WEIGHTED VOTING: A MATHEMATICAL ANALYSIS FOR INSTRUMENTAL JUDGMENTS

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The purpose of this essay is to clarify a particular question of political justice by means of a mathematical analysis and thus to demonstrate the utility of such inquiry for political philosophy. Traditionally political philosophy has had two tasks: one is the determination of what justice is; the other is the determination of whether or not particular instruments are appropriate for realizing given conceptions of justice. Except for a modern Pythagorean who seeks justice in formal symmetry, mathematics is not likely to be useful in the former task. But it can be useful in the latter task by reason of the clarification it permits.

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The particular subject of this volume, representation, is well-suited for the demonstration of this utility because representation is not an end-in-itself, but a means to achieve some other goal. The system of representation creates a policy-making body. But, since any relatively small body--whether representative or not--can make policy, the purpose of representation is not primarily to create a body, but rather to create one that involves a large electorate in the policy-making process. And the way the electorate is involved expresses some conception of justice, e.g. preference for hereditary aristocracy, preference for equality, etc. Whether the system of representation expresses the given conception of justice well or badly is an instrumental problem: And if a mathematical analysis can be shown to clarify it, then the utility of formal methods will be demonstrated for that case and by analogical inference for other instances of instrumental judgment as well.

The particular subject of this paper, weighted voting in legislatures, is also well suited for a demonstration of the utility of mathematical methods inasmuch as the consequences of weighted voting are not immediately obvious. Until the recent efforts at reorganization of the representative system as required by the rulings in Baker v Carr and Reynolds v Sims, the United States had had very little experience with weighted voting, and hence scholars have had no occasion to analyze its consequences. Recently,

however, we have had some experiments with weighted voting and, if the analysis in this essay is correct, the social circumstances suggest that the legislators involved completely misunderstood its consequences. Paradoxically, those whom its advocates say are its main beneficiaries, are in fact those who are probably hurt by it, while those whom its opponents say are its main victims are in fact those who probably benefit from it. To the elucidation of that paradox, we now turn.

I

The moral proposition expressed in the Supreme Court's insistence on "one person, one vote" is that each citizen ought to have—through the medium of his representative—the same chance as every other citizen to influence the outcomes of the legislative process. The technical problem of realizing this moral proposition in institutions is that of finding a device that preserves the equality of citizens when their influence is funneled through their representatives.

It has been assumed that legislative technicians have two variables they can adjust to retain this equality in spite of the funnel: the size of districts and the vote of representatives. One alternative is to make both districts and votes exactly equal. In this case, if there are n legislators, then each legislator's district has the same number of citizens as each other district, and each legislator has 1/n of the total votes in the legislature. This is the system we have heretofore supposedly used almost exclusively in local, state, and national legislatures, despite some gross variations in the size of districts and a surprisingly frequent occurrence of multiple member districts. The other alternative, so called weighted voting, is to make both districts and votes unequal but proportionate to each other. Thus, if there are s votes in the legislature, if there are m citizens

in the area for which the legislature sits, and if a specific district contains a given fraction, km, of the citizens, then the legislator from that district has ks of the votes in the legislature. It is on the basis of the assumption that these alternatives are equivalent that weighted voting is advocated.

The difficulty with this assumption is that what counts in the legislature for preserving the equality of citizens' influence is not the weight of the legislators' votes but the actual influence a given weight has over the legislative outcomes. If it turns out that weight and influence are different things, then weighted voting cannot preserve the equality of citizens as influence is funneled through the legislators.

In order to discover whether or not weight and influence are the same, we need a measure of influence that is independent of weight. Such a measure is immediately at hand in the form of the Shapley–Shubik power index, which is a specialization of Shapley's value for n–person games.[1] This index is based on the following considerations:

1. For each weighted vote, one wants to know what chance the legislator who casts it has of influencing an outcome

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[1] L. S. Shapley and Martin Shubik, "A Method for Evaluating the Distribution of Power in a Committee System," 48 American Political Science Review (1954) 787–792; L. S. Shapley, "A Value for n–Person Games," Annals of Mathematics Study No. 28 (Princeton, Princeton University Press, 1953) pp. 307–17.

of the legislative process.

2. Influence over an outcome involves, at the very minimum, membership in a winning coalition, i.e., being on the winning side on a division. Certainly one who is on the losing side is not ordinarily thought to have much influence.

