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The distribution among the members of formal control over a collectivity's actions is an important element of constitutions. The exercise of such control is mediated by decision rules, which may be very complex, involving preliminary decisions by sub-collectivities within the larger collectivity. Framers of constitutions pay very special attention to the distribution of control and to the form of the decision rules, for these determine in large part the power of various members of the collectivity as well as the power of the collectivity vis-a-vis the members. Illustrations that give some intuitive feel for this are easily found. The bicameral form of the U.S. Congress, with equal representation of states in one, and equal representation of individuals in the other, and with a requirement that legislation be affirmed by both bodies, is one example -- because there were, and are, two types of members of the United States, individual citizens as members and states as members. This paradoxical state of affairs, in which a federal body has as members both the constituent states and the members of these constituent states, was arrived at as one aspect of the resolution of the problem of sovereignty of states vs. sovereignty of the nation, and obviously reflects the outcome of a struggle between states with many persons and states with few, a struggle documented by numerous political historians.

Another example of a decision rule reflecting the different status of individual members and giving a heuristic sense of the power of the

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collectivity to act is the United Nations Security Council. The Security Council originally had eleven members, including five permanent, taking action upon a positive vote of seven of the eleven members, not a bare majority of six, but only under the condition that the positive votes include those of each permanent member. This innocuous-sounding procedure masks two points that become obvious upon some further examination of the rule: first, that the permanent members have allocated no power to the Security Council whatsoever, because each retains a veto; and second, that the relative control of the body's action by the permanent and temporary members is very skewed indeed, since no action can be taken without the vote of a permanent member, while an action can be taken without the votes of as many as four of the six temporary members.

Despite the importance of these formal structures for controlling the action of collective bodies, little theoretical work has been devoted to their analysis. In this paper, I will develop modes of analysis that offer some answers to the questions posed by constitutions, providing some mathematical analysis of situations which, though they can be stated quite precisely (as they are in constitutions) have hardly been subject to such analysis.

The Shapley-Shubik Measure of Power

There has been one major approach to the problem of describing the power of members of a collectivity by analysis of the decision rule, a measure proposed by Shapley and Shubik (1956). Subsequent to that work, there have been several applications made of the Shapley-Shubik measure, and numerous discussions of the measure, but little serious analysis of the problem of collectivity control and collectivity power other than by applications of this measure.

I propose here to examine in some detail both the problem of the distribution of control of the actions of a collectivity by its members and the problem of the power of the collectivity to act. One useful way to begin this is by examination of the Shapley-Shubik measure, which has some remarkable properties, and which is unique in these properties. In presenting a measure for the power of a member in a game, or any collectivity with a well-defined decision rule, Shapley and Shubik propose a measure that has the following three properties:

(a) The measure of a member's power inheres in his structural position in the collectivity defined by the rules, and in any relabelling, the procedure will reassign to members power measures according to the structural position, and not the new labels assigned to them.

(b) The measures of power are additive: if total control over the actions of the collectivity is counted as unity, then the sum of the measures of power of members will sum to one. If $\phi_i(N)$ is the measure of power of individual i in collectivity N, and the members of the collectivity are labelled i=1, ..., n, then

$$\phi_1(N) + \ldots + \phi_n(N) = 1$$

(c) If an individual has a given amount of power in one collectivity x, and a given amount of power in another collectivity y defined over the same members but with a different constitution (as an example, the power in the House of Representatives and in the Senate of an individual citizen from a given sized congressional constituency in a state of a given size), then his power in the combined collectivity (e.g., the Congress) is the sum of his power in the two separate collectivities.

The remarkable point about the measure developed by Shapley as a measure of the value of a game to a player and used by Shapley and Shubik as a measure of power in a collectivity is that it is the only measure fulfilling these conditions. Yet this remarkable fact should be noted and set aside; the crucial question about measures of power in a collectivity (or control of the collectivity's actions) should derive from considerations of the internal structure of the measure, and whether this structure corresponds to what one believes ought to be entailed by a measure of power. Consequently, I will examine the internal structure of this measure -- as, of course, many have done before now -- preliminary to discussing more generally the questions of control of a collectivity's action by its members and the power of the collectivity itself to act.

As a way of introducing consideration of the Shapley-Shubik measure, its background is important. Shapley developed the measure as a measure of the value of a game to its players -- reasoning that the expected

value of the game for a player is the value that he can expect to get out of it, and that this in turn is directly dependent on his value to the winning of the game -- that is, his contribution to the coalitions that constitute winning coalitions. The measure is more general than this, because it allows for the fact that in a game, the outcome may be something other than a situation in which the winning coalition wins all -- but that is not important for the application to power in collectivities. What is important is that the measure was first obtained as a measure of the value a player can associate with a given game, and that it is based on the important identity between value and power: the value a player associates with a game, the value that he can expect to get from it, is precisely what he can expect to realize from the game, which in turn is his power to affect the outcome of the game.*

The origin of the Shapley-Shubik measure of power is important because it gives some sense of the motivation behind the measure, and its intended original purpose. The measure was then adapted as a measure of power in a collectivity by setting the overall value of the game as 1, and determining that a coalition received the value of the game (normalized to unity as indicated) if the coalition was sufficient, according to the rules of the collectivity, to obtain passage of an action by the collectivity. This general orientation is somewhat different, as indicated above, from the usual problem of power in a collectivity, for the usual problem is not one in which there is a division of the spoils among the winners, but rather the problem of controlling the action of the collectivity. The action is ordinarily one that carries its own

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*This measure of value assumes a game in which there is full possibility of dividing the winnings among the members of the winning coalition, a division that will take place according to the contribution of each member to winning (in a sense to be defined shortly). This assumption is not valid for collectivities in which a proposed action is to be accepted or rejected, as in political bodies, for each action carries a fixed distribution of consequences. Thus as a measure of the "value of the game" to members of a collectivity, the Shapley measure would not be valid. However, this difference between political actions to be decided by a collective decision, and games in which there is a division of the winnings among the members of a winning coalition does not affect the validity of the measure as a measure of power.

consequences or distributions of utilities, and these cannot be varied at will, i.e., cannot be split up among those who constitute the winning coalition.* Instead, the typical question is the determination of whether or not a given course of action will be taken or not, that is, the passage of a bill, a resolution, or a measure committing the collectivity to an action.

The Shapley measure of value, converted by Shapley and Shubik to a measure of power, is in mathematical form, given by the following expression:

$$\phi_i(N) = \sum \frac{(s - 1)!(n - s)!}{n!} \delta_i(S) \tag{1}$$

where the summation is taken over all subsets of the collectivity of which i is a member,

s is the size of the subset or coalition S of which i is a member,

n is the size of the collectivity as a whole, and

$\delta_i(S)$ is 1 if the subset S including i is a winning coalition but is not a winning coalition without i, and zero otherwise.

Let us consider the total collectivity. In this collectivity, there are n! different orders in which members can cast votes. Consider for each of these orders that each vote is cast in favor until exactly the necessary number of votes for passage is obtained, and then all the remainder are cast against. If we were to list all these sequences, in each one the member whose vote is the last positive vote can be designated as the determining member. In a simple majority voting system in a collectivity with an odd number of members, this would be the member casting the (N+1)/2 vote. Since each member is in this position the same number of times, the power of each is 1/n.

*Certainly it is the case that in some collective actions, such a division of the spoils does in fact occur. A convention has some of these attributes, for there are spoils to be distributed among those delegations that support the winner, and particularly those delegations that cast the deciding ballots in favor of the winner. But this is an unusual case, in which there is a winning nominee, who does have spoils to distribute.

