NONCOOPERATIVE EXCHANGE WITH A CONTINUUM OF TRADERS

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ABSTRACT

Price formation and trade in a large exchange economy is modelled as a non-atomic noncooperative game in two contrasting ways: (1) with fiat money, with borrowing and bankruptcy permitted, and (2) with a commodity money and no borrowing. Results relating the noncooperative (Nash) equilibrium with the competitive (Walras) equilibrium are obtained for each model, and some special cases are considered. The basic problem of measurability of the strategy selection when there is a continuum of players is also considered, and a way of resolving it is proposed.

CONTENTS

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1. Introduction

A new game-theoretic approach to price formation and trade in exchange economies has recently been opened through the use of a special kind of noncooperative market game, as formulated by Shubik, Shapley, Schmeidler, and others (see Refs. [1-8]). In these games, a distinguished commodity is used as a "money" for making bids and payments at decentralized trading posts. In the first paper of this set, Shubik [1] gave an example involving two types of traders and two commodities, and he showed that if there is enough money in the game, then the prices and allocations at type-symmetric noncooperative (Nash) equilibria will converge, as the number of traders of each type increases, to competitive prices and allocations. His conjecture, soon verified, that this convergence is a quite general phenomenon, raises the possibility that a consideration of models with a continuum of traders would be fruitful.

In this paper we do not go into the economic considerations that lie behind this kind of noncooperative game (whose ancestry may be traced back to Cournot); these have been amply spelled out in [2] and [3] and elsewhere. Our present purpose is to explore some of the consequences, both substantive and methodological, of formulating these markets with a *non-atomic measure space* of economic agents. We shall

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concentrate on conditions for *equivalence* between the noncooperative equilibria (NE) and the competitive equilibria (CE). For better perspective, we shall use two contrasting models--one with a fiat money and one in which money has intrinsic worth. In a later paper we intend to consider some questions of existence, convergence, and finiteness for the NEs in these same models.*

The first of these models, treated in Secs. 3 and 4, is a non-atomic variant of one due to Shubik and C. Wilson [7] (see also [8]), in which fiat money is used for bidding and trade. Credit is unlimited and interest-free, but excessive borrowing is inhibited by *bankruptcy penalties* that are deducted from the payoffs of traders who are caught short when accounts are settled. For this model, we show (Theorem 1) that the NEs and CEs are in agreement under quite general conditions, which however require the economy to be "competitively bounded" (in a sense to be defined) and the bankruptcy penalties to be sufficiently steep. This result is in a sense a non-atomic counterpart to a result of Postlewaite and Schmeidler [5], who treat a very similar noncooperative trading model from the finite, asymptotic standpoint.

The second, contrasting model (Secs. 5 and 6) is a non-atomic version of Shubik's [1] in which the commodity used for bidding is valuable in itself and no borrowing is allowed. We show (Theorem 2) that if this commodity is plentiful and is distributed in a not-too-skewed manner, then any NE will find *most* of the traders behaving competitively--i.e., optimizing on their fixed-price Walrasian budget

*See also the paper of Jaynes, Okuno, and Schmeidler [6], who have considered a related non-atomic model.

sets. Moreover, if there are only finitely many trader types, then (Corollary 1) any type-symmetric NE will have *almost all* traders behaving in this fashion, so that the NE will be a CE as well.

Finally (Sec. 7), we take up a general methodological question that arises whenever one attempts to represent a non-atomic game in "strategic form," i.e., as a payoff distribution described as a function of the selection of strategies. The question, in brief, is how to arrange matters both technically and conceptually so that the supposedly independent decisionmakers will act in a jointly measurable way, ensuring that the integrals describing the outcome will be well defined.

2. The Underlying Non-Atomic Trading Economy $\mathcal{E}$

Let $\{T, \mathcal{C}, \mu\}$ be a non-atomic measure space of traders, where $T \equiv$ the set of traders, $\mathcal{C} \equiv$ the σ-algebra of coalitions, $\mu \equiv$ a non-atomic measure on $\{T, \mathcal{C}\}$. Trade occurs in m commodities. We shall denote by $\mathbb{R}^m$ the Euclidean space of dimension m, and by Ω^m the non-negative orthant of $\mathbb{R}^m$. Vectors in Ω^m represent commodity bundles or price vectors. For any $y \in \Omega^m$, y_j is the j^{th} component of y. 0 denotes the origin of $\mathbb{R}^m$, and also the number zero (the meaning is clear from the context). A set $S \in \mathcal{C}$ is called *null* if $\mu(S) = 0$; otherwise it is called non-null. The phrase "almost all traders" means all traders except for a null set.