3. Membership of a given legislator in a winning coalition is significant for influencing the outcome only if the coalition is minimal, or if it has a winning subcoalition that is minimal and contains that legislator. Here a minimal winning coalition is defined as a coalition that would cease to be winning if any member were subtracted from it. The rationale of this consideration is obvious: if a given legislator could have voted the other way without affecting the ability of the coalition, or of any sub-coalition, to win, then his influence is nil.

4. The uniquely influential position in a minimal winning coalition is that of the member who was added last, in a chronological sense. This is conventionally known as the <u>pivot</u> position. The paramount importance of this position may not be completely obvious, but the following model may make it clear: Let there be both supporters and opponents of a bill and some members who are undecided. Let the manager of the bill persuade by various inducements (such as rewriting the bill, promising votes on another bill, etc.) enough of the undecided to support the bill so that the supporters become a minimal majority. In such

circumstance, when the vote occurs, some of the undecided who were not approached by the bill's manager may nevertheless vote for the bill, perhaps because they have concluded that their vote will make a good impression on their constituents. One can say that the original supporters and especially those undecideds who become supporters for a consideration have influenced the outcome. But it is very hard to say the same thing about the undecideds who become supporters without solicitation, i.e., those who joined after the pivot. These latter have not influenced the form of the bill which passes, nor have they traded influence on this bill for influence on another one. All the unsolicited supporters have gained is, if anything, some personal advantage for themselves, which can hardly be regarded as the transmission of voters' influence. So we shall assume that their support is without significance for our purpose.

Similarly, we may discount the influence of the supporters who preceded the pivot. Typically, the main inducement the manager of a bill offers to the undecideds is a modification of the bill into something they can support. Presumably the original supporters were satisfied with the bill as originally introduced, so that each modification lessens the desirability of the bill for them. Hence the undecideds who have become supporters after modification are more satisfied with the modified

form of the bill while the original supporters are less satisfied. Carrying this argument to extremes, the single legislator most satisfied is the last one added to the minimal winning coalition, that is, the pivot.

This argument holds, of course, even if the currency used to pay the undecideds is promises about votes on other bills, promises about patronage, etc. Arrange the undecideds on a scale of hostility to the bill as originally introduced and consider the problem of the manager of a bill who is soliciting their support with promises, etc. Presumably the least hostile will become a supporter for the lowest price (i.e., the smallest promise), the next least hostile for the next higher price, etc. Since we can assume the manager of the bill has limited resources, we can assume also that he will buy the least hostile undecided first, the next least hostile second, etc. Thus clearly the highest price he must pay goes to the legislator in the pivot position. This is not to say that only the pivotal legislator is influential. Rather it is that the pivotal legislator is significantly more influential than any other on a given roll call.

The model on which this conclusion is based may seem unrealistic if one looks only at the outcomes of roll calls as they appear in the printed record with its alphabetical list of yeas and nays, for the pivot is not labeled, and one cannot tell which voters on the winning side were

original supporters, persuaded undecideds, or unsolicited undecideds. But in the actual history of the formation of the coalition of original supporters and persuaded undecideds on any important roll call, there is always a manager for each side and managers seldom seek more votes than their count of the house tells them is necessary to win. And it is in that more historical sense that one affirms influence to be concentrated at the pivotal position.

In many cases, of course, the legislators' decisions will not be so precisely ordered in time, nor their views on the legislation so precisely ordered in a "scale of hostility," that an exact identification of the pivotal man would be possible, even if the full history of the bill could be known. But this is not a serious drawback, since we are ultimately concerned not with votes on individual bills, but with the distribution of power inherent in the voting system itself. Those legislators who are assumed to have just missed the pivot position, in a doubtful case, will nevertheless be recognized in the power index, to the extent that other, equally-likely arrangements (in time or in viewpoint) exist in which they would be pivotal.

5. Adherents may come to a coalition in any order, and all orders are equally probable.[2] This is admittedly a "ceteris paribus" assumption but there is a strong reason

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[2] A variant assumption could be adopted at this point, in which all divisions between yeas and nays are considered as equally likely, rather than all orderings of members. Although the numerical values obtained are slightly different, we know of no significant qualitative differences that would

for adopting it when one is analyzing a constitutional provision like the method of legislative voting. After a legislature is elected, and the political viewpoints, personal groupings, and even the bargaining abilities of the members are known, one might be able to make a fairly accurate forecast of what minimal winning coalitions will occur and which legislators will be pivots—i.e., hold the "balance of power"—on the important bills. But before elections, the expectation of any particular winning coalition or of any particular losing coalition is quite uncertain.[3] Furthermore, in any particular session, legislators are faced with diverse issues, each of which in a poorly disciplined system is likely to produce a different set of supporters and opponents and to arrange the opponents and undecideds differently on a scale of hostility. And when one is writing an apportionment statute which is expected to last through many elections, the uncertainty is compounded.