If we count the number of times member i is in this position, and divide by the total number of sequences, this will give the proportion of times he was in a winning position. This is the measure of power, $\phi_i(N)$. It is possible now to reverse the sequence of voting. Then if his positive vote made him the determining member for the positive coalition in the forward sequence, his negative vote would put him in the determining position for a winning negative coalition in the reverse sequence. Thus whether considering positive or negative coalitions, the proportion of times his vote is the determining one is the same, and is $\phi_i(N)$. Obviously the sum of $\phi_i(N)$ over all n members will be one, since in every sequence there is one and only one member whose vote is deciding. Almost as obviously, the member's power in two subcollectivities sum to give his power in the larger collectivity: Consider a collectivity's decision to be the result of decisions by two (or more) subcollectivities. Then if we write these n! sequences, it is again possible to designate the member's vote that determines the outcome in the larger collectivity, even though it does so by determining the outcome of the subcollectivity. Thus for the collectivity as a whole, there are n measures, $\phi_i(N)$, which sum to one and are measures of individual power. But the proportion of times that subcollectivity N_j (with n_j members) is determining is simply the sum of the n_j values of $\phi_i(N)$ for its members. Thus the measure of power of the subcollectivity as a whole is $\phi(N_j)\left[=\sum_{i \epsilon N_j} \phi_i(N)\right]$, and the measure of the member's power in the subcollectivity is $\phi_i(N)$, the same as his measure of power in the larger collectivity.*

These considerations show that the Shapley-Shubik measure has the properties indicated earlier in conditions (ii) and (iii), though of course they do not show that it is the only measure with these properties. But it is useful to examine some implicit assumptions of the

*If the subcollectivity's power is normalized to one, then his measure of power in the subcollectivity is $\phi_i(N)/\phi(N_j)$, which is the same measure as would be obtained by applying Eq. (1) to the subcollectivity N_j rather than to the collectivity N.

measure. It counts each order equally, and since all persons are equally likely to be in a given position, and the vote in a given position is positive in exactly half of the n! orders, it assumes each member is equally likely to vote positive or negative. Secondly, if we look at the list of the n! orders that have been written down, and pick out one in which member i is the determining voter, and regard as the set S (with s members including himself) this particular coalition, then there are other orders in the list in which he is still in the determining position for this same coalition S, but the other members of S and the members of the losing coalition are ordered differently. Altogether, there are (s-1)!(n-s)! such orders, for this same coalition S, with him in the determining position. Each of these orders is counted once in forming the measure.

This last property of the measure produces some curious results that can best be seen by example. Thus I will apply this to an example treated by Shapley and Shubik, the original U.N. Security Council, to show the results of this property. There are two types of members, five permanent, with a veto, and six temporary, without, and the voting rule is taken with a majority of seven or more and no vetoes. Since there are only two types of members, I will not show all 11! orders, but only those for which #5 (the last-numbered permanent member) and #11 (the last-numbered temporary member) are determining. There happen to be 7,939,680 different orders in which #5 is determining, so that I will not list all these, but will merely indicate which orders they include. Those orders for which #5 is determining are:

composition of coalitions positive (determining) negative	number of different orders in these positive and negative coalitions
123467(5)891011	6!4! = 720·24 = 17,280
123468(5)791011	17,280
123469(5)781011	17,280
1234610(5)78911	17,280
1234611(5)78910	17,280
123478(5)891011	17,280
123479(5)681011	17,280
1234710(5)68911	17,280
1234711(5)68910	17,280
123489(5)671011	17,280

1234810(5)67911	17,280
1234811(5)67910	17,280
1234910(5)67811	17,280
1234911(5)67810	17,280
12341011(5)6789	17,280
1234678(5)91011	$7!3! = 5040\cdot 6 =$ 30,240
1234679(5)81011	30,240

and 18 more combinations each involving 3 different members from the set of 6 temporary members

12346789(5)1011	$8!2! = 40320.2 =$ 80,640
123467810(5)911	80,640
	80,640 each (x13)

and 13 more combinations each involving 4 different members from the set of 6 temporary members

1234678910(5)11	$9!1! = 362{,}880\cdot 1 =$ 362,880
1234678911(5)10	362,880
	362,880 each (x4)

and 4 more combinations each involving 5 different members from the set of 6 temporary members

123123467891011(5)	$10!0! = 3{,}628{,}800\cdot 1 =$ 3,628,800

Altogether, this gives the following tabulation for member #5:

6-4	coalitions	15	different	coalitions,	17,280	orders	for	ea. =	259,200
7-3	"	20	"	"	30,240	"	"	" =	604,800
8-2	"	15	"	"	80,640	"	"	" =	1,209,600
9-1	"	6	"	"	362,880	"	"	" =	2,237,280
10-0	"	1	"	"	3,628,800	"	"	" =	3,628,800
		57	different	coalitions					7,939,680 different orders

For member #11, the only coalitions in which he is a determining member are those which include all five permanent members and one temporary member, as follows:

123456(11)78910	$6!4! = 720 \cdot 24 = 17{,}280$
123457(11)68910	17,280
123458(11)67910	17,280
123459(11)67810	17,280
1234510(11)6789	17,280
5 different coalitions	86,400 different orders

According to the Shapley-Shubik measure, the relative power of member #5 and member #11 is 7,939,680:86,400, or over 90:1, since this is the ratio of the number of differently-ordered coalitions in which these two are the determining members. Adding all the orders in which the five permanent members are determining and all those in which the six temporary members are determining, we can arrive at the proportion of power held by the permanent five and the temporary six. This is .987 for the five, and .013 for the six, a result arrived at by Shapley and Shubik in their paper.

This disparity arises from two sources: the greater number of coalitions in which #5 (or the other permanent members) is a determining member, and the greater number of different orders for a given coalition, for most of the coalitions in which #5 is a determining member. For example, in the coalition of the whole, in which #5 is a determining member, there are 210 times as many different orders as in the one type of coalition in which #11 is the determining member. Thus, if for example, #5 was a determining member only in this coalition, and #11 was a determining member only in a single coalition of the 6-4 type, then #5 would have a power index 210 times that of #11, even though both were determining in only one distinct coalition. Such a disparity seems inappropriate, for it is implicitly based on the assumption that the probability that a coalition will form is proportional to the number of different orders that can exist in that coalition. It seems more reasonable to assume that each distinct coalition has the <u>same</u> probability of forming. The difficulty, of course, in doing this is that if each distinct coalition is counted only once, with no attention to the order of its members, then there may be several members who could cast the determining vote. The question then arises, to whose credit should it be counted?

Such questions as this arise with any measure of power other than the one which Shapley and Shubik propose. Its virtue lies in the mathematical properties it exhibits, which prevents this and other questions from arising. Nevertheless, some other questions must be asked about the measure, in view of its properties as just illustrated in the example. First, one might ask whether too high a price has been paid for the mathematical properties, since the resulting measure seems to have at least one unreasonable property, its dependence on the different orders of members of a coalition. Beyond that, however, it is useful to reexamine just what a concept of power should be, just whether it should have the properties taken implicitly as desirable, and whether the Shapley-Shubik measure does indeed have the properties which such a measure should have.

The Shapley measure was initially developed in a context of zero-sum and non-zero sum games, in which there is some "value" of the game, or good which constitutes the expected outcome if the game is rationally played. The Shapley measure of value is a measure of how that value could be expected to be divided among the players, given that each exercised rationally the power given him in the rules or constitution of the game. This realization of value would be arrived at through division of the winnings by the winning coalition.

But the situation posed by decisions in collective bodies is ordinarily quite different. The decision governs an <u>action</u> to be taken -- or not taken, depending on the outcome -- by the collectivity, an action that has a fixed profile of consequences for the members. Thus, since it is not a question of a battle over the division of spoils, as assumed by the Shapley value, one is not concerned directly with the power of members vis-à-vis one another. Instead, several other concepts are important. One is the power of the group to act. Such power may be very small, if there are very few vote outcomes that lead to action, or very large if most outcomes lead to action. Second, there is the power of the member vis-à-vis the acting collectivity; his power to <u>prevent</u> action of the collectivity, and his power to <u>initiate</u> action of the collectivity -- or, in short, his degree of control over the collectivity's actions, in both positive and negative directions. Now it may be useful

to compare members of the collectivity with regard to their power to initiate action or their power to prevent action -- but since each is exerting that power on the collectivity, the functional comparisons, which pertain directly to the collectivity's action, are between the collectivity's power to act and each member's power to initiate or prevent action.

The Power of Collectivities to Act

The power of collectivities to act, as provided by a set of constitutional rules, lies in the ease with which individual members' interests in collective action can be translated into actual collective action. If each member has a veto power over the collectivity's action, then this power to act is very small; if any member can initiate action, as in giving a fire alarm, the collectivity's power to act is very great. One might ask why members would devise a constitution that gave a collectivity only very little power to act; but the answer is obvious to anyone who has ever been a member of a collective body: The power of a collectivity to act is the power to act in accord with the aims or interests of some members, but often against the aims or interests of others. Thus for a collectivity of a given size, the greater the power of the collectivity to act, the more power it has to act against the interests of some members.

