GAME THEORY

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The theory of games might be called the mathematics of competition and cooperation. It is applied widely in economics, operations research, military science, political science, organization theory, and the study of bargaining and negotiation. First formulated in the 1920's, it did not become well known until the 1944 publication of a monumental work, Theory of Games and Economic Behavior, by the mathematician John von Neumann and the economist Oskar Morgenstern. Since then, many others have joined in extending and applying the theory.

Although the terminology of players, moves, rules, and payoffs might suggest a preoccupation with sports or recreation, the theory of games has seldom been of practical use in the playing of real games. This may be because the theory is based on idealized players, having clear motives and unlimited skill and calculating ability. Nevertheless, familiar parlor games help to illustrate certain ingredients of the theory. CHESS is a typical game of perfect information. Because there are no hidden moves, it is possible in principle to determine whether each board position is a win for White, a win for Black, or a draw; this is called the minimax value of the position. Moreover, an optimal pure strategy exists for each player: a complete plan of action that guarantees him the minimax value of the starting position, regardless of how his opponent plays. POKER, in contrast, is a game of imperfect information since decisions must be made without knowledge of the concealed cards. As a result, good play usually demands the use of purposely randomized behavior, or mixed strategies. For example, certain situations call for a bluff--not with certainty, but with a small probability. Poker also illustrates the possibility of there being no universally best way to play, even in principle. Indeed, when there are more than two players the theory predicts only a noncooperative equilibrium, in which each player's best strategy depends on the strategies adopted by the other players.

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The mathematical cornerstone of game theory is the Minimax Theorem, discovered by von Neumann in 1928. It asserts that every finite, zero-sum, two-player game has a minimax value if mixed strategies are allowed. This is illustrated in detail in Example 1 below. In this theorem, "zero-sum" means that any gain in payoff for one player represents an equal loss for the other. Many parlor games are zero-sum, but the "games" that are found in economics or operations research are usually not, since wealth may be created or destroyed.

The Minimax Theorem does not apply to nonzero-sum games or games with more than two players. Nevertheless, such games have a weaker form of solution, a "noncooperative" equilibrium in which no player, acting on the assumption that the other players' strategies are fixed, can gain anything by changing his own strategy. (Examples 2 and 3.) This theorem was proved in 1950 by John Nash, a mathematics student at Princeton, and these solutions are often called Nash equilibria.

The theory of cooperative games provides another approach to games with more than two players. It is concerned primarily with coalitions--- groups of players who coordinate their actions and perhaps even pool their winnings. A cooperative game can often be put into the form of a characteristic function, v(S), which expresses for each set of players S the amount they can get if they form a coalition excluding the other players. In an economic context, v(S) might represent the gross product achievable by an arbitrary subset of the national economy, or, in an industry model, the prospective profit of each set of firms acting as a cartel. In a political context, v(S) could be defined to be 1 for sets of legislators that have enough votes to pass a bill and 0 for sets that don't. Often, however, a single function v(S) is not enough to describe the essential worth of a coalition, and more complex mathematical forms must be employed.

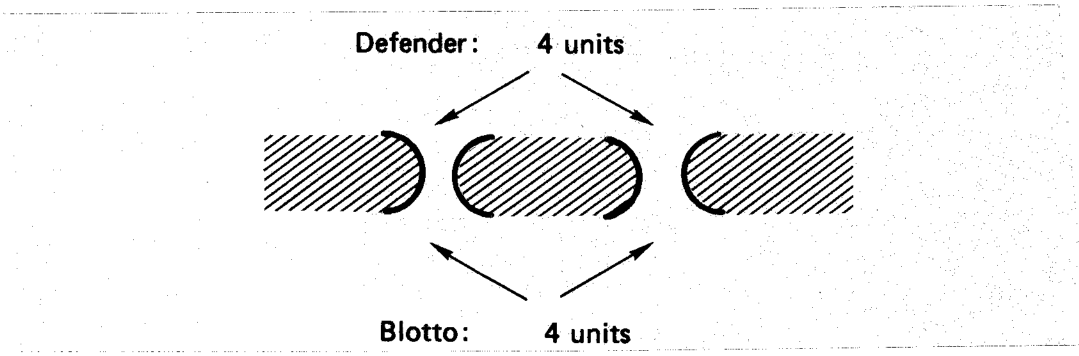
More than a dozen solution concepts for cooperative games have been introduced, serving different analytical purposes. One of them, named the core by Donald Gillies and Lloyd Shapley in 1953, is defined as the set of outcomes of the game that are "socially stable," in that no coalition has

