A CONCEPT OF STABILITY IN MANPOWER PLANNING

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1. INTRODUCTION

A number of manpower and personnel policy decisions in the Air Force logically depend upon knowledge of the way in which the skill composition of the force* changes over time. For example, the Directorate of Manpower and Organization at Headquarters USAF issues "skill ratio guides" which are used for estimating detailed manpower requirements. If these guides are established without taking into account the way in which the skill mixture of the force changes over time their use will not yield achievable skill distributions. As another and quite related example, DCS/P must decide on appropriate retention rates. (These decisions mean something when reenlistment is expected to exceed what may be a desirable level.) It is obvious that a knowledge of the relationship between retention rates and the ultimate skill distribution is essential for such decisions.

In this paper we address ourselves to the problem of predicting future skill composition as a function of present skill mixture, retention rates, etc. If retention rates, periods of enlistment, dates of separation, skill advancement rates, and the initial distribution of skills within the force are known, then the skill distributions in succeeding time periods may be computed. Now if the total force size is or becomes stabilized and the retention rates, etc., are essentially constant then it will always be found that these skill distributions will approach a stable distribution. The rapidity of approach to the stable condition will vary from case to case.

*We will use the expression "force" to describe any functional category which is under scrutiny. This could be an individual Air Force Specialty, career field, or broad functional area such as "highly technical", etc.
(In all cases that we have examined the stable distribution is essentially reached within five years.) In any event the stable distribution indicates the direction in which the skill composition of the force is moving.

The body of this paper illustrates a method for computing stable skill distributions. After a very general description of the method, it is used to obtain stable skill distributions for an extremely simple model. The solutions of the simple model are then compared with solutions of somewhat more realistic ones.

It should be emphasized that no attempt will be made in this paper to describe what skill mixes or retention rates should be sought. To do that would require analyses of the relationship between skill distribution and force effectiveness (among other things). In a later paper we will attempt to describe how the technique illustrated here should be combined with information concerning effectiveness in such a way as to offer insight into good retention goals.
II. A GENERAL DESCRIPTION OF THE METHOD

In this section we shall give a general description of the model upon which the subsequent steady state calculations are based. Let us restrict our attention to a specific functional category, such as the highly technical or non-technical category. Any member of this category will have associated with him a number representing the length of time, in years, that he has spent in the Air Force. This number will be an arbitrary integer ranging from one to the maximum length of time that any individual spends in the service, which we shall in this paper take as 20 years. If a given individual has \( N \) years of service behind him, then we shall say that he is in state \( N \). Therefore the experience mixture of a given functional category, at any time, can be described by the percentage of people in each state from 1 to 20, and also the total number of people. This is, of course, only verbally different from the customary way of describing an experience mixture.

Now, after the passage of a year, any person in state \( N \) will move to state \( N+1 \) (i.e., his length of time in the service will have increased by one year), unless, for one reason or another, he departs from the functional category that we are considering. For example, the transition from state 4 to state 5 will not be made by everyone. Assuming that the original enlistment period was four years, then transition from state 4 to state 5 will be made only by those airmen who reenlist. If the first-term reenlistment rate is \( \lambda (0 < \lambda < 1) \) then there is a probability of \( \lambda \) that a given airman who begins the year in state 4 will make the transition to state 5. On the other hand there will be a probability of \( 1 - \lambda \) that such a person leaves the service altogether.
There are, of course, other ways in which an airmen may leave the force that we are considering. For example, an airmen who makes the transition from state 4 to state 5 (i.e. reenlists after four years), and who reenlists for an additional four years, will have a certain probability of leaving the system from state 8. Losses may also be incurred by an annual attrition (illness, death) rate, regardless of the state in which the person may be found.

We shall introduce one additional idea before the steady state computation is described. We are interested in describing a situation in which the total number of people in the force remains constant from year to year. This means that the number of people in state 1, at the beginning of a year, is precisely equal to the number of people lost during the previous year. This condition would require a good deal of flexibility on the part of recruitment and training agencies, and can only be regarded as an approximation. We shall describe this phenomenon by saying that a transition has occurred from some higher state to state 1. For example a transition will be made from state 4 to state 5 with probability (the first term reenlistment rate) and with probability $1-\alpha$ a transition is made from state 4 to state 1.