The Shapley-Shubik index embodies all these considerations. It is a measure of the chance that a legislator

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arise from defining the power index in this way. See John F. Banzhaf III, "Weighted Voting Doesn't Work: A Mathematical Analysis" Rutgers Law Review, Vol. 19, (1965), pp. 317–43.

[3] In large bodies with a relatively low degree of party discipline (such as most legislatures in the United States), it is not possible to predict coalitions and pivots even after the election. See William H. Riker and Donald Niemi, "The Stability of Coalitions on Roll Calls in the House of Representatives," 54 American Political Science Review (1962) 58–65.

with a given weight, w, has to influence the outcomes of the legislative process when the outcomes are visualized as influenced by the means described in the foregoing model. Naming this chance the power, ϕ, of w (i.e., ϕ_w), the index is:

$$\phi_w = \frac{p(w)}{n!}$$

where n is the number of legislators and where "p(w)" signifies the number of permutations of n in which some particular legislator having weight w is the pivot.

To show that this index embodies all the previous considerations, we observe that the denominator on the right side, n! (read: "n factorial," that is, $1 \cdot 2 \cdot \ldots \cdot n$), includes consideration 5, inasmuch as n! is the number of possible permutations of n. Thus it takes equal account of all possible ways in which the legislators might be ranked in order of hostility to, or support of, a measure. The numerator, on the other hand, reflects considerations 1, 2, 3, and 4, which successively narrowed the meaning of influence from influence over outcomes (1), to membership in a winning coalition (2), to membership in a minimal winning coalition (3), to holding of the pivot position (4). In words the power index might be expressed thus:

$$\text{Power of a legislator of a given weight} = \text{the ratio: } \frac{\text{number of possible pivots for that legislator}}{\text{all possible arrangements of legislators in historically distinct coalitions}}$$

Since we are primarily concerned with the question of whether or not influence and weight are different things, it is also important to construct a measure of the relationship between "power," as it has just been defined, and the nominal voting strength, w. To this end we define the <u>power ratio</u>, ρ_w associated with the weight w, as

$$\rho_w = \phi_w \div \frac{w}{W},$$

where W is the total weight of all members. Thus, the power ratio is the power index of a legislator divided by his <u>relative</u> weight in the assembly. We note that the power indices of all the members must add up to 1, since there is exactly one pivot associated with each arrangement. Since the relative weights also add up to 1, we see that when weight and influence are identical the power ratio for each weight will be equal to 1. When weight and influence are different, however, some of the power ratios will exceed 1, and others will be less. Thus, the power ratio gives a quick indication of the "bias" inherent in a given weighted voting system.

It would be well to emphasize here that for a system that includes well-disciplined and long-lasting political parties, the foregoing discussion is inadequate. Not only would the cohesiveness of each party's voting behavior have to be measured, but we should have to inquire into the process whereby the party reaches its decision on how

to vote on a bill, whether to press for modifications, and so on. The decisive battles, if the parties are few and large, would be fought in the caucus rooms, or in party councils far removed from the formal law-making machinery. The theoretical tools of pivot analysis and the power index might still be applied, in principle, to such strong-party situations. The practical difficulty lies in the fact that the mathematical model of the process is no longer given to us ready-made, in the constitution or by-laws of the legislative body, and we are confronted with the problem of translating an informal or semiformal system of partisan procedures into formal terms.

Even with weak or unstable parties, there are, of course, some kinds of legislative influence that the power index does not directly reflect, e.g., the influence of legislative officers who are more likely to be managers than pivots. Yet officers are agents of members, and at the time of election of officers, members transmit some of their constituent's influence to officers. The index may be interpreted therefore as attributing to back benchers (though perhaps not in a precise way) some of the influence they have by election transmitted to house officers. One kind of influence it does not reflect, however, is the influence of nonelected leaders (e.g., committee chairmen in the House of Representatives, who achieve their special influence mostly by seniority).

