ON BALANCED GAMES WITHOUT SIDE PAYMENTS--
A CORRECTION

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The purpose of this note is to repair Lemma 6.3 of Ref. [1], which has belatedly been discovered to be false.** Fortunately, a somewhat more complicated result is available (Lemma 6.3b below) that can substitute for the faulty lemma and leave the rest of the argument of [1] intact.***

Lemma 6.3 asserted:

> "If π is in general position, then no (n-2)-dimensional affine set contains more than n - 1 members of the set $\{m_S(\pi) : S \in \mathcal{N}\} \cup \{m_N, m_{N-\{n\}}\}$."

([1], p. 18); and the following "proof" was offered:

> "If any n of the points mentioned lay in the same (n-2)-dimensional affine set, then the determinant composed of their coordinates would vanish, establishing a rational algebraic relationship among the $\pi_{S,i}$ ... [thereby contradicting the assumption that π is in general position]."

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**We are indebted to Louis Billera for first pointing out the difficulty and for making some suggestions toward its resolution. Naturally, the reader of this note should have a copy of [1] at hand.

***For clarity, a slight rewording of the proof of Lemma 6.4 is advisable: lines 11 and 12 on page 19 of [1] should read "...vertices would span an affine set K that meets $(m_N, m_{N-\{n\}}]$, in violation ...".

Unfortunately, the vanishing of this determinant need not imply anything at all about the $\pi_{S,i}$. For example, if we take $S_1 \cup S_2 \cup \ldots \cup S_n \neq N$, then the matrix of coordinates of the corresponding $m_{S_j}(\pi)$ has a zero column, making the determinant vanish independently of π. Less trivially, if we let $N = \overline{1234567}$ and take $\{S_j\} = \{\overline{12}, \overline{13}, \overline{23}, \overline{123}, \overline{456}, \overline{457}, \overline{467}\}$, then a determinant of the following form results:

$$\begin{vmatrix} + & + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & + & 0 & 0 & 0 & 0 \\ 0 & + & + & 0 & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & + & + & 0 \\ 0 & 0 & 0 & + & + & 0 & + \\ 0 & 0 & 0 & + & 0 & + & + \end{vmatrix}$$

which again vanishes independently of π.

As we shall see, this kind of counterexample can be avoided if we restrict the S_j to be members of some minimal π-balanced subset $\mathcal{B}$ of $\mathcal{N}$, a condition that is satisfied anyway in all applications of the lemma. There is another kind of counterexample, however, involving the point $m_{N-\{n\}}$; indeed, the determinant of coordinates vanishes identically whenever m_N, $m_{N-\{n\}}$, and $m_{\{n\}}(\pi)$ are among the points selected. This difficulty will be circumvented by treating the point $m_{N-\{n\}}$ in a special way in the new lemma.

We now proceed to the repair job. If A is any real matrix, then $\rho(A)$ will denote the rank of A. If A is square, then $\det(A)$ will denote the determinant of A and A_{ij} the "ij^{th} minor" of A, that is, the determinant of the submatrix of A that suppresses row i and column j, multiplied by $(-1)^{i+j}$. The following preliminary lemma will be useful.

Lemma 6.3a. *Let* $A = (a_{ij})$ *be an arbitrary* $p \times p + 1$ *real matrix and let* $\mathbf{A}$ *denote the* $p + 1 \times p + 1$ *matrix consisting of* A *with a row of 1's subjoined. Suppose that*

$$\rho(A) = \rho(\mathbf{A}) = p.$$

Then indices i_1, j_1, j_2 *exist, with* $i_1 \neq p + 1$ *and* $j_1 \neq j_2$, *such that* $a_{i_1j_1}, a_{i_1j_2}, \mathbf{A}_{i_1j_1}, \mathbf{A}_{i_1j_2}$ *are all nonzero*.

Proof. Since $\rho(A) = p$, there is a j_1 such that $\mathbf{A}_{p+1,j_1} \neq 0$. Expanding $\det(\mathbf{A})$ on column j_1, we have

$$\det(\mathbf{A}) = \sum_{i=1}^{p} (\pm a_{ij_1}\mathbf{A}_{ij_1}) \pm \mathbf{A}_{p+1,j_1}.$$

Since $\rho(\mathbf{A}) = p$, this determinant vanishes; hence there is an $i_1 \neq p + 1$ such that $a_{i_1j_1}\mathbf{A}_{i_1j_1} \neq 0$. Similarly, expanding $\det(\mathbf{A})$ on row i_1 yields a $j_2 \neq j_1$ such that $a_{i_1j_2}\mathbf{A}_{i_1j_2} \neq 0$. Thus, $a_{i_1j_1}, a_{i_1j_2}, \mathbf{A}_{i_1j_1}, \mathbf{A}_{i_1j_2}$ are all nonzero, as claimed.