Let us recapitulate. At any time the experience mixture is described by the fraction of people in states 1 through 20. Let us denote the fraction of people initially in state $i$, by $p_i (0)$. In one year there is a probability $p_{ij}$ of making a transition from state $i$ to $j$. For example $p_{4,5} = \alpha$ and $p_{4,1} = 1-\alpha$. Most of these transition probabilities will be zero, except for transitions to higher experience levels, or transitions into the state 1.
After one year the fraction of people in state \( i \) will be given by

\[
P_j(1) = \sum_{i=1}^{20} p_i(0) p_{ij},
\]

and generally the fraction of people in state \( i \) after \( n \) years is given by

\[
P_j(n) = \sum_{i=1}^{20} p_i(n-1) p_{ij}.
\]

It may be shown that as \( n \) becomes larger and larger the force composition \((p_1(N), \ldots, p_{20}(N))\) approaches a steady state composition, \( p_1, \ldots, p_{20} \), which may be obtained by solving the equations

\[
P_j = \sum_{i=1}^{20} p_i p_{ij}.
\]

The steady state composition has the property that once it is reached, it will repeat itself from year to year, unless the total force size is changed, the enlistment period, or the retention rates vary. Even if the steady state is not reached for a number of years, it may still be regarded as an indication of the direction in which the system is moving.
III. APPLICATION OF THE METHOD TO A MODEL

In this section the basic model described in Section II is applied, and certain variations are discussed.

A simplified version of the typical pattern of gain, loss and advancement in an Air Force specialty or career field or broad functional area is depicted in Figure 1 (p. 7). Each of the circles represents a year or state. The first represents students; the second 3-level airmen; the third and fourth to 5-level airmen; the fifth through the twentieth to 7-level airmen. All airmen move from states 1 to 2, 2 to 3, and 3 to 4. The airmen moving from state 4 to 5 are governed by the retention rate, or assuming all are eligible for reenlistment, the first-term reenlistment rate. All airmen move from states 5 to 6, 6 to 7, 7 to 8, and 8 to 9. Those moving from 9 to state 10 correspond to the second-term reenlistment rate. All airmen move from 10 to 11, 11 to 12, etc. From state 20, men are lost to the system. The other losses, represented by dotted lines on Figure 1, are generated by first-term non-reenlistees (between states 4 and 5) and second-term non-reenlistments (between 9 and 10). Since what is sought is a stable or steady state solution all losses are replenished, through entry into state 1. The stable skill distribution solution in terms of first-term retention rate appears in the Appendix on page 16.

If the second-term reenlistment rate is assumed as constant at .85, Fig. 2 (p.8) describes the stable skill mixes associated with all first-term retention rates. For example, at a first-term retention rate of 25 per cent, the skill mix representing equilibrium will be composed of 13 per cent students, 13 per cent 3-level airmen, 26 per cent 5-level airmen, and 48 per cent 7-level airmen. If the first-term reenlistment rate was .50, 9 per
Fig. I—Basic model
Fig. 2 — Stable force skill distribution vs first-term re-enlistment rate
cent of the force would be student, 9 per cent 3-level, 18 per cent 5-level, and 64 per cent 7-level.

The model we have described abstracts from a rather formidable body of detail which prevails in the real world. Since this paper is meant to be only illustrative, this did not seem undesirable. However, it remains to be demonstrated how elements of "reality" can be introduced into the model. We will do this by testing the sensitivity of the stable condition results of the basic model to changes, each of which adds an element of reality. This exercise will serve two purposes: (1) it will demonstrate more fully how a model of this type is constructed and put to use, and (2) it will indicate (it turns out) that a good deal of added complication in the form of closer approximations to reality does not significantly alter the steady state result.

VARIATION 1

The first variation introduces at the end of a first enlistment three alternative actions instead of two. Where formerly the possibilities at this point included either reenlistment* for five years or no reenlistment, this adaptation departs from the simple averaging of the two reenlistment alternatives (implied in the 5-year reenlistment period) and treats them explicitly and separately.