the power to force an improvement for its members. This means, if there is a characteristic function v, that the total payoff to each set S must be at least v(S). (Example 4, below.) Unfortunately, if coalitions are too strong the core may fail to exist, and if they are too weak the core may include a great variety of different outcomes and not provide a satisfactory solution of the game. But in many important economic applications the core does give a sharp solution, closely related to the classical equilibrium of supply and demand, and so core theory has become part of the equipment of the mathematical economist.

Another solution for cooperative games, called the Shapley value, is defined as a weighted average of the marginal contributions $v(S)-v(S-\{p\})$ of a player p to the various coalitions S he might join, the weighting factor being $(s-1)!(n-s)!/n!$. Unlike the core, the Shapley value is always well-defined and unique. But it is not enforceable by strategic action; it merely indicates what a player should expect to get *a priori*. The Shapley value may be used when a fair appraisal of a set of conflicting claims is sought, e.g., as a basis for an adjudicated settlement. (Example 4.) It may also be applied to voting models, where it expresses the probability of each voter being pivotal in the alignment of support on an issue, assuming that all alignments are equi-probable. (Example 5.) A measure of voting power is thereby obtained, known as the Shapley-Shubik power index. For example, in the "game" of enacting federal legislation, the President (through his veto) holds roughly 1/6 of the total power according to this index, while the House and Senate divide the rest.

The following models, though highly simplified, will serve to illustrate many of these concepts and to suggest the wide variety of possible applications.

1. COLONEL BLOTTO is ordered to attack through the mountains. His situation resembles a football quarterback's:



A breakthrough at either pass will bring victory, but requires a 2 : 1 local superiority. How should Blotto deploy his forces?

In the following payoff matrix, "1" denotes victory, "0" defeat, and "x:y" the strategy of sending x units to the left and y to the right:

Blotto's strategy:	Defender's strategy: 4:0	3:1**	2:2**	1:3**	0:4
4:0*	0	0	1	1	1
3:1	1	0	0	1	1
2:2*	1	1	0	1	1
1:3	1	1	0	0	1
0:4*	1	1	1	0	0

Note that no pure strategy is effective for either side. Blotto should choose equally at random one of the three strategies marked (*); he will then break through with probability $\frac{2}{3}$ against any defense. The enemy commander should similarly mix the three strategies marked (**); his chance of stopping Blotto will then be at least $\frac{1}{3}$. The minimax value is $\frac{2}{3}$.

2. TWO PRISONERS are interrogated in separate cells. Each faces a 10-year sentence if the other gives evidence against him, but otherwise just 3 years on a lesser charge. The prosecutor has promised a 1-year reduction, however, in exchange for testimony against the other prisoner:

		2nd Prisoner:	
		testify	be silent
1st Prisoner	testify	(9, 9)	(2, 10)
	be silent	(10, 2)	(3, 3)

For each prisoner, "testify" is a dominating strategy--the best choice whatever the other does. Hence, (9, 9) is the the unique Nash equilibrium, and seems the only "rational" outcome for this noncooperative game. But the apparently irrational principle of "honor among thieves" leads to a distinctly better outcome (for the thieves!), namely (3, 3).

Games of this kind set up a confrontation between individual and collective rationality, and are often studied experimentally for the insights they afford into social behavior.

3. THREE CHILDREN engage in a "truel" with snowballs. They throw simultaneously at 10-second intervals, and are not very accurate. Abner's chance of scoring a hit on any one toss is only 4%, Becky's is 3% and Chuck's is 2%. They can throw at either opponent, but must fall down when hit. The truel continues until only one (or perhaps none) remains standing.