Given a fixed space of traders, the specific data of the economy $\mathcal{E}$ are their initial endowments and their preferences. The endowment of trader t is written a^t; we assume that $a : T \to \Omega^m$ is a measurable function and that $\int_T a_j^t d\mu > 0$ for $j = 1, \ldots, m$. The preferences are given by utility functions $u^t : \Omega^m \to \mathbb{R}^1$; we assume that for each $t \in T$, $u^t(y)$ is continuous, concave, and nondecreasing in y. If $u^t(y)$ is strictly increasing in y_j we shall say that t *desires* j, and if y is such that $u^t(y) = \max \{u^t(y') : y' \in \Omega^m\}$ we shall say that y *satiates* t.

An *allocation* is a measurable function $x : T \to \Omega^m$ with $\int_T x d\mu = \int_T a d\mu$, describing the result of a redistribution of the commodities among the traders. A *competitive equilibrium* (CE) is an ordered pair (p, x) where $p \in \Omega^m$ is interpreted as a price vector and x is an allocation such that, for almost all t, the bundle x^t is optimal in the *budget set* $B^t(p)$, defined by

$$B^t(p) = \{y \in \Omega^m : p \cdot y \leqq p \cdot a^t\}. \tag{1}$$

In other words, for almost all $t \in T$ we have

$$\begin{cases} x^t \in B^t(p), \text{ and} \\ u^t(x^t) = \max \{u^t(y) : y \in B^t(p)\}. \end{cases} \tag{2}$$

We shall call an allocation x [price p'] *competitive* if there is a price p [allocation x'] such that (p, x) [(p', x')] is a CE. We shall call a CE (p, x) "tight" if (2) holds for *all* $t \in T$ for which the maximum can actually be attained, and call it "normalized" if $|p| = 1$, where $|\ |$ denotes the Euclidean norm. Of course, if (p, x) is a CE then so is (Kp, x), for any $K > 0$. We do not formally exclude CEs with $|p| = 0$, but we note that this can happen only in the special case where there is an allocation that satiates almost all traders. Also note that if (p, x) is any CE, then there is an $\tilde{x}$, agreeing almost everywhere with x, such that $(p, \tilde{x})$ is a tight CE.

Let us now recall the notion of "shadow prices of income"* at a CE. Suppose (p, x) is a CE. Then almost every trader's bundle x^t maximizes the concave function $u^t(y)$ over the convex set $B^t(p)$. A *shadow price for t* is a nonnegative real number λ^t such that x^t also solves the unconstrained problem of maximizing

$$u^t(y) + \lambda^t\,(p \cdot a^t - p \cdot y)$$

over all y in Ω^m. If any such number exists, there is a smallest one; denote it by $\lambda^t((p, x))$. Otherwise, set $\lambda^t((p, x)) = \infty$. Note that $\lambda^t\,((Kp, x)) = \frac{1}{K}\lambda^t((p, x))$ for all $K > 0$. Assuming that $\mathcal{E}$ has at least one CE with $|p| > 0$, we define

*Henceforth abbreviated "shadow prices."

(3) $\hat{\lambda}^t[\mathcal{E}] = \sup \{\lambda^t((p, x)) : (p, x) \text{ is a normalized, tight CE of } \mathcal{E}\}$,

and we call $\mathcal{E}$ *competitively bounded* if it has at least one CE, but none with $|p| = 0$, and if $\hat{\lambda}^t[\mathcal{E}]$ is finite for almost all $t \in T$.

Competitive boundedness is not a very satisfactory condition in itself, as it relates to the solutions rather than to the data of the model. However, we can impose more direct conditions on the u^t and a^t that will guarantee competitive boundedness. Conditions of smoothness and strict convexity of preferences, as in [9] or [10, pp. 168-176], will ensure that the economy is "regular" and so will have only a finite (odd) number of normalized, tight CEs. If in addition the gradient* of u^t is bounded, for almost all t, then the Kuhn-Tucker theorem** will provide us with finite shadow prices at each CE and hence a competitively bounded economy. Alternatively, instead of bounding the utility gradients, we could require that each commodity have a nonnull set of traders who desire it (thereby making the competitive prices positive and the budget sets compact) and that $a^t \neq 0$ for almost all t (making the budget sets full-dimensional), then again the Kuhn-Tucker theorem will yield finite shadow prices at each CE. Other, more complicated conditions that deal separately with each commodity could also be formulated.