Reenlistment is possible for four years or for six years—and, of course, non-reenlistment remains an alternative. All other elements of the basic model are duplicated. Figure 3 (p. 10) depicts this first variation. The ratio of reenlistments between 4-year and 6-year is assumed fixed at .15 to .85 which corresponds to current experience.

*It is assumed that everyone is eligible for reenlistment, hence reenlistment and retention are synonymous.
Fig. 3—Variation one
The stable solution appears in the Appendix on page 17. The stable
skill mix at .25 first-term retention is: students 12.6 per cent; 3-level
12.6 per cent; 5-level 25.2 per cent; and 7-level 49.6 per cent. This
compares with 13 per cent, 13 per cent, 26 per cent, and 48 per cent,
respectively, for the basic model at the corresponding reenlistment rate.
Hence, it is apparent that this refinement alters the solution little.

VARIATION 2

The second variation involves applying an annual attrition factor of
2 per cent. All other features are the same as basic model. Figure 4
(page 12) depicts this case. The solution appears in the Appendix on page 18.
The .25 first-term reenlistment rate stable condition is: student 15 per
cent, 3-level 14 per cent, 5-level 28 per cent, and 7-level 43 per cent.
The effect of the annual attrition loss was, as might have been expected,
to reduce the 7-level component of the force vis-a-vis the other components.
The reason for the 5 percentage point drop in 7-level can be traced to the
fact that in the basic model of the 20 states after state 9, no losses were
suffered by 7-level save retirement at the end of the twentieth state. With
annual attrition the 7-level (composed of 16 of the 20 states) suffers
relatively more than the other components (composed of only 4 of the 20 states).
Inclusion of an attrition factor would appear warranted on the basis of its
considerable impact on the stable solution.

VARIATION 3

Here an early reenlistment alternative is offered to first-term airmen
at the end of their second year. This involves two alternates at the end of
the second state: reenlistment for five years or continuation of the first
four-year commitment. At the end of the first four years those who did not
Fig. 4—Variation two
choose early reenlistment are offered a reenlistment opportunity. Figure 5 on page 14 depicts this model. Note that losses to the system now occur at the end of year 7 as well as in years 4, 9, and 20.

To make this variation comparable with the basic model the early reenlistment level (at the end of state 2) and the regular first-term enlistment level (at the end of state 4) equalled an overall first-term rate of .25 in the basic model. For this computation and stable condition solution see page 20 in Appendix. At the .25 first-term reenlistment level (which in this case amounted to .135 early and .135 regular first-term reenlistment), the steady state yields the following skill mix: students 13.2 per cent, 3-level 13.2 per cent, 5-level 26.4 per cent, and 7-level 47.2 per cent. The solution is only part of a percentage point from the basic model solution.

VARIATION 4

Variation 4 is essentially the same as the basic model, but with the rigid assumptions regarding the skill mix implications of each state eliminated. The solution sought is not how many students, 3-level, etc., but rather how many who have been in 0 - 1 year, 1 - 2 years, etc. This "experience" distribution stable condition for all reenlistment rates appears in Figure 6 on page 15. Note that any functional area or job may be described by use of this chart simply by sketching a line roughly parallel to the curves and corresponding to (1) the length of time for formal training (the distance between this line and the X-axis would correspond to what we have called student), (2) the average length of time required for advancement from the 3-level to the 5-level (the distance between this line and the previous one would measure the relative number of 3-level airmen), (3) etc.
Fig. 5 — Variation three
Fig. 6 — First-term re-enlistment rate vs stable experience distribution
APPENDIX

In this section we shall describe the mathematics used in obtaining the stable distribution for the basic model and its variations.

BASIC MODEL (SEE FIGS. 1 AND 2)

Let $p_1$, $p_2$, ..., $p_{20}$ represent the stable distribution, that is $p_i$ is the fraction of the total force in state $i$. We shall represent the first-term retention rate by $r$, and assume that the second-term retention rate is .85. We have the following equations:

$$p_4 = p_3 = p_2 = p_1,$$
$$p_5 = rp_4,$$
$$p_9 = p_8 = p_7 = p_6 = p_5,$$
$$p_{10} = .85 p_9,$$
$$p_{20} = p_{19} = \ldots = p_{10}.$$

These equations permit us to express all of the $p_i$'s in terms of $p_1$, and if we use the fact that $p_1 + p_2 + \ldots + p_{20} = 1$, we obtain

$$p_1(4 + 5r + 11 \times .85 \times r) = 1$$
or

$$p_1 = \frac{1}{4 + 14.35r}.$$ Similar expressions may be obtained for the other $p_i$'s.