This failure aside, however, it seems that the Shapley-Shubik index embodies the decisive source of influence in voting bodies. It would give us added confidence in this assertion if we had some empirical evidence that this was the case. Unfortunately such evidence is hard to come by. One of the authors has attempted to assess whether or not members of the French Assembly who moved from party to party in 1953-54 displayed an effort to improve their influence as measured by the index (applied in this case to parties, not individuals). The outcome of the test was equivocal, although it was apparent that the smaller parties, which have typically less power than weight, were more unstable than the larger ones.[4]

In spite of the absence of hard evidence on its adequacy, the index seems to capture, better than any other available measure, our intuitive notions of power.[5] Consideration of simple numerical examples, where all the possibilities can be seen and kept in mind, tends to reinforce the intuitive appeal of the model, by providing many instances where the power indices satisfactorily straighten out the distorted picture given by the raw weights.

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[4] William H. Riker, "A Test of the Adequacy of the Power Index," 4 <u>Behavioral Science</u> (1959), 120-31.

[5] William H. Riker, "Some Ambiguities in the Notion of Power," 58 <u>American Political Science Review</u> (1964), 341-49.

II

The interesting feature of the power index for the present consideration is that influence as measured by the power index is often, indeed usually, quite different from weight. One extreme and dramatic example often repeated in the literature is this: Assume a three-person committee with 50 votes for A, 49 votes for B, and 1 vote for C, wherein 51 votes is required to win. There are three voters and hence $1 \cdot 2 \cdot 3 = 6$ possible arrangements, which are here listed with the pivotal member indicated by a superscript of an asterisk:

AB*C	BCA*
AC*B	CA*B
BA*C	CBA*

In this case the power indices are $\phi_{50} = 4/6 = 2/3$, $\phi_{49} = 1/6$, $\phi_1 = 1/6$. Here A and C have more power than (relative) weight, and B has less, the power ratios being $\rho_{50} = 1.33$, $\rho_{49} = .34$, $\rho_1 = 16.67$.

In the previous example, the least weighted member gains influence at the expense of the medium weighted one. More frequently, however, the least weighted loses. An extreme example is the five member committee weighted (12, 6, 6, 4, 3), in which 16 can win. Here, the least weighted member, 3, is a dummy, since he could not figure

in any minimal winning coalition. For this member to pivot, some combination of some of the other four members must sum to 13, 14, or 15 so that, when 3 is added, a minority becomes a majority. But no subset of (12, 6, 6, 4) sums to 13, 14, or 15. So the three-vote member is powerless and the indices are $\phi_{12} = 1/2$, $\phi_6 = 1/6$, $\phi_4 = 1/6$, $\phi_3 = 0$.

Most real weighted voting systems that have been examined do not display quite these extremes, but in few of them are power and weight identical. In Nassau County, New York, some of the members of the Board of Supervisors are actually dummies who never pivot.[6] In the Electoral College, the most heavily weighted states have a power ratio of about 1.05, which means that they have about five per cent more chance to pivot than their weight would indicate, while the least weighted states have a power ratio of about .97, or about three per cent less power than relative weight.[7] These are not startling variations, although they may have some practical significance. In the French National Assembly in 1953-54, among approximately sixty distributions of weights of parties for which the

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6 John F. Banzhaf, "Weighted Voting Doesn't Work: A Mathematical Analysis," 19 Rutgers Law Review (1965), 317-41. The 1964 weights are given as (31, 31, 28, 21, 2, 2); the corresponding indices are 1/3, 1/3, 1/3, 0, 0, 0.

7 Irwin Mann and L. S. Shapley, "The *a priori* Voting Strength of the Electoral College" in Martin Shubik, ed., Game Theory and Related Approaches to Social Behavior (New York, John Wiley & Sons, 1964).

indices were calculated, four showed that the smallest parties had power ratios of less than .80, and in one instance the ratio was less than .60.[8]

Of course, power and weight can turn out to be exactly porportional (that is, exactly equal for each legislator so that all power ratios are 1), although it seems that such cases are relatively rare. In a marginal note we list all the exactly proportional systems we have discovered or had brought to our attention.[9] Although we do not know a general rule to construct such systems and therefore cannot estimate their likelihood, we do know that they must meet at least one highly restrictive condition. A necessary condition for power and weight to be proportional is that the sum of the weights, when they are expressed in whole numbers without a common factor, must be a divisor of n!.