We may develop a measure of the power of a collectivity to act by using some, but not all, of the ideas behind the Shapley-Shubik measure. That is, we retain the idea of examining all possible subsets, but we count a distinct subset only once, not according to its number of possible different orders. The subset S has the value $v(S) = 1$ if the subset, under the rules of the constitution, is sufficient to make the collectivity act, that is, sufficient to pass a positive action. Otherwise, the value of the subset, $v(S)$, is zero.

In a collectivity of n members, there are 2^n distinct partitions of the collectivity into two subsets, that is, 2^n different subsets favoring the action (including the subset with no members, and the coalition with all members), and for each of these, a complementary subset consisting of the remaining members. For collectivities of five and six members, lists of the distinct partitions are included as Appendix A,

for reference use in later examples. Thus a natural measure of the power of the group to act under the constitutional rules is merely the sum of v(S) over all distinct subsets or coalitions S, divided by 2^n:

$$A(N) = \frac{\Sigma v(S)}{2^n} \tag{2}$$

If a collectivity operated under a decision rule in which each member has a veto, that is a unanimity decision rule, the power is at a minimum for this size collectivity, since only the coalition of the whole will give action: $A(N) = 1/2^n$. If the collectivity has an odd number of members, and the decision rule is a majority rule, then exactly half the coalitions will be equal to or greater than the member required for action, $(n+1)/2$, and $A(N) = 1/2$. In the case of the U.N. Security Council, with a veto for five members and a rule of 2 out of 6 for the other six, there are 57 coalitions out of 2048 in which the collectivity will act, and $A(N) = 57/2048 = .0278$.

In a collectivity with an even number of members, where $n/2+1$ positive votes are necessary for action, then the power of the collectivity to act is slightly less than 1/2, since all the coalitions with exactly n/2 members fail, whereas it would be necessary for half of them to pass if A(N) were to equal 1/2. Since there are $n!/(\frac{n}{2}!\frac{n}{2}!)$ distinct coalitions with exactly half the members, then for an even-membered group with a majority rule,

$$A(N,n/2 + 1) = \frac{1}{2} - \frac{1}{2^{n+1}}\left(\frac{n!}{\frac{n}{2}!\frac{n}{2}!}\right) \tag{3}$$

More generally, if the decision rule requires a positive vote by m out of n members for passage of an action, then the power of the collectivity is the sum of all those combinations of n members, taken m or more at a time, divided by 2^n:

$$A(N,m) = \frac{\sum_{k=m}^{n} \frac{n!}{k!(n-k)!}}{2^n} \tag{4}$$

It is interesting to write the expression for the power of the United States to act through legislation. If legislation must pass a committee in the House of Representatives of n_1 members, a committee in the Senate of n_2 members, then pass both bodies by a majority and obtain the President's signature, or pass both by a two-thirds vote without the President's signature, then the total number of subsets or coalitions is $2^{n_1+n_2+100+435+1}$. If the power of the two committees to act, as calculated by Eq. (4), is $A(N_1)$ and $A(N_2)$ respectively, then the power of the nation to act through legislation is:

$$A(N) = \frac{A(N_1)A(N_2)}{2^{n_3+n_4}} \left[\frac{1}{2}\sum_k \frac{n_3!}{k!(n_3-k)!} \sum_j \frac{n_4!}{j!(n_4-j)!} + \frac{1}{2}\sum_h \frac{n_3!}{h!(n_3-h)!} \sum_r \frac{n_4!}{r!(n_4-r)!} \right] \tag{5}$$

where the limits of summation are, for k and j,

$$\frac{n_3+1}{2} \le k \le n_3 \ , \ \frac{n_4+1}{2} \le j \le n_4,$$

and for h and r,

$$\frac{2n_3}{3} \le h \le n_3 \ , \ \frac{2n_4}{3} \le r \le n_4,$$

and where n_3 and n_4 are the sizes of the House and Senate.

Equation (5) includes, in the first term within brackets, all those partitions of Senate and House from a majority vote up to the total, multiplied by 1/2, which is the power to act of a collectivity con-

sisting of one person, in this case the President, plus all those partitions of Senate and House from a 2/3 majority to unanimity, which are sufficient for passage without the President's assent.

The concept of the power of a collectivity to act can be applied also to those situations in which a public good, or a public bad, can be supplied by only one or a few members of a collectivity. For example, if one person in a community can pollute a stream, then the power of the community to act in polluting the stream is obtained by summing all those partitions except those in which no members are positive:

$$A(N,1) = 1 - \frac{1}{2^n} \quad . \tag{6}$$

Thus, unless n is very small, A(N,1) will be close to 1.0. This reflects the difficulties that arise in preventing the creation of public nuisances: the "action of the collectivity" can be initiated by one person, or a few, and does not require assent from the other members of the community. In contrast, the decision rule for those collective actions that can restrain the public nuisance is ordinarily at least a majority vote, a rule that gives the collectivity much less power to act.

However, only a portion of the total problem is treated by examining the power of the collectivity to act. The power of individual members to prevent action or to initiate action must be examined as well. It is to the first of these that I now turn.

The Power of Members to Prevent Action

The Shapley-Shubik measure derives a member's power from the number of ordered subsets in which he is the determining member. In the modified context under consideration here, his power as an individual member vis-à-vis the collectivity should derive from the number of subsets (distinct in membership, not in ordering) in which he is a determining member in initiating or preventing the group's action. Since we have just examined the collectivity's power to act, it is natural to examine

his power to prevent its action. Following the conception described above, and labelling individual i's power to prevent action $P_i(N)$, it is:

$$P_i(N) = \frac{\sum_S [v(S) - v(S - [i])}{\sum_S v(S)} , \tag{7}$$

that is, his power to prevent action is the number of subsets in which he is a determining member, divided by the number of subsets in which the collectivity acts. For the special case in which all members have identical roles, and there is a decision rule with m out of n votes necessary for collective action, $P_i(N)$ is a function only of n amd m. In this case, $P_i(N)$ can be expressed as $P_i(n,m)$. In this simple case, a reasonably direct way of calculating $P_i(n,m)$ can be given, from the fact that the numerator of Eq. (7) is the number of partitions in which his defection would convert passage to failure, and in a simple collectivity with n members and m required for passage, this is the number of combinations of the other n-1 members partitioned so that exactly m-1 of them are positive, or in other words, the combination of n-1 things taken m-1 at a time, or $\frac{(n - 1)!}{(m - 1)!(n - m)!}$. The denominator is the sum of the number of combinations of n things taken k at a time, for k from m to n. Thus

$$P_i(n,m) = \frac{\frac{(n - 1)!}{(m - 1)!(n - m)!}}{\sum_{k=m}^{n} \frac{n!}{k!(n - k)!}} ,$$

or simplifying, (8)

$$P_i(n,m) = \frac{1}{n \sum_{k=m}^{n} \frac{(m - 1)!(n - m)!}{k!(n - k)!}}$$

In the case of the U.N. Security Council, the number of partitionings in which the Council acts is 57, and the number in which any of the first five members is determining is also 57, so that

$$P_1(N) = \frac{57}{57} = 1.0.$$

The power of any of the six temporary members to prevent action is calculated similarly: There are five coalitions out of the 57 in which any one of the temporary members can be deciding, so that

$$P_6(N) = \frac{5}{57} = .088.$$

In each case, $P_i(N)$ can be conceived as a fraction of $A(N)$: It is that proportion of all partitions in which the collectivity acts that member i can prevent action. If the proportion is 1.0, as it is when member i has veto power, the collectivity's power to act is exactly counterbalanced by the individual member's power to prevent action. If it is less than one, the member's power to prevent action is only that fraction of the collectivity's power to act.