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*Or the "superdifferential" (= set of supergradients), if we do not have differentiability.

**See e.g., Theorem 3.17 in [11, p. 52].

3. Trade With Unbounded Credit

To cast the trading economy $\mathcal{E}$ in the form of a noncooperative game $\Gamma(\mathcal{E})$, we must formulate strategy sets for the traders and a market mechanism that translates the strategies chosen into a trading outcome. As in [2] or [3] we shall suppose that there are m *trading posts*, one for each commodity. A trader is permitted to put up commodities for sale in the trading posts, and at the same time he can put up amounts of *money* for purchasing them. In the present model this is worthless paper money--for example, checks or IOUs written by the traders, or trading scrip borrowed from a bank at no interest. There is no limit on the amount of money a trader can spend, but if at the end of trading he is "in the red," i.e., did not receive enough from sales to cover his purchases, then he is declared bankrupt and must suffer a penalty. We model bankruptcy not as a detailed repossession procedure, but simply as a disutility that depends on the size of the shortfall.* On the other hand, if our trader ends up with a surplus of the paper money, he has no utility for it.

The formal treatment is as follows. The *strategy set* Z^t of trader t is given by

$$Z^t = \{s^t = (b^t, q^t) : b^t \in \Omega^m, q^t \in \Omega^m, q_j^t \leqq a_j^t\}.$$

Here b_j^t represents the *bid* of trader t at the j^{th} trading post and q_j^t the *quantity* of commodity j that he offers for sale. Given a

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*The disutility may be explained as the cost of having to liquidate other assets, not represented explicitly in the model, or as the forfeiture of some collateral that the trader may have put up as security, or as the cost of obtaining a longer-term loan, etc. Note that the creditors of a bankrupt <u>do</u> receive payment--there is no "domino" effect.

selection of strategies by the traders, how are the markets cleared? In the finite models of [2] and [3] this is straightforward: the price at each trading post is simply the ratio between the total amount bid and the total amount offered. But we are beset by a fundamental difficulty in the non-atomic case. The totals would of course be the integrals $\int_T b_j^t d\mu$ and $\int_T q_j^t d\mu$, but these make no sense unless the functions $b : T \to \Omega^m$ and $q : T \to \Omega^m$ are measurable. It is not clear how to justify this requirement. Why should independent decisionmakers behave in a jointly measurable way? Fortunately, it is possible to ignore this question if we are interested only in describing equilibrium selections, which we may *define* to be measurable. Nevertheless, it is disturbing, and in the long run perhaps unsound, to work with a game in which the strategies chosen do not always define an outcome. We shall therefore return to this question in Section 7, where a model for noncooperative but measurable strategic selection in non-atomic games will be suggested.

For the moment let us assume the necessary measurability and write, for short, $\int b_j$ for $\int_T b_j^t d\mu(t)$, etc. Then the trading price for the j^{th} commodity can be expressed as the ratio of bids to offers at the j^{th} trading post:

$$p_j = \begin{cases} \int b_j / \int q_j & \text{if } \int q_j > 0 \\ \infty & \text{if } \int q_j = 0. \end{cases} \tag{4}$$

The commodities and money are then disbursed by the trading post according to these prices, i.e., in proportion to the traders' respective bids and offers. Letting $x^t \in \Omega^m \times \mathbb{R}^1$ stand for the $(m + 1)$-vector that trader t obtains, we have

$$(5)\qquad \begin{cases} x_j^t = a_j^t - q_j^t + b_j^t/p_j & \text{for } j = 1, \ldots, m \\ \\ x_{m+1}^t = -\sum_{j=1}^{m} b_j^t + \sum_{j=1}^{m} p_j q_j^t, \end{cases}$$

where x_{m+1}^t is his final holding of money.* We must now define his utility for this outcome.

Let $\underline{x}^t$ denote the projection of x^t on Ω^m, i.e., $\underline{x}^t$ is the m-vector of real commodities obtained by t. The direct utility of this bundle is $u^t(\underline{x}^t)$, but we must add a penalty term. At the moment we take this to be of the linear form $\lambda^t \min [0, x_{m+1}^t]$, where $\lambda^t > 0$; later we shall see that any "harsher" penalty will also suffice.** The utility of the final outcome x^t to trader t is then given by

$$U^t(x^t) = u^t(\underline{x}^t) + \lambda^t \min [0, x_{m+1}^t].$$

Finally, his payoff, expressed in game theory style as a function of the strategy selection $s \in Z = \times_{t \in T} Z^t$ (assumed measurable), is

$$\pi^t(s) = U^t(x^t),$$

where x^t derives from s according to (4) and (5) above.