The fraction of the force in the student category is given by $\frac{1}{4 + 14.35r}$; the fraction in the '30' category by $\frac{1}{4 + 14.35r}$; the fraction in the '50' category by $\frac{2}{4 + 14.35r}$; and the fraction in the '70' category by $\frac{14.35r}{4 + 14.35r}$. 
VARIATION 1 (SEE FIG. 3)

In this variation we permit the possibility of a choice between a six or four year term for the first reenlistment. We represent, as before, the first term reenlistment rate by \( r \), but we assume in addition that 85% of the first term reenlistments are for six and the remaining 15% are for four years. The second term reenlistment rate for those who have chosen a six year enlistment is assumed to be 100%; for those who have chosen the four year enlistment, the second term reenlistment rate is assumed to be 85%.

Since an airman who has selected a six year reenlistment has a probability of remaining in the service different from that of an airman who has selected a four year reenlistment, it is necessary to differentiate these two categories. This is illustrated in Figure 3, where the states 5, 6, \( \cdots \)-10 refer to airmen who have selected a first reenlistment of four years, and the states 5', 6', \( \cdots \)-10' refer to those airmen who have selected a first reenlistment of six years. Since we are assuming that all reenlistments after state 10 occur at the rate of 100%, there is no necessity to continue this distinction after state 10.

We are interested in determining the twenty-six numbers \( p_1 \cdots p_{20} \) and \( p_5', \cdots p_{10}' \), which characterize the stable distribution. These numbers satisfy the following equations.

\[
\begin{align*}
    p_4 &= p_3 = p_2 = p_1 \\
    p_5 &= .15r \cdot p_4 \\
    p_8 &= p_7 = p_6 = p_5 \\
    p_{10} &= p_9 = .85 \cdot p_8 \\
    p_{5'} &= .85r \cdot p_4 \\
    p_{10'} &= p_{9'} = \cdots = p_{5'}
\end{align*}
\]
\[ P_{11} = P_{10} + P_{10}' \]
\[ P_{20} = p_{19} = \ldots = P_{11} \]

We may, as before, express each of the \( p_i \)'s in terms of \( p_1 \), and using the fact that the sum of the \( p_i \)'s is one, we obtain
\[ p_1 (4 + .6r + .255r + 5.1r + 9.775r) = 1 \quad \text{or} \]
\[ p_1 = \frac{1}{4 + 15.73r}. \]

The other \( p_i \)'s may now be expressed in terms of \( p_1 \), and the stable distribution determined.

The fraction in the student category and '30' category are each given by \( \frac{1}{4 + 15.73r} \); the fraction in the '50' category by \( \frac{2}{4 + 15.73r} \); and the fraction in the '70' category by \( \frac{15.73r}{4 + 15.73r} \).

Solving for a first-term reenlistment rate \( r \) of .25, students and '30' category is equal to .125; '50' category is equal to .252; and '70' category is equal to .496. This solution is not quite comparable to the \( r = .25 \) case of the basic model since the second term reenlistment in this case is slightly higher than in the latter. However, correcting for that factor would yield a solution only slightly different from the above.

**VARIATION 2 (SEE FIG. 4)**

In this variation we introduce an annual attrition factor which takes the form of a fixed percentage loss from each state, \( p_1, p_2, \ldots, p_{20} \). In other respects the model is identical to the basic model.