<u>The Power of Members to Initiate Action</u>

By similar considerations, the power of individuals to initiate action may be conceived. The number of partitions in which the individual is determining is the number in which he can, by changing his direction of action, change a negative collective action to a positive one, and thus initiate action. As a measure of the member's power to initiate action, it should be compared to the total number of partitions in which no action occurs.

$$I_i(N) = \frac{\sum_S (v(S + [i]) - v(S))}{\sum_S (1 - v(S))} . \tag{9}$$

For the special case in which all members have identical roles, and the collectivity has a decision rule with m of n required for action, $I_i(N)$ can be written as a function of n and m, $I_i(n,m)$. The method for calculation of $I_i(n,m)$ may be found by considerations similar to those used for finding $P_i(n,m)$. The numerator of Eq. (9) is the number of partitions in which individual i's addition would convert failure to passage, and in the simple collectivity, this is merely the number of combinations of the other n-1 members partitional with exactly m-1 positive, or $\frac{(n-1)!}{(m-1)!(n-m)!}$. The denominator in this case is the total number of partitions of the whole collectivity in which fewer than m members are positive, or $\sum_{k=0}^{m-1} \frac{n!}{k!(n-k)!}$. Thus after simplifying,

$$I_i(n,m) = \frac{1}{n \sum_{k=0}^{m-1} \frac{(m-1)!(n-m)!}{k!(n-k)!}} \tag{10}$$

In the case of a permanent member of the U.N. Security Council, the numerator is as for $P_i(N)$, and the denominator is 2048 - 57, to give

$$I_1(N) = \frac{57}{2048 - 57} = .0286 ,$$

and for the temporary members,

$$I_6(N) = \frac{5}{2048 - 57} = .0025 .$$

Thus the power to initiate action is small for the permanent as well as the temporary members. The permanent members of the security council, in devising this constitution, in effect sacrificed the power of the collectivity to act (A(N)), as well as their own power to initiate action, and in return kept complete power to prevent action by the collectivity.

Example: Change in U.N. Security Council Rule.

The United Nations Security Council changed its size and its voting rule, increasing the total size to 15 (10 instead of 6 temporary members), and changing the voting rule from 7/11 with no vetoes to 9/15 with no vetoes. It is useful to examine changes in the power of the Council and of the permanent and temporary members by this action. Calculations are below:

$$A(N^*) = \frac{\sum_{k=4}^{10} \frac{10!}{k!(10-k)!}}{2^{15}} = \frac{848}{32,768}$$

$$= .0259$$

$$P_1(N^*) = 1.0$$

$$P_6(N^*) = \frac{84}{848} = .099$$

$$I_1(N^*) = \frac{848}{32,768 - 848} = .0266$$

$$I_6(N^*) = \frac{84}{32,768 - 848} = .0026$$

Table 1 compares the structure of power before and after the change. It is evident that the change has been relatively minor. The power of the Council to act has reduced very slightly, as has the power of permanent members to initiate action. The power of the temporary members to prevent action has increased slightly, and their power to initiate action has increased even more slightly.

Since there is a larger number of temporary members, 10 rather than 6, one might be tempted to add their power to prevent or initiate action in the two cases, and compare the sums, which would show a greater difference. This temptation must be resisted, however, because the sum of

Table 1

POWER STRUCTURE OF U.N. SECURITY COUNCIL

Before and After Expansion		Before Change 5+6=11 : 7/11	After Change 5+10=15 : 9/15
Power of Council to act	$A(N)$	.0278	.0259
Power of permanent member to prevent action	$P_i(N)$	1.0	1.0
Power of temporary member to prevent action	$P_6(N)$	.088	.099
Power of permanent member to initiate action	$I_i(N)$	.0286	.0266
Power of temporary member to initiate action	$I_6(N)$	.0025	.0026

members' power has no operational meaning. Each member's power is power vis-à-vis the collectivity's action, and it must be maintained as such. Only later, after having examined the power of subcoalitions within the collectivity, will we be in a position to discuss the power of more than one temporary member.

Example: A simple five-man collectivity with differing decision rules.

Some sense of the way in which these measures of collectivity power and member power vary can be obtained by examining their variation as the decision rule changes from one in which a single member can initiate action (as in the case of creation of a public nuisance, or turning in a fire alarm) to one in which a single member can veto action.

Case I: One man necessary to initiate action:

$$A(N,1) = \frac{2^5 - \frac{5!}{0!5!}}{2^5} = \frac{31}{32}$$

$$P_i(N) = \frac{1}{31}$$

$$I_i(N) = \frac{1}{32 - 31} = 1$$

Case II: Two men necessary to initiate action:

$$A(N,2) = \frac{\sum_{k=2}^{S} \frac{5!}{k!(5-k)!}}{2^5} = \frac{26}{32} = \frac{13}{16}$$

$$P_i(N) = \frac{\frac{5!}{2!3!} \cdot \frac{2}{5}}{26} = \frac{4}{26} = \frac{2}{13}$$

$$I_i(N) = \frac{4}{32 - 26} = \frac{2}{3}$$

Case III: Three men necessary to initiate action:

$$A(N,3) = \frac{\sum_{k=3}^{S} \frac{5!}{k!(5-k)!}}{2^5} = \frac{16}{32} = \frac{1}{2}$$

$$P_i(N) = \frac{\frac{5!}{3!2!} \cdot \frac{3}{5}}{16} = \frac{6}{16} = \frac{3}{8}$$

$$I_i(N) = \frac{6}{32 - 16} = \frac{3}{8}$$

Case IV: Four men necessary to initiate action:

$$A(N,4) = \frac{\sum_{k=4}^{S} \frac{5!}{k!(5-k)!}}{2^5} = \frac{6}{32} = \frac{3}{16}$$

$$P_i(N) = \frac{\frac{5!}{4!1!} \cdot \frac{4}{5}}{6} = \frac{4}{6} = \frac{2}{3}$$

$$I_i(N) = \frac{4}{32 - 6} = \frac{4}{26} = \frac{2}{13}$$

Case V: Five men necessary to initiate action:

$$A(N,5) = \frac{\frac{5!}{5!0!}}{2^5} = \frac{1}{32}$$

$$P_i(N) = \frac{1}{1} = 1$$

$$I_i(N) = \frac{1}{32 - 1} = \frac{1}{31}$$

The tabulation of group and member power with these five decision rules shows the symmetry between rules which require m and n-m+1 votes for passage.

Table 2

POWER STRUCTURE WITH DIFFERENT DECISION RULES
IN COLLECTIVITY OF FIVE MEMBERS

		Number of members necessary for passage				
		1	2	3	4	5
Power of collectivity to act	$A(N,k)$	$\frac{31}{32}$	$\frac{13}{16}$	$\frac{1}{2}$	$\frac{3}{16}$	$\frac{1}{32}$
Power of member to prevent action	$P_i(N)$	$\frac{1}{31}$	$\frac{2}{13}$	$\frac{3}{8}$	$\frac{2}{3}$	1
Power of member to initiate action	$I_i(N)$	1	$\frac{2}{3}$	$\frac{3}{8}$	$\frac{2}{13}$	$\frac{1}{31}$

Example: Acting power of the United States

The acting power of the United States can be calculated through use of Eq. (5). If the House and Senate Committees involved (N_1 and N_2 in Eq. (5)) use a majority decision rule of an odd number of numbers for passing a bill out from committee, then $A(N_1) = A(N_2) = 0.5$. The quantity $\frac{1}{2^n}\sum\frac{n!}{k!(n-k)!}$, where the sum is taken over $m > n/2$, is 0.5, and the same sum for $m \geq 2n/3$ is .000437 for the Senate, and 1.6×10^{-12} for the House. Thus the full description of the power to act is

$$A(N) = 0.5 \cdot 0.5[0.5 \cdot 0.5 \cdot 0.5 + 0.5 \cdot .000437 \cdot 1.6 \times 10^{-12}] = .03125.$$

This power to act is in effect equivalent to the power of a simple collectivity of five members in which each member had a veto, leading to one affirmative partition out of 32. The effect on the measure of acting power of the House and Senate override of the presidential veto is negligible. The reason for this is that the proportion of partitions in which more than 2/3 of members are positive is very small, only about .0004 in the Senate, and 1.6×10^{-12} in the House. Thus the acting power of the U.S. in legislation is not at all affected by the existence of the veto override.

The actual power of the country to act is, of course, affected by this rule, because the probability of the Senate and House having more than 2/3 of their members in favor of an action is not accurately given by the proportion of partitions in which more than 2/3 are positive. This derives from the fact that the members' votes are not cast independently with probability 0.5, as this model implicitly assumes. There is a high degree of interdependence.