*If $p_j = 0$ or ∞, we define both b_j^t/p_j and $p_j q_j^t$ to be 0. (See Remark 1 in the next section.)

**Remark 2, below.

We now have a game in strategic (or "normal") form. With the rest of the data fixed, it depends on the penalty parameters λ^t, i.e., on the function $\lambda : T \to \Omega^1$, and so we shall denote it by $\Gamma_\lambda(\mathcal{E})$, or simply Γ_λ. A *non-cooperative equilibrium* (NE) of this non-atomic game* is a measurable $s_* \in Z$ such that, for almost all t,

$$\pi^t(s_*) = \max_{s^t \in Z^t} \pi^t(s_*|s^t),$$

where $(s_*|s^t)$ is the same as s_* except that s_*^t is replaced by s^t. An *NE allocation* is an allocation $x_* : T \to \Omega^m \times \mathbb{R}^1$ which is the final outcome produced at some NE s_* according to (4) and (5).

There is always a trivial NE of Γ_λ, namely the collection of strategies in which each trader bids and offers nothing. No one wants to be the first to enter an inactive trading post. Other, semi-trivial NE may also exist where some subset of the trading posts are inactive. But we shall focus our attention on *active NEs*, i.e., those which produce positive total bids and offers at every post. An *active NE allocation* of Γ_λ is an allocation produced at an active NE of Γ_λ. Note that an NE allocation being active does not imply that any net trade necessarily takes place.

We are now ready for our main result.

THEOREM 1. *Every active NE allocation of $\Gamma_\lambda(\mathcal{E})$ that leaves almost all traders unsatiated is competitive for $\mathcal{E}$, provided that the penalties λ^t are positive. Moreover, if $\mathcal{E}$ is competitively bounded and if each commodity j is desired by a non-null set of traders, then every competitive allocation of $\mathcal{E}$ is an active NE allocation of $\Gamma_\lambda(\mathcal{E})$, provided that $\lambda \geqq \hat{\lambda}[\mathcal{E}]$:*

*Compare Schmeidler [12].

<u>Proof</u>. Let $p \in \Omega^m$ and $x \in \Omega^m \times \mathbb{R}^1$ be the price and final outcome at an active NE. It is directly verified that $\int \underline{x} = \int a$ and that $\int x_{m+1} = 0$. Suppose $x^t_{m+1} > 0$ for t in a non-null set $S \in \mathcal{C}$. Then there is a non-null set of unsatiated traders who could improve their individual payoffs by increasing their bids, contradicting the definition of NE. Hence $x^t_{m+1} \leqq 0$ for almost all t. But since $\int x_{m+1} = 0$, this implies that $x^t_{m+1} = 0$ for almost all t. It is now immediate that $(p, \underline{x})$ is a CE.

Suppose now that $\mathcal{E}$ is competitively bounded with each j desired by a non-null set. Let x be a competitive allocation. Then there is a normalized p such that (p, x) is a CE. We must have $p > 0$, by the desirability assumption, as well as $p\cdot x^t = p\cdot a^t$ for almost all t. Let $\tilde{\lambda} : T \to \Omega^1$ be a set of shadow prices at this CE; note that $\tilde{\lambda}^t \leqq \hat{\lambda}^t[\mathcal{E}]$ for almost all t.

Turning now to the game, define the strategies $s^t_* = (b^t, q^t)$ as follows:*

$$(6) \qquad \begin{cases} b^t_j = \max\,[p_j x^t_j - p_j a^t_j,\ 0] \\ \\ q^t_j = \max\,[a^t_j - x^t_j,\ 0]. \end{cases}$$

(Thus, we have $b^t_j q^t_j = 0$ for all j.) Since $p > 0$, it is obvious that the selection $s_* = \{s^t_* : t \in T\}$ produces the price vector p and the allocation x. We claim that s_* is an NE of Γ_λ for any $\lambda \geqq \hat{\lambda}[\mathcal{E}]$.