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[8] William H. Riker, "A Test of the Adequacy of the Power Index," 4, Behavioral Science (1959), 120-31.

[9] Particular cases are (5, 3, 3, 1), with seven votes required to win; (9, 9, 4, 4, 4) with 16 to win; (43, 43, 43, 43, 43, 20, 20, 20, 20, 20) with 158 to win. There are also the following classes of examples, in each of which n denotes the number of members: (x, 1, 1, ..., 1) where $x = b + \{b^2-b)/(n-b)\}$ and b is any whole number less than $\sqrt{n}$ (for example, (2 2/3, 1, 1, 1, 1) with 3 to win); (1, 1, 1, y, y, ..., y) where $y = 2 + \{1/(n-3)\}$ and n is even and greater than 4 (for example, (1, 1, 1, 2 1/3, 2 1/3, 2 1/3) with 5 to win); and (1, 1, 1, z, z, ..., z) where $z = 2 + \{1/(n-1)\}$ and n is odd and greater than 3 (for example, (1, 1, 1, 2 1/4, 2 1/4) with 4 to win). In all these classes, the terms appearing in brackets {} do not affect the outcome, but serve only to equalize the power ratios. Finally there are all cases generated from these particular cases and classes of cases by multiplication of each weight by a constant.

(If fractional weights appear, then their least common denominator multiplied by the sum of the weights must be a divisor of n!. Thus, in the case (1, 1, 1, 2 1/3, 2 1/3, 2 1/3) the sum is 10 and the LCD is 3; the product of these numbers is 30, which is a divisor of 6! = 720.) The reason for this condition is that the indices are expressible as ratios of integers of which the denominator is n!. If, therefore, weight is to equal power, relative weights must also be expressible as a ratio of which the denominator is n!. This necessary condition is in itself quite restrictive, but the actual number of cases of identical power and weight is even smaller than this condition indicates, for it is not a sufficient condition. This is easily shown by the counter example of the set of weights (3, 2, 1) which sums to n! but has the following indices: $\phi_3 = 2/3$, $\phi_2 = 1/6$, $\phi_1 = 1/6$.

It seems clear, therefore, that power and weight are not usually identical and are often quite different. This fact indicates that the assumption on which weighted voting is built is false. It is not the case that unequally weighted legislators (whose weights are, however, proportionate to the sizes of their districts) always or even occassionally funnel citizen's influence into legislation without distortion. Hence one might conclude that weighted voting cannot satisfy the Court's insistence on "one person, one vote." Such a conclusion would be somewhat premature,

however, because it may well be that the distortion induced by weighting is less than the distortion which courts will come to allow as a result of variations in the size of districts which are supposed to be equal.[10] We therefore turn to an examination of the degree of bias generated by weighted voting.

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[10] Just how much variation in district size the Supreme Court will allow is now uncertain, for in Reynolds v. Sims, 377 US 533 at 578, the Court announced its intention of proceeding on a case-by-case basis rather than of laying down guidelines. In that same case, however, Mr. Chief Justice Warren frequently observed that for one voter to have twice the influence of another was "inconceivable."

III

For large bodies (e.g., over twenty members) with weighted voting, in which there are many different weights and no weight close to a majority, a good approximation to the power indices is that they are proportional to w/W–w), where w is the weight of a given member and W is the total vote. The constant of proportionality is somewhat less than 1; being at most (n–1)/n where n is the number of members. Since (W–w) is close to W when w is small, the smaller members will have power ratios approximately equal to that constant of proportionality, and hence power indices slightly less than their relative weights: The larger members will have power ratios correspondingly greater than 1. This indicates a general bias in favor of the more heavily weighted members. But it also places an upper limit on the degree of bias, for it means that the <u>spread</u> between the power ratio of the largest member and that of the smallest member is about equal to the difference between their relative weights, and hence will seldom exceed the relative weight of the largest member. (In the Electoral College, which is the largest weighted "legislature" for which the actual computations have been carried out, this spread amounts to about 8%, as already remarked. This proves to be a rather small bias when compared with the intentional advantage given the smaller states by the framers, in the form of two extra votes to each state regardless of population.[11]

[11] Mann & Shapley, op. cit.

An analysis of a recently-proposed system of weighted voting for the New Mexico State Senate yields similar results, as shown in Table 1. Thirty-seven districts were proposed, with votes in the Senate ranging from 1 to 32 in proportion to population, the total vote being 509. The power ratios reveal a small but systematic bias in favor of the larger districts. The total "spread", however, is only 6.1%, which cannot be considered a very serious distortion of the intended allocation of power among the senators.