An estimate of the interdependence, and the effective number of independent numbers in the House and Senate could be obtained by estimating the variance of p (where p is the proportion of members voting positive) over all bills, and then equating this to the binomial variance, pq/n, for unknown n. An approximation to this may be obtained by use of an empirical regularity that has been observed in many

political elections in countries with two-party systems and single-member constituencies: the variance of p over different constituencies is about .0187. Setting this equal to pq/n, and letting $p \approx .5$, gives $n \approx 13$. Thus the constituencies act as if they are composed of 13 independent members, each with the same probability (probabilities alike, within and between constituencies) of voting for a given party.

If 13 is taken as the "effective" size of the House and Senate, assuming that they behave with the same degree of interdependence as do those voting bodies that have been studied (and the widespread uniformity of this variance, .0187, suggests that they do), then the acting power added by the veto override is no longer negligible. The proportion of partitions in a simple collectivity of size 13 that are greater than 2/3 positive (9 or more positive) is .1334. Thus the estimate of A(N) with this modification (but still assuming the House and Senate act independently) is:

$$A(N) = 0.5 \cdot 0.5[0.5 \cdot 0.5 \cdot 0.5 + 0.5 \cdot .1334 \cdot .1334]$$

$$= .0335$$

Thus according to this calculation, the veto override adds acting power of .0022, or contributes .0022/.0335 = .066 of the total acting power of the nation in legislation.

The Power of Combined Forces

In initiating action of a collectivity, or in preventing a collectivity from acting, the individual's power is not limited to his unilateral action. Nor is it the case that his power can merely be added to that of other members to obtain their combined power. For example, in the U.N. Security Council with 11 members, any five of the temporary members can combine to block action, the set of five having exactly the same power to prevent collective action as any one permanent member. Yet the sum of the individual power of five temporary members prevent action is not 1.0, but .44. Even less could one add, for this purpose, the earlier measure of Shapley and Shubik, despite

its apparent property of "additivity." For example, the sum of the power of five temporary members, by that measure, is only .011, compared to a power of .197 for a permanent member, although the power of such a bloc to prevent action is in fact identical to that of the permanent member.

In order to determine the power of a given subset C, it is necessary to treat them as if they were a single entity and analyze the power $P_c(N)$ and $I_c(N)$ just as was done by use of Eqs. (7) and (9) for $P_i(N)$ and $I_i(N)$. The relevant equations are:

$$P_c(N) = \frac{\sum_S (v(S) - v(S - C))}{\sum_S v(S)} , \qquad (11)$$

$$I_c(N) = \frac{\sum_S (v(S + C) - v(S))}{\sum_S (1 - v(S))} . \qquad (12)$$

where C is a given subset of members of N.

The power of a subset C of which i is a member always includes his individual power, so that the preventive power he gains by joining such a bloc is given by $P_c(N) - P_i(N)$. He gains this power, of course, at the cost of giving up some autonomy over his own vote, as I have discussed elsewhere [1968]. His power <u>within</u> the subset C, to induce it to prevent or initiate action of the larger collectivity, may be calculated from considerations similar to those with which we began: the power of the subset C to prevent action of the larger collectivity with him as a member minus the power of the subset C without him as a member to prevent such action is the basis of his power to induce the subset to take a negative position, and similarly for his power to induce the coalition to initiate action. This power is likely to be reflected in the decision rule within this subset for determining its bloc vote, although the actual decision rule used may reflect this relative power only imperfectly.

It is useful to examine one property of $P_c(N)$ as C increases in size, approaching N. It is clear that when C includes all n members

of the collectivity N, then it is a determining member for every partition of N. It is able to initiate action in all those partitions for which $v(S) = 0$, and able to prevent action for all those partitions in which $v(S) = 1$. Thus $P_N(N) = 1$, and $I_N(N) = 1$. As a consequence, then for any member i with $P_i(N) < 1$ or $I_i(N) < 1$, or both, there is an increase in $P_c(N)$ from $P_i(N)$ to 1 as C (including i) increases in size, and an increase in $I_c(N)$ from $I_i(N)$ to 1 as C increases in size. If all members other than i are alike, as in a simple committee or other voting body with undifferentiated members, then this increase can be described simply as a function of the size of C. If the members are differentiated, as in the U.N. Security Council, or in federal legislation in the U.S., $P_c(N)$ and $I_c(N)$ change as a function of the number of members of each type. It is useful to examine $P_c(N)$ and $I_c(N)$ in examples of both these kinds.

<u>Example</u>: The power of blocs in a simple five-man collectivity.

In the simple five-man collectivity examined in an earlier illustration, we may examine $P_c(N)$ and $I_c(N)$ for subsets C varying in size from 1 to 5, and for each decision rule. A tabulation may be made, as shown in Table 3, by examining the list of partitions of five-members collectivities given as Appendix A.

As Table 3 illustrates, there is a symmetry between the power to prevent action and the power to initiate action under decision rules which require m and n - m + 1 votes respectively, for action to occur. That is, this example suggests that in a simple collectivity with decision rule requiring m votes for collective action,

$$I_c(n, n - m + 1, j) = P_c(n, m, j).$$

Comparison of Eqs. (13) and (14) given below will show that this equality holds for simple collectivities of any size. Figure 1 shows the increase in power to prevent or initiate action under different decision rules, as the size of the subset C increases from 1 to 5.

<u>Example</u>: Three kinds of coalitions within the U.N. Security Council.

In the U.N. Security Council, there are three different kinds of coalition possible: within the set of permanent members, within the

Table 3

Numerator and Denominator for $P_c(n,m)$ for n=5, m=1,2,3,4,5, and Size of Subset C=1,2,3,4,5

Number Necessary for Passage (m)	Number of Winning Partitions	Number of Winning Partitions by Removal of Specific Subsets C				
		i	i,j	i,j,k	i,j,k,l	i,j,k,l,m
1	31	1	3	7	15	31
2	26	4	10	18	26	26
3	16	6	12	16	16	16
4	6	4	6	6	6	6
5	1	1	1	1	1	1

Numerator and Denominator for $I_c(n,m)$ for n=5, m=1,2,3,4,5, and Size of Subset C=1,2,3,4,5

Number Necessary for Passage (m)	Number of Losing Partitions	Number of Losing Partitions That Win by Addition of Specific Subsets C				
		i	i,j	i,j,k	i,j,k,l	i,j,k,l,m
1	1	1	1	1	1	1
2	6	4	6	6	6	6
3	16	6	12	16	16	16
4	26	4	10	18	26	26
5	31	1	3	7	15	31

set of temporary members, and between one or more permanent members and one or more temporary members. For convenience using the list of partitions of five and six member collectives given as Appendix A, the number of partitions in which coalitions of various kinds can block or initiate action can be found by inspection. The tabulation is given in Table 4.

Table 4, and accompanying Figs. 2 and 3, show the power of U.N. Security Council members (11-member council) as given by the decision rule, in all possible coalitions. The possibility of coalition, either with other permanent members or with temporary members, does not affect the power of permanent members to prevent action, for that power is already absolute, through the veto. It greatly affects, however, the power of temporary members to do so, for that power rises rapidly as the number of temporary members in a coalition increases from one to five (and of course it increases to 1 if a coalition includes a permanent member).