First observe that since (p, x) is a CE with shadow prices $\tilde{\lambda}^t$, we have

$$u^t(x^t) = \max_{y \in \Omega^m} \,[u^t(y) + \tilde{\lambda}^t(p\cdot a^t - p\cdot y)]$$

*See Remark 3, below.

for almost all t. But $\lambda^t \geqq \tilde{\lambda}^t$ and $p\cdot a^t - p\cdot x^t = 0$, hence

$$u^t(x^t) = \max\,[u^t(y) + \lambda^t(p\cdot a^t - p\cdot y) : y \in \Omega^m,\ \ p\cdot a^t - p\cdot y \leqq 0\,].$$

Moreover, since $\lambda^t \min\,[0,\ p\cdot a^t - p\cdot y] < \tilde{\lambda}^t(p\cdot a^t - p\cdot y)$ whenever $p\cdot a^t - p\cdot y > 0$, we get

$$u^t(x^t) \geqq \max\,[u^t(y) + \tilde{\lambda}^t(p\cdot a^t - p\cdot y) : y \in \Omega^m,\ \ p\cdot a^t - p\cdot y > 0]$$

$$\geqq \max\,[u^t(y) + \lambda^t \min\,[0,\ p\cdot a^t - p\cdot y] : y \in \Omega^m.\ p\cdot a^t - p\cdot y > 0]$$

for almost all t. Thus we obtain

$$u^t(x^t) = \max\,[u^t(y) + \lambda^t \min\,[0,\ p\cdot a^t - p\cdot y] : y \in \Omega^m]$$

for almost all t, which translates easily into:

$$\Pi^t(s_*^t) = \max_{s^t \in Z^t} \Pi^t(s_*^t | s^t)$$

for almost all t. So we have an NE.

Of course, there is no assurance that this NE, defined by (6), is active; indeed, it is entirely possible that $x_j = a_j$ for some j, meaning that the CE we started with happens to call for no trade in some commodities. A simple trick gets around this, however.* Since $\int a > 0$ by assumption, we can find for each j a positive number ε_j and a non-null set S_j such that $a_j^t \geqq \varepsilon_j$ for all $t \in S_j$. If any trading post j should prove to be inactive, then we merely modify (6) so that $b_j^t = p_j\varepsilon_j$ and $q_j^t = \varepsilon_j$ for all $t \in S_j$. This is feasible and does not disturb the final outcome x. Moreover, no trader outside

*Or see Remark 3, below.

S_j would gain by entering the market for j, since p_j is a competitive price, nor would the traders in S_j have anything better to do individually than to buy back what they just sent in. So the equilibrium is preserved and, repeating this modification for other j's if necessary, we obtain the desired active NE.* Q.E.D.

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*The issue of inactive trading posts is not so easily disposed of in other variants of the game--e.g., when the rules require that $b_j^t q_j^t = 0$ for all t, j. Inactive trading posts in an NE are of two basically different types: those that possess a "virtual price" at which no one would want to buy or sell, even if goods or bids were available, and those that do not possess such a price; see the discussion of this point in [2] or [3].

4. Example and Remarks

The various conditions given in Theorem 1 may be better understood if we consider the following example, in which the NE/CE "equivalence" breaks down completely in the presence of satiation and competitive unboundedness. Let $m = 1$, and let the trader space be $T = [0, 1]$ with Lebesgue measure. Let $a^t = 3$ for $0 \leqq t \leqq 1/2$ (the "rich" traders) and let $a^t = 1$ for $1/2 < t \leqq 1$ (the "poor" traders). For all traders, let

$$u^t(x) = \begin{cases} x - x^2/4 & \text{for } 0 \leqq x \leqq 2, \\ 1 & \text{for } 2 \leqq x < \infty \end{cases}$$

(see the figure), and let $\lambda^t \equiv L > 0$.

There are many NEs in this game. To describe one simple family of them, let y be a number satisfying $0 \leqq y < 1$ and consider the following strategy selection:

$$\text{rich } t: \begin{cases} b^t = 0 \\ q^t = y \end{cases} \qquad \text{poor } t: \begin{cases} b^t = \dfrac{y(1 - y)}{2L} \\ q^t = 0. \end{cases}$$

The market price will then be

$$p = \frac{1 - y}{2L}.$$

The rich traders will wind up with $3 - y$ units of the commodity, plus some worthless money. Being satiated, they are obviously optimal