Lemma 6.3b. *If* π *is in general position and if* $\mathcal{B}$ *is* π*-balanced with* $|\mathcal{B}| = n$, *then every* n *members of the set* $\{m_S(\pi) : S \in \mathcal{B}\} \cup \{m_N\}$ *are linearly independent. Moreover,* *if* K *is the affine set spanned by any* $n - 2$ *members of* $\{m_S(\pi) : S \in \mathcal{B}\}$, *then* $K \cap (m_N, m_{N-\{n\}}]$ *is empty.*

Proof. Let $S_1, \ldots, S_n$ be the members of $\mathcal{B}$, and define the $n \times n$ matrix $B = (b_{ij})$ by

$$b_{ij} = \begin{cases} \pi_{S_i,j} & \text{if } j \in S_i, \\ 0 & \text{if } j \notin S_i. \end{cases}$$

The rows of B are nonzero multiples of the corresponding vectors $m_{S_i}(\pi)$; hence if the latter were not linearly independent then $\det(B) = 0$. This would mean that the π-balancing weights for $\mathcal{B}$ were not unique, and hence that some proper subset of $\mathcal{B}$ would also be π-balanced. Accordingly, let $\mathcal{B}^*$ be a *minimal* π-balanced subset of $\mathcal{B}$ with $|\mathcal{B}^*| < n$, and let $p = |\mathcal{B}^*|$. Then we can find a $p \times p + 1$ submatrix A of B with $\rho(A) = p$ (or $\mathcal{B}^*$ would not be minimal) and $\rho(\underline{A}) = p$ (or $\mathcal{B}^*$ would not be π-balanced). Lemma 6.3a now provides indices i_1, j_1 such that both $b_{i_1 j_1}$ and the corresponding minor in $\underline{A}$ are nonzero. Hence, in the equation $\det(\underline{A}) = 0$ the variable $\pi_{S_{i_1 j_1}}$ appears with a nonzero coefficient, giving us a nontrivial rational algebraic relationship among the $\pi_{S,i}$. Since this is forbidden by hypothesis, the vectors $m_{S_i}(\pi)$ must be linearly independent.

Next, suppose that some $n - 1$ of these vectors, together with m_N, were linearly dependent. Let A be the matrix consisting of the corresponding $n - 1$ rows of B. Then

$\rho(A) = \rho(\underset{\sim}{A}) = n - 1$. Hence Lemma 6.3a could be applied and would yield exactly the same contradiction as before. This completes the proof of the first sentence of Lemma 6.3b.

Finally, denote by L the line through the points m_N and $m_{N-\{n\}}$, and note that L also passes through $m_{\{n\}}(\pi) = m_{\{n\}} = (0, \ldots, 0, 1)$. Without loss of generality, let K be spanned by $m_{S_1}(\pi), \ldots, m_{S_{n-2}}(\pi)$. Define

$$\underset{\sim}{A} = \begin{bmatrix} b_{11} & \ldots & \ldots & b_{1n} \\ \ldots & & & \ldots \\ b_{n-2,1} & \ldots & \ldots & b_{n-2,n} \\ 0 & \ldots & 0 & 1 \end{bmatrix}$$

The independences already established reveal that $\rho(A) \geqq n - 2$ and $\rho(\underset{\sim}{A}) \geqq n - 1$. Since obviously $\rho(A) \leqq \rho(\underset{\sim}{A}) \leqq \rho(A) + 1 \leqq n$, there are three possibilities to consider.

(i) : $\rho(A) = n - 2$, $\rho(\underset{\sim}{A}) = n - 1$. Then $(0, \ldots, 0, 1)$ is a linear combination of the first $n - 2$ rows of $\underset{\sim}{A}$. This means that $m_{\{n\}}$ is actually an affine combination of $m_{S_1}(\pi), \ldots, m_{S_{n-2}}(\pi)$, and so is an element of K. Hence L and K intersect at $m_{\{n\}}$. But this is the only point of intersection, since L is a line and contains points (e.g., m_N) that we know are not in K. In particular, the interval $(m_N, m_{N-\{n\}}]$ (which does not include $m_{\{n\}}$) is disjoint from K.

(ii) $\rho(A) = \rho(\underset{\sim}{A}) = n - 1$. Then Lemma 6.3a once again leads to a forbidden relationship among the $\pi_{S,i}$. (Note that the i_1 of the lemma cannot be $n - 1$, since row i_1 has two nonzero entries. This is the only way in which we use j_2.)

(iii) $\rho(A) = n - 1$, $\rho(\underset{\sim}{A}) = n$. Then K and L do not intersect, since every point in L is a linear combination of the last two rows of $\underset{\sim}{A}$, while every point in K is a linear combination of the first $n - 2$ rows of $\underset{\sim}{A}$. This completes the proof.

The original Lemma 6.3 in [1] was used explicitly only in the proof of Lemma 6.4 (twice), and implicitly only in the proof of Lemma 6.5. An inspection of these proofs reveals at once that the present Lemma 6.3b is adequate for each of these applications.

REFERENCE

[1] Shapley, L. S., "On balanced games without side payments," P-4910, The Rand Corporation, Santa Monica, California, September 1972.