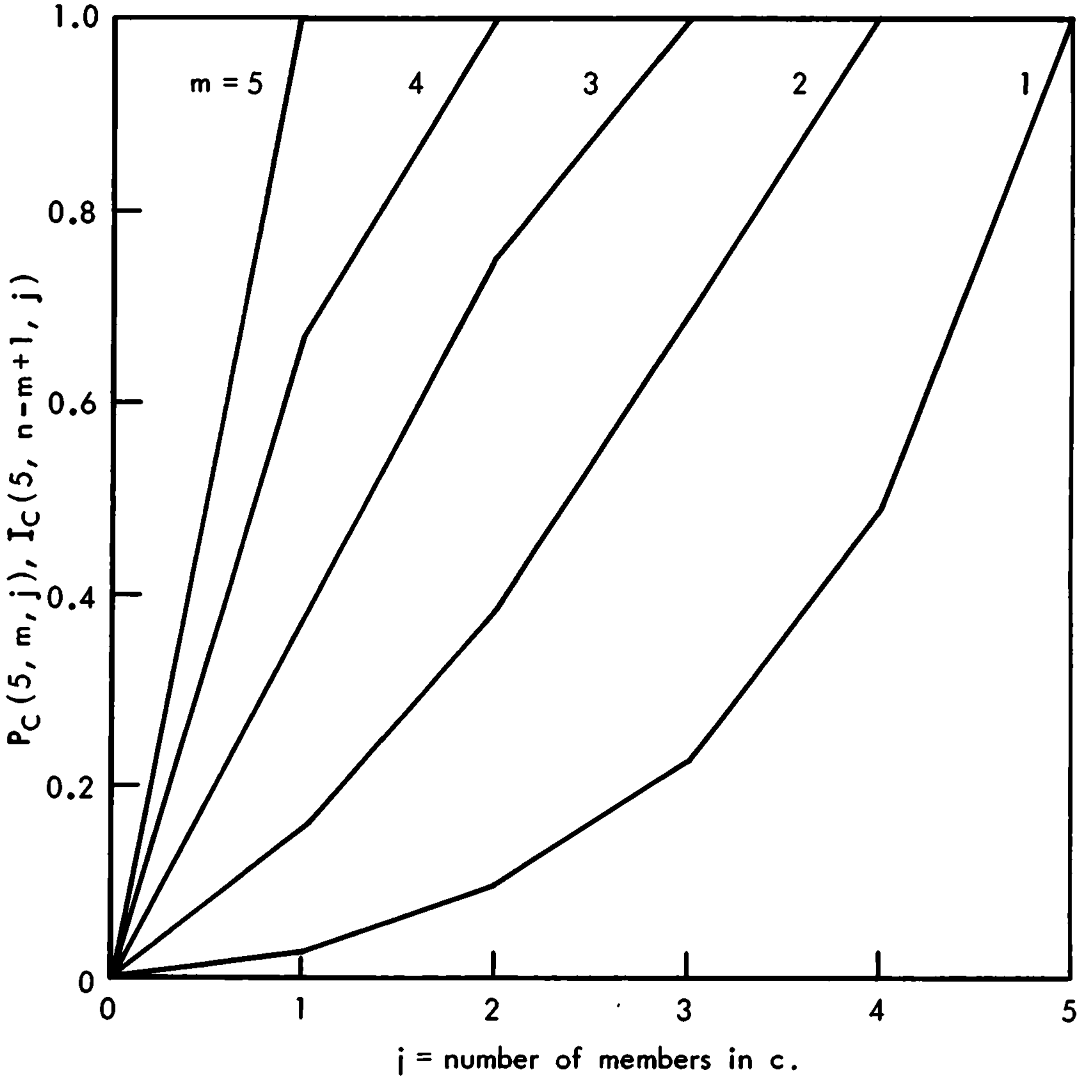
1.0
0.8
0.6
0.4
0.2
0
m = 5
4
3
2
1
$P_C(5, m, j), I_C(5, n-m+1, j)$
0
1
2
3
4
5
j = number of members in c.

Figure 1

Table 4

Numbers of Partitions of U.N. Security Council (11 Members) in Which Coalitions of Different Composition Can Prevent Action or Initiate Action

Number of Passing Partitions	Number of Failing Partitions
57	1991

a. Coalitions Within the Set of Six Temporary Members

Size of C	No. of Partitions Blocked	No. of Partitions Initiated
1	5	5
2	13	7
3	22	7
4	41	7
5	57	7
6	57	7

b. Coalitions Within the Set of Five Permanent Members

Size of C	No. of Partitions Blocked	No. of Partitions Initiated
1	57	$1 \cdot 57 = 57$
2	57	$3 \cdot 57 = 171$
3	57	$7 \cdot 57 = 399$
4	57	$15 \cdot 57 = 855$
5	57	$31 \cdot 57 = 1767$

c. Coalitions Between Permanent and Temporary Members (Only initiation is shown because one permanent member can block all 57 passing partitions)

Number of Temporary Members	Number of Permanent Members 1	2	3	4	5
1	62+5=67	$3 \cdot 62+5=191$	$7 \cdot 62+5=439$	$15 \cdot 62+5=935$	$31 \cdot 62+5=1927$
2	64+7=71	$3 \cdot 64+7=199$	$7 \cdot 64+7=455$	$15 \cdot 64+7=967$	$31 \cdot 64+7=1991$
3	71	199	455	967	1991
4	71	199	455	967	1991
5	71	199	455	967	1991
6	71	199	455	967	1991

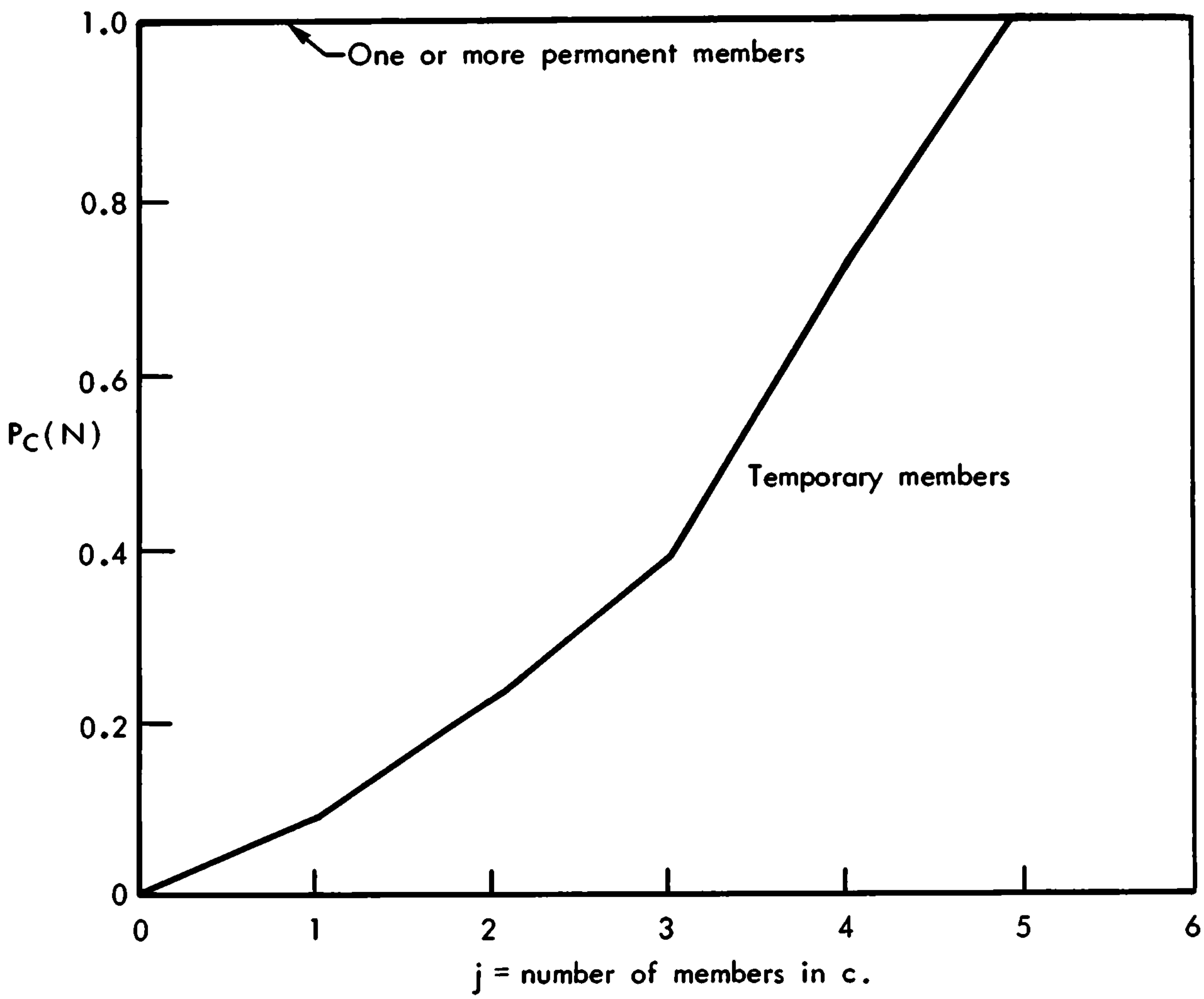
1.0
0.8
0.6
0.4
0.2
0
$P_C(N)$
One or more permanent members
Temporary members
0
1
2
3
4
5
6
j = number of members in c.