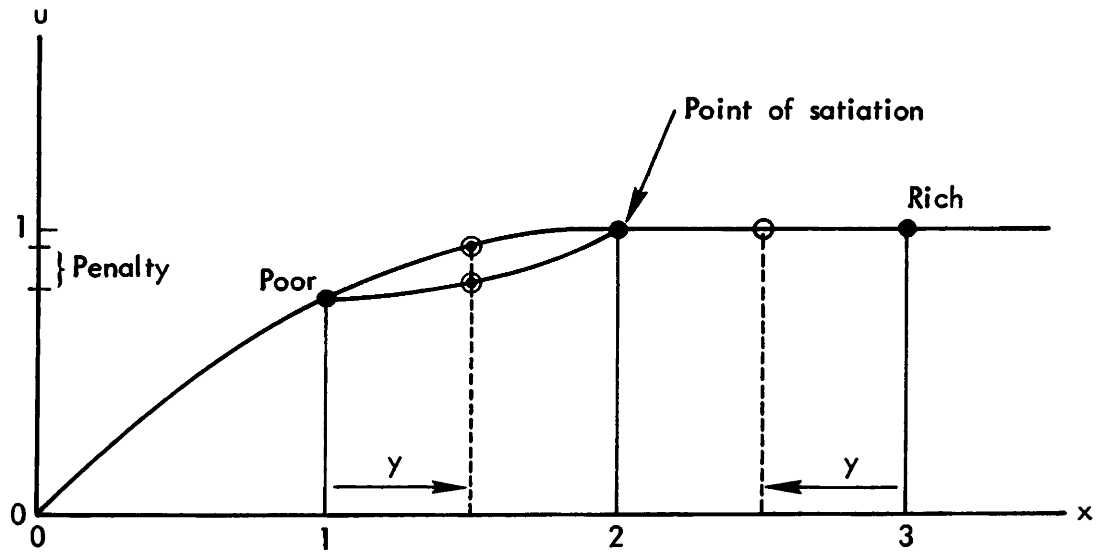

Fig. 1

w.r.t. the given selection. The poor traders will be in the "penalty zone," but will have $1 + y$ units of the commodity. At that level of consumption, their direct marginal utility is $1 - (1 + y)/2 = (1 - y)/2$. Since this is equal to pL, the penalty rate for additional purchases, they too find themselves optimal w.r.t. the given selection.* So we have an NE for any y in the designated range. Note that for $y = 0$ the NE is inactive.

The CE situation is rather trivial, since there is only one good. (We could easily add more goods without changing matters much, however.) There are essentially just two CEs. One is given by $p > 0$ and $x^t \equiv a^t$; it corresponds to the inactive NE at $y = 0$. The other is given by $p = 0$ and $x^t \equiv 2$. This CE corresponds to the limit NE as $y \to 1$. But taking $y = 1$ does not actually yield an NE, because no money is bid. In fact, $x^t \equiv 2$ is not an NE allocation for any Γ_λ with $\lambda > 0$, even if λ^t is not constant--this is the consequence of the competitive unboundedness due to $p = 0$. On the other hand, none of the active NEs at $0 < y < 1$ yield competitive allocations--this is the consequence of the satiation of the rich traders.

Remark 1.

How are we to interpret a market that is half active? Say, for example, we have $\int q_j > 0 = \int b_j$. We would expect that an individual trader could "make a killing" here, acquiring a lot of goods for just a small positive bid b_j^t. This is in fact how the finite-trader versions work. But in our non-atomic setting, the individual traders are only

*Their final utility turns out to be $(3 + y^2)/4$ (independently of the penalty coefficient L), as shown by the lower curved segment in the figure.

allowed to play with infinitesimals. Numbers like a_j^t, b_j^t must be treated as *densities*, not as real quantities of goods or money, and it would be meaningless to set $x_j^t = \int q_j$, since this equates a density with an integral. It would be more realistic to set $x_j^t = \infty$ if t is the only bidder, but this creates other problems. So in our formal model, we arbitrarily award an individual entering on the "short" side of a half-active market exactly nothing. This introduces a kind of spurious stability into what ought to be an extremely unstable situation. But, fortunately, it does not create artificial, half-active NEs, as there is instability from the other side of the market: indeed, the traders on the "long" side will always prefer to withdraw their goods or bids, since they are getting nothing in return.

Remark 2

Theorem 1 continues to hold if we replace the linear penalties λ^t by any set of "harsher" penalties. Precisely, given $\lambda: T \to \Omega^1$, let Φ denote a family of penalty functions $\Phi^t : \Omega^m \times \mathbb{R}^1 \to \Omega^1$, and let $\Phi \succ \lambda$ mean that $\Phi^t(x^t) \geqq \lambda^t|x_{m+1}^t|$ if $x_{m+1}^t < 0$ and $\Phi^t(x^t) \equiv 0$ if $x_{m+1}^t \geqq 0$, for all t. Let $\Gamma_\Phi(\mathcal{E})$ denote the unbounded-credit game derived from $\mathcal{E}$ in which the payoff to t is $u^t(\underline{x}^t) - \Phi(x^t)$. Then in both parts of Theorem 1 we may replace $\Gamma_\lambda(\mathcal{E})$ with $\Gamma_\Phi(\mathcal{E})$, $\Phi \succ \lambda$. The proof presents no difficulty; note that the functions Φ^t need not be continuous, or even monotonic in x_{m+1}^t.