Flukish exceptions to the general rule just described for obtaining the indices approximately, do exist. For example, the case (12, 6, 6, 4, 3) previously discussed can be generalized to any number of members, thus: (... ... ..., 6, 6, 4, 3). If the unspecified weights are all multiples of six, and if their total is a multiple of twelve, then the smallest member will always be a dummy: $\phi_3 = 0$. Moreover (as the reader can easily verify in the five-member example), the four-vote member is as powerful as a six-vote member: $\phi_4 = \phi_6$, which makes for a considerable spread in their power ratios. Though possible, a gross distortion of this kind is extremely unlikely to occur in a real situation.

Such exceptions aside, it is safe to assume that the bias given by difference of the weights of the members is not very great so long as one considers only the weights

TABLE 1

Weight	Power Index (%)	Power Ratio
32	6.43	1.023
31	6.22	1.021
29	5.79	1.016
28	5.58	1.014
27	5.37	1.012
24 (4 seats)	4.74	1.006
23 (3 seats)	4.53	1.004
21	4.12	0.999
20 (2 seats)	3.92	0.997
17	3.31	0.991
13 (2 seats)	2.51	0.984
10	1.92	0.978
9 (2 seats)	1.73	0.976
8	1.53	0.974
7 (3 seats)	1.34	0.972
5 (2 seats)	0.95	0.969
4	0.76	0.967
3 (6 seats)	0.57	0.965
2	0.38	0.963
1 (2 seats)	0.19	0.962

and powers of the members. But the Supreme Court asks us to look beyond the legislature, and to consider also the influence of the citizens.[12] And this introduces complications.

The Supreme Court seems to ask that each citizen have the same chance as each other citizen to influence legislative outcomes. This suggests that power indices should be calculated for citizens rather than legislators. And for this calculation we have two quite different models available, according to whether the representative is regarded as a Burkean legislator, who acts for the whole area represented without consideration of the particular interests of his constituency, or as a delegate, who on each issue acts only on behalf of his constituency. We will call the first model the free agent representative and the second model the delegate representative.

In the free agent case, where representatives supposedly seek to satisfy the general interest, it is difficult to say that the citizens have any power at all, for they have abdicated to their legislators. But, since citizens do elect legislators, we can conventionally assume that each citizen has a power index of ϕ_w/d where w is the weight of

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[12] "And the right of suffrage can be denied by a debasement or dilution of the weight of a citizen's vote just as effectively as by wholly prohibiting the free exercise of the franchise." Mr. Chief Justice Warren in Reynolds v Sims, 377 US 533, at 555.

his representative in the legislature and d is the size of his district. In this case, therefore, the power of citizens can vary only as much as the power ratios of the legislators, provided that the weights are assigned in proportion to district size. And, as we have already indicated, this is not usually a great variation.

At the other extreme, where legislators act simply as "funnels" for their constituencies' decision, as, for example, in the Electoral College, the calculation of citizens' power indices is much more complex, but the result is quite startling. In general, the citizen in a small district proves to have a much smaller power index than the citizen of a large district. Without involving ourselves in excessive mathematical detail, let us see why this must be so.

Consider the following model: An election in which there is a single, overriding issue — e.g., which party is to rule — is held for a weighted-vote legislature with weights proportional to the sizes of the different districts. The process of coalition-formation — i.e., campaigning — may be thought of as assembling supporters for a "platform", which can be modified and reformulated to attract increasing numbers of supporters. (It is easy to visualize this in the electoral college, where the platform, in the present sense, is essentially just the presidential candidate and his running mate. In the kind of legislative system we are

now imagining, where the legislators are mere ciphers transmitting the popular will, the analogous thing to the presidential ticket is not so much the several legislative candidates, as the bundle of opinions they carry, or the program that they advocate.) The election is decided by the fact that one of the competing bundle of opinions obtains the support of a winning coalition of the whole electorate, that is, more than fifty per cent of the voters in enough districts to make up a weighted majority in the legislature. For simplicity of exposition, we are assuming a unicameral system, without executive veto.