Figure 2

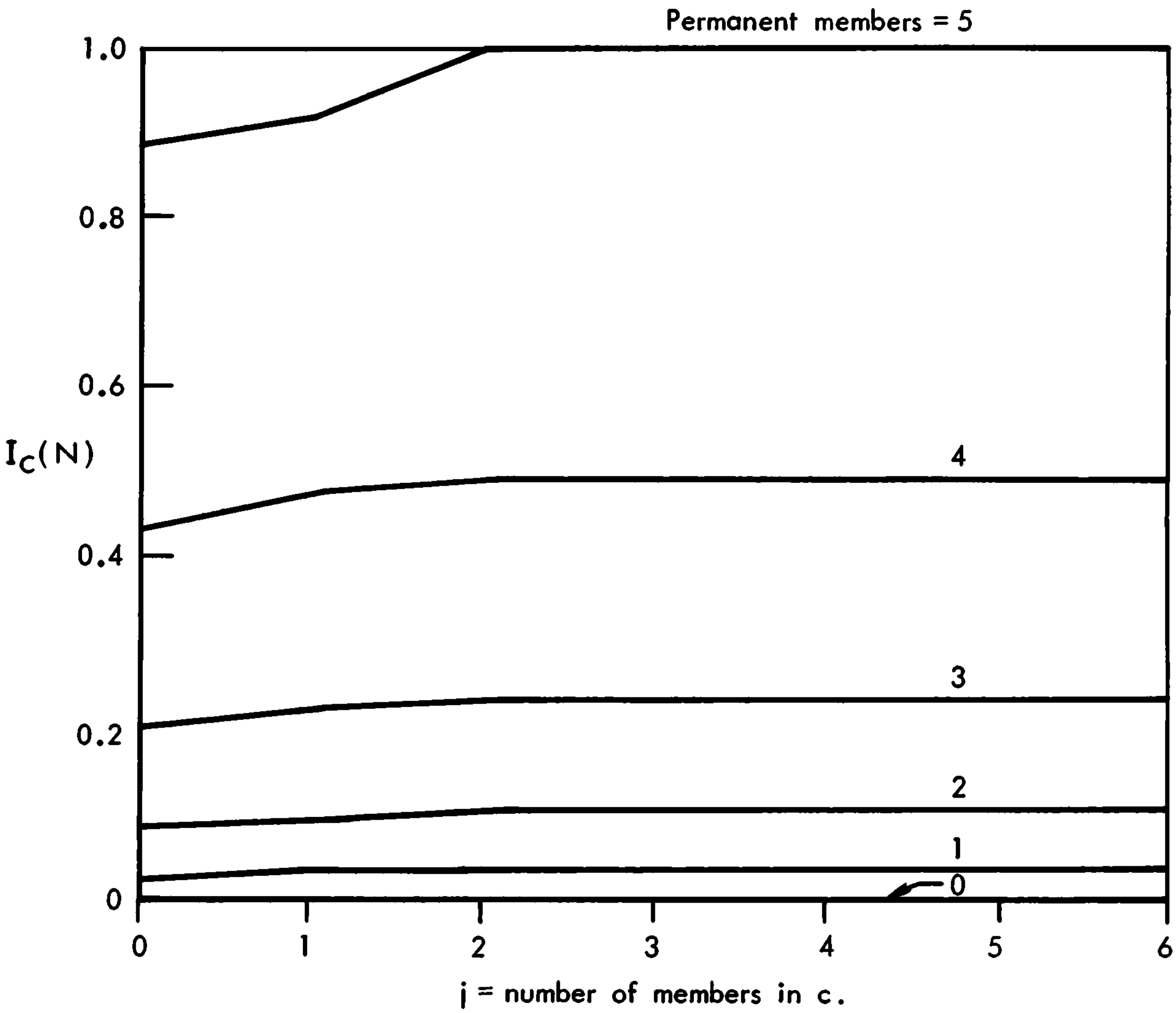
Permanent members = 5
1.0
0.8
0.6
0.4
0.2
0
$I_C(N)$
4
3
2
1
0
0
1
2
3
4
5
6
j = number of members in c.

Figure 3

The possibility of coalition among the permanent members increases sharply their power to initiate action, which rises to 0.886 with five permanent members, in the absence of any temporary members. The addition of one or two temporary members increases slightly the power of the coalition, as the slight rises in the lines of Fig. 3 shows. Overall, it is clear that the temporary members gain little power through coalition with one another except for purposes of preventing action, where their power in combination is strong. They gain much power through coalition with permanent members, but because they add little to the permanent members' power, their power within that coalition, to determine its position, will be very small. For example, in a coalition between one temporary and one permanent member, the temporary member gains power to initiate action in 62 partitions (67-5), while the permanent member gains power to initiate action in only 10 (67-57). The temporary member gains the power to block action in 52 partitions (57-5), while the permanent member gains no preventive power. As a consequence of this difference in value of the two members to the coalition, the decision rule within the coalition should give differential power to the two.

The Value Added by Coalition

Since membership in a coalition reduces a member's autonomy of action in casting his vote, unless he has complete control of the coalition, it is useful to ask the circumstance under which it will be to the benefit of all members of a potential coalition to join. This circumstance will prevail when the power of the coalition is greater than the sum of the power of the members individually--for if that is the case, some decision rule can be found such that the fraction of control over the collectivity's action held by each member, multiplied by the coalition's power, is greater than his individual power in the absence of coalition, for every coalition member.* The condition for a

*This addition of power to the power of the members assumes, however, that the appropriate decision rule is used by the coalition. For a coalition may have, as a coalition, a high amount of preventive or initiating power, but that power not be transmitted to the members through an inappropriate decision rule in the coalition.

coalition to be mutually beneficial in preventing collectivity action is:

$$P_c(N) > \sum_{i \in C} P_i(N) \ ,$$

and the condition for a coalition to be mutually beneficial in initiating collectivity action is

$$I_c(N) > \sum_{i \in C} I_i(N) \ .$$

This condition, for P or I, holds in Figs. 1 or 2 wherever the curve of $P_c(N)$ or $I_c(N)$ is concave upward. There is value added (i.e., power added) by a coalition of temporary members of the Security Council for coalition size up through five members, by the accelerating increase of $P_c(N)$ with increase in size of C. There is power added for initiating action by coalitions among permanent members of the Council. And there is a slight amount of power added for initiating action by coalitions between permanent members and one or two temporary members, as Table 4 shows. Figure 1 and Table 3 show that in the simple collectivity of 5 members, power to prevent action is gained by coalitions when m = 1 or 2, and power to initiate action is gained by coalition when m = 4 or 5.

From considering these examples, it appears that a more direct way of calculating $P_c(N)$ and $I_c(N)$ may be devised when the collectivity is a simple one with all members undifferentiated. In this case, with a decision rule requiring m positive votes for action, the power of a coalition of j members to prevent and initiate action may be labelled $P_c(n,m,j)$ and $I_c(n,m,j)$ respectively, since P and I are functions solely of n, m, and j. The power of the coalition to prevent action may be calculated by considering the number of partitions in which the collective action fails if the votes of the coalition members are missing from the positive side and entered on the negative side, minus the number of partitions in which the collective action fails before this change is made. This difference divided by the total number of

partitions in which collective action succeeds $[\sum_S n(S)]$, is the power of the coalition to prevent action. The numerator is simply the number of failures contributed by defection of the coalition, and the denominator is the number of affirmative partitions, which is $\sum_{k=m}^{n} \frac{n!}{k!(n-k)!}$.

