-t)$ is statistically identical with the process $n(t)$. But if $n(t)$ is a fixed sample function, and $r(t)=n(-t)$, then the births of $n(t)$ coincide with the deaths of $r(t)$.

Consider a queue of type M/M/s (Poisson input, s counters, exponential service time, first come, first served). If $n(t)$ is the sum of the number of customers on queue, plus those being served, then $n(t)$ is a birth and death process in which customers' arrivals correspond to births, and departures to deaths. We restrict ourselves to the case of unsaturated queues in "equilibrium." This can, of course, always be achieved by letting $n(0)$ be an appropriate random variable. By considering $r(t)=n(-t)$, the following is now clear:

THEOREM 2. a) <u>The sequence of departure times form a Poisson process</u>. b) <u>The value of</u> $n(t)$ <u>is independent of all past departure times</u>. c) If t_o <u>is a departure time, then</u> $n(t_o+0)$ <u>is independent of all past departure times</u>.

Suppose that customers, after departing from a first queue of type M/M/s, enter a second multiple-counter queue, where they are served first come, first served, with exponential service time. Such a combination of two tanden queues will be referred to as a σ-system. It is clear, by Theorem 2b, that if $n(t)$ and $n'(t)$ refer, respectively, to the first and second queues of a σ-system, then $n(t)$ and $n'(\tau)$ are independent, $\tau \leq t$. This was first proved in the special case $s=1$, $t=\tau$, by Jackson [2.]

In what follows, the term <u>waiting time</u> will be used to refer to the time elapsed between a customer's arrival and departure, the service time included. Let T and T' denote a customer's waiting time at the first and second queues of a σ-system, respectively.

THEOREM 3. <u>If</u> $s=1$, <u>then</u> T <u>and</u> T' <u>are independent</u>.

Proof. Let n be the number of customers at the first queue the instant after a customer C departs, and let n' be the number of customers C finds at the second queue (customers being served included). As a corollary of Theorem 2c, n and n' are independent. Let

$$A(t;k) = \Pr\left\{ T < t \mid n' = k \right\}.$$

n is the number of Poisson events of density λ that occurred during the waiting period T. We have

$$\Pr\left\{ n=j \mid T=t,\ n'=k \right\} = e^{-\lambda t}(\lambda t)^j/j! \ .$$

Therefore

$$E\left\{ z^n \mid n'=k \right\} = \int_0^\infty e^{\lambda t z}\, e^{-\lambda t} dA(t;k), \qquad |z| < 1.$$

Now the left side is independent of k. Therefore A(t;k) does not depend on k. Hence n', and, consequently also T', are independent of T.

In connection with this work, the author has benefited from discussions with A. W. Marshall and T. E. Harris.

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