In certain cases, discontinuous penalties are useful. For example, competitive boundedness may fail only because at some CE the utilities are infinitely steep on the budget sets of some traders t with $p \cdot a^t = 0$, giving them in effect infinite shadow prices for income.

But if we introduce penalties discontinuous at $x^t_{m+1} = 0$ for these traders, and if the economy is otherwise competitively bounded and the desirability condition holds, then again every CE allocation will be an NE allocation.

Remark 3

Instead of the strategies (6), we could have used the strategies

(7) $$\begin{cases} b^t_j = p_j x^t_j \\ q^t_j = a^t_j \end{cases}$$

in the proof of Theorem 1, since they produce an NE with the same prices and allocation. This might be considered a simplification, since the problem of inactive markets is avoided. It also shows that Theorem 1 holds equally well for the variant of the game in which all goods are required to be cleared through the market; such a "sell-all" rule (see Sec. 5) might be appropriate for situations where assets are "monetized" in each time period (i.e., evaluated at prevailing market prices), or in which the traders for some reason do not enjoy physical possession of their initial endowments, but merely have claims on the proceeds of sale (see the discussion in [3]). Technically, the restriction $q^t_j \equiv a^t_j$ also serves to cut down the great multiplicity of NEs that arise in the unrestricted version--an indeterminacy that is especially bothersome in the finite-trader case where the multiple solutions, unlike (6) and (7), do not fall into equivalence classes that yield the same prices and final allocations.*

*Another way to attack the multiplicity of NEs is by the "either-or" rule $b^t_j a^t_j \equiv 0$, mentioned in an earlier footnote. But this has the possible conceptual disadvantage of granting to coalitions or partnerships a strategic freedom that is denied to single players.

5. Trading Using a Commodity Money, With No Credit

The rules of the game in our second model are generally as before, but with three important differences: (1) a real commodity is now identified as money (or "cash") and used for bidding; (2) every trader is required to put up *all* of his non-money holdings for sale; and (3) no credit is allowed. For convenience we shall number the commodities up to m+1, with "cash" being the $(m + 1)^{st}$. The initial endowments are now given by a measurable function $a : T \to \Omega^{m+1}$ and the utility functions are now $u^t : \Omega^{m+1} \to \mathbb{R}^1$, $t \in T$.

In addition to our previous assumptions on u^t, we shall require that the cash commodity, unlike the fiat money of the previous model, is always desired to a certain extent, even in comparison with other commodities that may be held in very small amounts. Precisely, we shall require:

ASSUMPTION A. *For any* $\delta > 0$ *and for each* $j=1, \ldots, m$, *there is a number* $B(\delta)$ *such that*

$$\frac{u_j^t(x)}{u_{m+1}^t(x)} < B(\delta)$$

holds for all $x \in \Omega^{m+1}$ *with* $x_j \geqq \delta$ *and for all* $t \in T$.

Here $u_j^t(x)$ denotes the j^{th} partial derivative of u^t evaluated at $x \in \Omega^{m+1}$, if u^t is differentiable there. More generally, it denotes any *supergradient* of the function u^t (regarded as a concave function of the single variable x_j), evaluated at that point. Under this assumption

we shall show in Theorem 2 that if there is a "large" amount of money in the market, initially distributed in a "non-skewed" manner, then the active NE allocations are "nearly" competitive. Of course, these terms in quotation marks will all have to be given precise meanings.

First let us set up the noncooperative game $\Gamma(\mathcal{E})$. Since all goods are automatically put up for sale, the strategies now consist only of bids b_j^t, $j = 1, \ldots, m$, sent to the m trading posts. Moreover, with no credit available, a trader may use only the cash on hand:

$$Z^t = \{b^t \in \Omega^m : \sum_{j=1}^{m} b_j^t \leqq a_{m+1}^t\}.$$

This rule eliminates any bankruptcy problems and also eliminates any need for a central clearing house to settle the accounts. Prices are formed and the goods and money delivered exactly as before (compare (4), (5)): That is, $p \in \Omega^m$ and $x^t \in \Omega^{m+1}$ are defined by

$$p_j = \int b_j / \int a_j$$

$$x_j^t = \begin{cases} b_j^t/p_j & \text{if } p_j > 0 \\ 0 & \text{if } p_j = 0^* \end{cases}$$

for $j = 1, \ldots, m$; and

$$x_{m+1}^t = a_{m+1}^t - \sum_{j=1}^{m} b_j^t + \sum_{j=1}^{m} p_j a_j^t.$$

The payoff to player t is the utility of his final bundle, $u^t(x^t)$. The NEs and active NEs of this game are defined exactly as before.