In the process of building up a coalition, the citizens are assumed to join (i.e., decide to vote for it) in random order. But the order in which majorities in the various districts occur will not be purely random. To see this, suppose at some point exactly thirty per cent of the population supports a coalition. Since its supporters join in random order, each district will have about thirty per cent of its population in the coalition. But this will not be exact. The larger districts will, usually, have closer to thirty per cent than the smaller, simply because they represent larger samples of the total population. In other words, whatever the overall percentage is at any one time, the district percentages will be dispersed around that number, with the smaller districts having the larger standard deviation.

Thus, as the coalition approaches winning size, adding citizens drawn at random from the population, the various districts will begin to swing over and give the coalition majority support, but not all at once. Because each district is a unique sample of the total population, some districts will come in sooner than others. And they do not do it in purely random order because of the differences in standard deviation. The big districts will tend to come in near the middle of the list. This definitely enhances their chance of pivoting. In consequence, a big district has distinctly more power than the power index of its delegate in the legislature would indicate. A new bias, under the delegate representative model, is therefore superimposed on the bias already discussed; both favor the larger districts at the expense of the smaller.

The magnitude of this effect is surprisingly large. An estimate using techniques of probability theory, supported by numerical computations for some particular cases, indicates that the citizens' power indices are multiplied by a factor proportional to the square root of their district populations when we pass from the free-agent model to the delegate model. For example, if district A has 10,000 citizens and its delegate has one vote in the legislature, and if district B has 40,000 citizens and its delegate has four votes, then a citizen of B has roughly twice the chance of being pivotal in an election as a

citizen of A. If district C has 250,000 citizens, then its citizens are individually five times as powerful (under the assumptions of the delegate model) as those of A. A very considerable advantage accrues to the citizens of large districts.[13]

The principle behind this "square root" bias in two-step voting systems can be expressed in a few words independently of our detailed definition of power. The influence of the citizens is most strongly felt and exercised in close elections. But if the overall election is close, then all other things being equal a large district is more apt to be closely divided than a small district, because the larger district is a better sample of the total population.

To compare the effects of the two models — free agent and delegate representative — we present an example of a ten seat assembly, in which the legislators have one vote for each 100 citizens represented. (Table 2. We have purposely made the total population a round number, and the district sizes perfect squares, in order to make the relationships amongst the numbers in the table more apparent.)

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[13] This conclusion can be supported by an independent argument (which is closely related to the variant described in footnote 2). Let each citizen assume that all other citizens vote purely at random, and let him ask what the chance is of his vote deciding the outcome. This number is of course very small, since it requires essentially a tie vote in his district, coupled with a nearly evenly divided legislature. However, this small number again proves to be approximately proportional to the square root of the size of the citizen's district.

TABLE 2*

		Powers of Legislators			Powers of Citizens		
Size of Dist'ct	Votes in Assembly	Ideal case	Free Agent case	Delegate case	Ideal case	Free Agent case	Delegate case
400	4	.04	.0333	.0178	.0001	.0000833	.0000445
400	4	.04	.0333	.0178	.0001	.0000833	.0000445
400	4	.04	.0333	.0178	.0001	.0000833	.0000445
400	4	.04	.0333	.0178	.0001	.0000833	.0000445
900	9	.09	.0921	.0737	.0001	.0001023	.0000819
900	9	.09	.0921	.0737	.0001	.0001023	.0000819
900	9	.09	.0921	.0737	.0001	.0001023	.0000819
1600	16	.16	.1504	.1606	.0001	.0000940	.0001003
1600	16	.16	.1504	.1606	.0001	.0000940	.0001003
2500	25	.25	.2897	.3866	.0001	.0001159	.0001546
Col # 1	2	3	4	5	6	7	8

* Cols. 6, 7, 8 are Cols. 3, 4, 5 divided by Col. 1. Col. 4 gives the calculated power indices of the legislators in the assembly. (Their power ratios may be found by multiplying Col. 7 by 10000.) Col. 8 gives the approximate power index of a typical citizen in each district, in the two-step voting model; it was obtained by multiplying Col. 7 by the square root of the district size and then rescaling, so that the sum of the powers of all citizens would total 1.

In this example, there are 10,000 citizens, so that under the ideal of equality embodied in the phrase 'one man, one vote,' the ideal power index for citizens would be .0001. It is apparent that neither interpretation of weighted voting achieves this ideal, and that in the delegate case it departs from it widely and consistently. (The irregularities that appear in the free agent case are due to the small size of the assembly and the even smaller number of different weights. A larger example would show a more consistent pattern, as in Table 1, with only moderate departures from the 'ideal' case.)