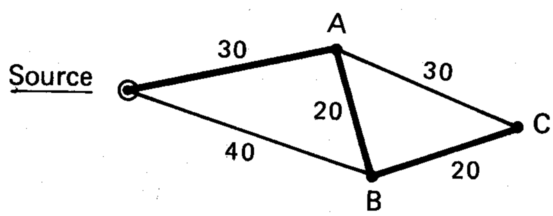
This simple "game of survival" has not one, but several Nash equilibria. The obvious one is to throw always at your more accurate opponent. The calculations are not difficult, but the result is paradoxical: Chuck wins 37.6% of the time, Becky 32.3%, and Abner 28.8%. The "fittest" individual is the least likely to survive!

A more sophisticated solution is the following: Abner's plan is to throw first at Becky, but to shift to Chuck if Chuck ever throws at him.

Becky still throws at Abner, but Chuck, deterred by Abner's threat of retaliation, throws only at Becky. As in the other solution , no one can gain by changing his/her strategy, given the strategies of the other two. But the outcome is quite different: Abner 43.5%, Chuck 36.0%, Becky 19.2% (In each case there is a small probability of no winner, due to simultaneous hits.)

There are many other solutions, both noncooperative and cooperative. Selecting the right solution concept in any multiplayer game generally requires looking beyond the physical or legal constraints and considering psychological and sociological factors like the credibility of threats, the firmness of commitments, and the ethics of collusion.

4. THREE COOPERATING TOWNS wish to tie into a nearby power source. The possible transmission links and their costs are as shown below:



The minimum-cost connecting system (heavy lines) costs 70 units. How should this cost be apportioned among the towns?

Calculating the minimum cost for each subset of towns fills out the characteristic function for this cooperative game: $v(ABC) = -70$, $v(AB) = -50$, $v(AC) = -60$, $v(BC) = -60$, $v(A) = -30$, $v(B) = -40$, $v(C) = -60$, and $v(\emptyset) = 0$. According to the Shapley value, towns A, B, C should be assessed 15, 20, 35 units, respectively. Moreover, this solution has the desirable core property, since no town or set of towns is asked to pay more than an independent system would cost.

5. NASSAU COUNTY (NY) has six Supervisorial districts, of greatly different size. In an effort to equalize representation throughout the county, board members are assigned different voting weights, corresponding to the populations of their districts. The weights assigned in 1964 were

[31, 31, 28, 21, 2, 2], and a majority of 58 out of 115 was required to pass a measure.

Though it wasn't immediately noticed, this weighted majority game had three "dummy" players, with no power at all. Even the Supervisor from North Hempstead, with 21 votes, could never be pivotal. The Shapley-Shubik power indices are easily seen to be (.333, .333, .333, 0, 0, 0), since any two of the "Big 3" make a majority.

In 1971 the required majority was raised from 58 to 63, keeping the weights the same. This had the drawback of making deadlocks possible (e.g., on a 62-53 vote), but it did eliminate the "dummy" problem, as each members was now essential to at least one winning coalition. The new power indices are (.283, .283, .217, .117, .050, .050)--in better accord with the original intent. Power index analysis is now widely used in the design of voting systems.

RECOMMENDED FURTHER READING: G. Owen, Game Theory, Saunders, 1966 (an elementary mathematical text); M. Davis, Game Theory, Basic Books, 1970 (a nontechnical introduction); J. D. Williams, The Compleat Strategyst, McGraw-Hill, 1966 (a popular "primer" on matrix games); J. McDonald, The Game of Business, Doubleday, 1977 (case histories from the business world); A. Rapoport, Fights, Games, and Debates, University of Michigan Press, 1960, J. Brams, Game Theory and Politics, The Free Press, 1975 (political theory and applications); R. Dawkins, The Selfish Gene, Oxford University Press, 1976 (a layman's account of the "game" of evolution and survival).