The number of partitions in which the action fails, out of all the 2^n partitions is merely $\sum_{k=0}^{m-1} \frac{n!}{k!(n-k)!}$. The number of partitions of n - j numbers in which the action fails is $\sum_{k=0}^{m-1} \frac{(n-j)!}{k!(n-j-k)!}$, but each of these represents 2^j partitions in the original set of partitions, so that

$$P_c(n,m,j) = \left[2^j \sum_{k=0}^{m-1} \frac{(n-j)!}{k!(n-j-k)!} - \sum_{k=0}^{m-1} \frac{n!}{k!(n-k)!}\right] \frac{1}{\sum_{k=m}^{n} \frac{n!}{k!(n-k)!}}$$

or simplifying by combining terms under the summation sign,

$$P_c(n,m,j) = \left[\sum_{k=0}^{m-1} \frac{2^j(n-j)!(n-k)\ \ldots(n-j-k+1) - n!}{k!(n-k)!}\right] \cdot \frac{1}{\sum_{k=m}^{n} \frac{n!}{k!(n-k)!}} \tag{13}$$

From similar considerations, $I_c(n,m,j)$ may be calculated, recognizing that the partitions in which the collective action passes are those in which $m \le k \le n$, and the j numbers are now arbitrarily entered on the positive side (so that only m - j others are necessary for passage). The equation for $I_c(n,m,j)$ is:

$$I_c(n,m,j) = \left[2^j \sum_{k=m}^{n} \frac{(n-k)!}{(k-j)!(n-k)!} - \sum_{k=m}^{n} \frac{n!}{k!(n-k)!}\right] \frac{1}{\sum_{k=0}^{m-1} \frac{n!}{k!(n-k)!}}$$

Or simplifying by combining terms under the summation sign,

$$I_c(n,m,j) = \left[\sum_{k=m}^{n} \frac{2^j (n-k)!k \ldots (k-j+1) - n!}{k!(n-k)!}\right] \frac{1}{\sum_{k=0}^{m-1} \frac{n!}{k!(n-k)!}} \qquad (14)$$

Both these formulas, $P_c(n,m,j)$ as given by Eq. (13), and $I_c(n,m,j)$ as given by Eq. (14), are valid only over certain ranges of coalition size. If $j > n - m$, then the coalition can block all actions, and $P_c(n,m,j) = 1$. If $j \geq m$, then the coalition can initiate action against any opposition, and $I_c(n,m,j) = 1$. Equations (13) and (14) are valid for all coalitions smaller than these, that is, $j \leq n - m$ for $P_c(n,m,j)$, and $j < m$ for $I_c(n,m,j)$.

Changes in Power of the Collectivity to Act as a Result of Coalition

The power of the collectivity to act, A(N), is determined by the decision rule of the collectivity, as shown in an earlier section. But when two or more members act as a single member through coalition, this may change the power of the collectivity to act, depending on the decision rule used by the coalition itself. For example, in the U.N. Security Council, if a coalition forms between three permanent members, so that they agree to vote alike, and they agree to decide internally without a veto, by a majority of two out of three, then this reduces the effective number of permanent members to three, and reduces the total number of partitions of the Security Council vote to 2^9, or 512.* The

*The majority rule here is used in illustration, because by counting each partition equally, one implicitly assumes equal probabilities of a member voting positively or negatively. If a "member" consists of a coalition C, and its probability of voting positively, as given by A(C), is not 1/2, then the different partitions must be weighted by A(C) or 1 - A(C) depending on whether the coalition C is positive or negative in that partition.

number of affirmative partitions remains at 57, so that the acting power of the collectivity becomes 57/512, or four times its value when no coalitions existed. If, on the other hand, five of the temporary members (the optimum number for increase in preventive power) formed a coalition using a majority rule for voting within the coalition, then the Security Council in effect is reduced to seven members, one with no power and the other six with equal power. There are 2^7, or 128, partitions, and two of them (the first six members positive, and the seventh either positive or negative) are affirmative partitions, so that A(N) = 2/128 = 32/2048, which is less than the acting power of 57/2048 without that coalition. This reduction in acting power of the collectivity is directly due to the power added by the coalition of five members as discussed in the preceding section. It follows from the fact that the coalition members, in determining a decision rule within their coalition, can create a decision rule more restrictive than the 2 out of 6 decision rule that obtains in the absence of coalition. The majority rule, 3 out of 5, is more restrictive, and in effect reduces the number of affirmative partitions from 57 to 32.

It appears, then, that the power of coalition lies in the power of the coalition members to determine for themselves an internal decision rule that differs from the decision rule that implicitly obtains for the coalition when its members are acting individually. When the coalition's decision rule is more restrictive on affirmative action than the implicit decision rule when they act individually, the power of the total collectivity to act will decrease as a result of the coalition. When it is less restrictive (as in the example of a majority rule governing a coalition of three permanent members who have individual vetoes in the absence of coalition), then the power of the total collectivity to act will be increased.

Behavioral Power

This analysis is concerned wholly with the study of formal power as given by the constitutional rules of a collectivity. Yet even in the analysis of formal power, assumptions about members' behavior is implicitly made. By the device of counting each partition of the collectivity once, and adding the number of partitions in obtaining measures of

power, it is implicitly assumed that each member has equal probability of voting for or against a collective action (except in the case of formal coalition, in which case their vote is completely dependent, voting as a bloc). This is appropriate for the analysis of formal power as given by a constitution, that is, for an analysis of organizational rules. It does not, however, provide a basis for behavioral prediction of the collectivity's action, when further information exists about the members. In particular, the two assumptions made in the analysis of formal power, equal probabilities of positive and negative votes by each member, and independence of votes among members, may be empirically investigated, and the collectivity's action predicted by the use of such information.*

The way in which this may be done in the case of differing probabilities of positive and negative votes is evident by recognizing that each of the partitions counted once in the present analysis is a partition in a binomial distribution which has a weight of $p^k(1 - p)^{n-k}$, where k is the number of positive votes in that partition, and p is the probability of a positive vote from a member. If all members have $p = 1/2$, then each partition has equal weight, and the above analysis holds. If all have the same p, but $p \neq 1/2$, then each partition should be weighted by $p^k(1 - p)^{n-k}$, where k is the number of positive votes in the partition--or, what is the same thing, the probability of outcomes with m or more positive members may be obtained by using the binomial distribution, i.e., by replacing in the relevant equations those terms of the form $\sum_{k=m}^{n} \frac{n!}{k!(n - k)!}$ with $\sum_{k=m}^{n} \frac{n!}{k!(n - k)!}p^k(1 - p)^{n-k}$.

*Another implicit assumption of this analysis of formal power is the isolation of collective decisions from one another. Once this isolation is no longer assumed, then the possibility exists of trading control over one collective decision for control over another. But in the analysis of such systems of exchange, it is necessary first to carry out an analysis of formal power, as done here, in order to know exactly what constitutional power each member has as his initial resources in such an exchange market. The examination of such systems in the presence of variations in formal power of the kind examined above will be carried out in subsequent work.

If different members have different probabilities of voting positively, but all remain independent, a more detailed calculation must be made, weighting each partition separately. If p_i is the probability of a positive vote by member i, then each partition must be weighted by $\prod_{i \in S} p_i \prod_{j \notin S} (1 - p_j)$.

If there is interdependence among the votes of members of the collectivity--that is, something between total independence and a formal coalition--then the analysis becomes more difficult. If this interdependence is uniform, and in a simple undifferentiated collectivity, then it does not affect the relative power of individuals, but can affect the power of the collectivity to act, depending on the decision rule. In such a situation, stochastic models for interdependence of action, as discussed in Coleman (1964, Chapter 11), may be useful. For situations that are more complex, either in the formal structure of the collectivity or in the form of the interdependence among members, more complex models must be developed.

Appendix A

DISTINCT PARTITIONS OF FIVE- AND SIX-MEMBERED COLLECTIVITIES

S N-S

0/12345	1234/5	45/6123	1245/36
1/2345	1235/4	46/1235	1246/35
2/3451	1254/3	56/1234	1256/34
3/4512	1543/2	123/456	1345/26
4/5123	5432/1	124/563	1346/25
5/1234	12345/	125/634	1356/24
12/345		126/345	1456/23
13/452	0/123456	134/562	2345/16
14/523	1/23456	135/624	2346/15
15/234	2/34561	136/245	2356/14
23/451	3/45612	146/623	2456/13
24/513	4/56123	146/235	3456/12
25/134	5/61234	156/234	12345/6
34/512	6/12345	234/561	12346/5
35/124	12/3456	235/614	12356/4
45/123	13/4562	236/145	12456/3
123/45	14/5623	245/613	13456/2
124/53	15/6234	246/135	23456/1
125/34	16/2345	256/134	
134/52	23/4561	345/612	
135/24	24/5613	346/125	
145/23	25/6134	356/124	
234/51	26/1345	456/123	
235/14	34/5612	1234/56	
245/13	35/6124	1235/46	
345/12	36/1245	1236/45	

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