A few comments on this example are in order. Columns 3 and 6 are what the advocates of weighted voting think (or at least say) they are establishing, and column 6 is what would be achieved by equal districts with equal votes in the legislature. Columns 4 and 7 are what weighted voting actually establishes, if one accepts the free agent model, while columns 5 and 8 are what is established if one accepts the delegate model. Note in column 8 that the power of a citizen in the largest district is more than three times as great as the power of a citizen in one of the smallest districts.

We do not presume to decide whether the free agent model or the delegate model, or something in between, is preferred for American legislatures. Apparently sometimes one, sometimes the other fits the situation better.

We only note that the free agent model is peculiarly attractive to legislators and less attractive to citizens. Burke first set forth the free agent model in a speech to the electors of Bristol, <u>after</u> he had been elected to a seat which he <u>never</u> thereafter contested. Eulau et al found that in the four states they studied state legislators viewed themselves as free agents (or "trustees" in Eulau's terminology) much more frequently than as delegates.[14] Unfortunately Eulau had no independent way of observing their behavior so we do not know that their own role conceptions were actually carried through into action. And internal evidence in the findings suggests that the preference for the free agent model is simply a consequence of the conventional culture of legislatures in which the creative role of trustee is more highly valued than the more or less mechanical role of delegate. Contrasting with this data, Miller and Stokes found by interviewing congressmen and constituents simultaneously that congressmen in 1958 conformed to the delegate model in issues such as civil rights and to our formulation of the delegate model on social welfare issues, but they they tended to be free agents in foreign affairs.[15] This evidence suggests that

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[14] John Wahlke, Heenz Eulau, William Buchanan and LeRoy C. Ferguson, <u>The Legislative System</u> (New York, John Wiley and Sons, 1962), p. 281

[15] Warren E. Miller and Donald E. Stokes, "Constituency Influence in Congress," 57 <u>American Political Science Review</u> (1963), pp. 45-56.

neither model is entirely satisfactory in general but that on some occasions one fits and on other occasions the other fits.[16] We conclude, therefore, that if weighted voting occurs in large legislatures and if the free agent model is appropriate, then weighted voting introduces a mild bias in favor of the larger districts. On the other hand, for large legislatures, if the delegate model is appropriate, weighted voting introduces a very large bias in favor of the citizens of larger districts. At least in this latter circumstance, therefore, weighted voting is entirely inappropriate for achieving the ideal of "one person, one vote."

In the beginning of this essay we suggested a paradox: that those whom its advocates say are its beneficiaries are probably those who are hurt by weighted voting, while those whom its opponents say are its victims probably gain from it. This paradox can now be easily explained.

Typically the reason (whether publicly stated or not) for advocating weighted voting has been that the advocate wishes to save the seats of representatives from small districts which are likely to be consolidated under reapportionment into districts of equal size.[17] Furthermore,

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16 This is the conclusion reached also in Lewis A. Froman, Congressmen and Their Constituencies (Chicago, Rand McNally and Co., 1963).

17 In the report of the Mulligan committee, which proposed weighted voting in the New York legislature, it was argued that such voting "would provide the people in the less populous counties... an independent voice in the Assembly." Citizens Committee on Reapportionment, Report (New York, 1964), p. 38.

it has almost invariably been the citizens of large districts who have brought suits against the adoption of weighted voting in legislatures, presumably because they feared that their influence would be less under a system of weighted voting than under a system of districts of equal size. If the analysis in this essay is correct, however, both advocates and opponents have been wrong and each have taken positions contrary to their own best interests, which is the paradox we set out to explain. And the explanation appears to be quite simple: Both parties to the dispute have apparently assumed that the citizens and legislators from the smaller districts would have more power than citizens and legislators from larger districts simply because the former have more legislators than the latter. If, as we have argued, this assumption is false, then the paradox is explained.

Reverting to our initial purpose of showing the utility of mathematical analysis for political philosophy, we observe that the paradoxical behavior of both advocates and opponents demonstrates that our mathematically arrived at conclusions are not immediately obvious to commonsense. In this circumstance at least a mathematical analysis has made possible a complete reversal of judgment on the appropriateness of a particular instrument for realizing a given conception of justice. We believe it is quite likely that mathematical analysis might similarly clarify many

other problems of political philosophy especially when (as with the notions of equality, representation, freedom, liberalism, etc.) both the ideals and the instruments have some quantitative elements.