We are interested in determining the twenty numbers \( p_1, p_2, \ldots, p_{20} \) which characterize the stable distribution. If a first-term retention rate is
represented by \( r \), second-term retention rate is set at .85, and annual attrition is represented by \( a \), the elements of the stable distribution satisfy the following equations:

\[
\begin{align*}
    p_2 &= (1-a) p_1, \\
    p_3 &= (1-a) p_2, \\
    p_4 &= (1-a) p_3, \\
    p_5 &= r (1-a) p_4, \\
    p_6 &= (1-a) p_5, \\
    p_7 &= (1-a) p_6, \\
    p_8 &= (1-a) p_7, \\
    p_9 &= (1-a) p_8, \\
    p_{10} &= .85 (1-a) p_9, \\
    p_{11} &= (1-a) p_{10}, \\
    \text{etc. to } p_{20} &= (1-a) p_{19}
\end{align*}
\]

As before, these equations permit us to express all of the \( p_i \)'s in terms of \( p_1 \), and if we use the fact that \( p_1 + p_2 + \ldots + p_{20} = 1 \), we obtain

\[
p_1 = \frac{1}{\sum_{n=0}^{19} (1-a)^n + r \sum_{n=4}^{19} (1-a)^n + .85r \sum_{n=9}^{19} (1-a)^n}
\]

The fraction of the force in student category is \( p_1 \); the fraction in the '30' category \( (1-a) p_1 \); the fraction in the '50' category is \( (1-a)^2 p_1 \); the fraction in the '70' category is \( (1-a)^3 p_1 \); and the fraction of the force in the '70' category is therefore \( 1 - p_1 \sum_{n=0}^{3} (1-a)^n \).
Setting annual attrition at .02, and r at .25 we arrive at the following
stable distribution: students, .148; '30' category, .145; '50' category,
.275; and '70' category, .425.

VARIATION 3 (SEE FIG. 5)

In this variation we allow the choice of reenlistment after only
half (two years) of the first enlistment is served. Those who do not
elect early reenlistment may choose reenlistment after the full (four years)
first enlistment. We assume that both "early" and regular first-term
reenlistees sign up for a five year period, at the end of which time we
assume the same second-term retention rate, .85, applies to both groups.

Since the airmen who chose early reenlistment will come up for their
second reenlistment in their seventh year, they are differentiated from the
remainder of the group who come up for their second reenlistment at the
end of year 9. The former are designated in Figure 5 by states 3',--9'.
At state 10 the two groups merge since we assume their subsequent careers
have the same characteristics.

We are concerned with determining the twenty-seven numbers \( p_1, p_2, \ldots, p_{20}, p_3, p_4, \ldots, p_9 \), which characterize the stable distribution. If
"t" represents the early (after two years) reenlistment rate and "m"
represents the regular (after four years) reenlistment and the second
reenlistment takes place in both groups at rate of .85, these numbers
\( (p_1, p_2, \text{ etc.}) \) satisfy the following equations:

\[
\begin{align*}
p_2 &= p_1, \\
p_4 &= p_3 = (1-t) p_2, \\
p_9 &= \ldots = p_5 = m p_4,
\end{align*}
\]
\[ p_{71} = p_{31} = t p_2, \]

\[ p_{91} = p_{81} = 0.85 p_{71}, \]

\[ p_{20} = \ldots = p_{10} = 0.85 p_9 + p_9, \]

Again, we may express each of the \( p_i \)'s in terms of \( p_1 \), and using the fact that the sum of the \( p_i \)'s is one, obtain

\[ p_1 = \frac{1}{4 + 14.55 t + 14.35 m - 14.35 mt} \]

The fraction in both the student and '30' categories is

\[ \frac{1}{4 + 14.55 t + 14.35 m - 14.35 mt}; \]

the fraction in the '50' category is

\[ \frac{2 (1-t)}{4 + 14.55 t + 14.35 m - 14.35 mt}; \]

and that in the '70' category is

\[ 1 - \frac{4 + 2t}{4 + 14.55 t + 14.35 m - 14.35 mt}. \]

In order to compare this variation with the basic model we will want to choose values of \( m \) and \( t \) such that total first-term reenlistees is comparable with a given value of \( r \) in basic model. For this condition to be satisfied

\[ t + m (1-t) = r. \]

Satisfaction of the constraint can be achieved with many \( t \)'s and \( m \)'s. Let us arbitrarily assume that \( t = m \). Then if \( r = 0.25 \), it will be found that \( t = m = 0.135 \).

Substituting in our solutions for each category of airmen we find that the stable distribution at first-term reenlistment rates comparable to \( r = 0.25 \) is students and '30' category, 0.132; '50' category, 0.264; '70' category, 0.472.