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*See Remark 1 above.

Given a price vector $p \in \Omega^m$ at an active NE, we denote by $\tilde{p}$ the (m+1)-vector obtained by setting $\tilde{p}_j = p_j$ for $j = 1, \ldots, m$ and $\tilde{p}_{m+1} = 1$. At an active NE $b_* : T \to \Omega^m$, we define a trader t to be *interior* if $\Sigma_{j=1}^m b_j^t < a_{m+1}^t$, i.e., if he refrains from using all of his "cash" in the bidding.

<u>LEMMA 1</u>. *Suppose* (p, x) *are the prices and allocation at an active NE for* $\Gamma(\mathcal{E})$. *For almost all* $t \in T$, *if* t *is interior, then* x^t *is optimal for* t *on the budget set* $B^t(\tilde{p})$ *(See* (1)).

<u>PROOF</u>. Let $b_* : T \to \Omega^m$ be the NE, and let $u^t(x^t(b^t))$ be the payoff to t regarded as a function of his strategy b^t, keeping the other strategies fixed according to b_*. It can be shown without difficulty that this function is concave. Then b_*^t is the solution to

$$\text{Max}\left\{u^t(x^t(b^t)) : b^t \in \Omega^m, \sum_{j=1}^m b_j^t \leqq a_{m+1}^t\right\}.$$

By the Kuhn-Tucker theorem there is a multiplier $\sigma \geqq 0$ such that b_*^t also solves

$$\text{Max}\left\{u^t(x^t(b^t)) + \sigma^t(a_{m+1}^t - \sum_{j=1}^m b_j^t) : b^t \in \Omega^m\right\}$$

and such that

$$\sigma^t(a_{m+1}^t - \sum_{j=1}^m b_{*j}^t) = 0.$$

Since trader t is interior, the second factor is positive; hence $\sigma^t = 0$. But then b_*^t solves the unconstrained problem

$$\text{Max}\left\{u^t(x^t(b^t)) \ : \ b^t \in \Omega^m\right\},$$

which implies that $x^t(b_*^t)$ is optimal on $B^t(\tilde{p})$, as claimed. Q.E.D.

We wish to talk about markets in which there is a large amount of money, initially distributed in a "non-skewed" manner. For short, we shall denote the total amount of money $\int a_{m+1}$ by M. To make precise the idea that the distribution is not too skewed, let $0 < \sigma \leqq 1$ and define the function $a_{m+1} : T \to \Omega^1$ to be σ-fair if

$$\frac{1}{M} \int_S a_{m+1}^t d\mu(t) \geqq \frac{\sigma\mu(S)}{\mu(T)} \tag{8}$$

for all $S \in \mathcal{C}$. Intuitively, this means that any fraction f of the population has at least the fraction σf of the money.* Or, at the level of the individual trader, σ-fairness implies that the density a_{m+1}^t is at least $\sigma M/\mu(T)$ for almost all $t \in T$.

Keeping all other data of the market fixed, i.e., T, $\mathcal{C}$, μ, m, $\underline{a}$, and u, we denote by $\underset{\sim}{E}_{\sigma,M}$ the class of economies $\mathcal{E}$ in which the money distribution a_{m+1} is σ-fair and totals M. Our aim is to study the prices and allocations associated with the active NEs of the games $\Gamma(\mathcal{E})$, $\mathcal{E} \in \underset{\sim}{E}_{\sigma,M}$ when M is large, and to compare them with the competitive prices and allocations.

*Thus, σ is equal to (or less than) the derivative at the origin of the so-called *Lorenz curve*, which is commonly used to depict the degree of concentration in the holdings of a commodity throughout a population [13].

Let $\mathcal{E} \in \underset{\sim}{E}_{\sigma,M}$. If b_* is an active NE of $\Gamma(\mathcal{E})$, with prices p and allocation x, let $Q(b_*, \mathcal{E})$ denote the set of traders t who are *not* optimal on their budget sets $B^t(\tilde{p})$; this set is clearly measurable. If $\mu(Q(b_*, \mathcal{E})) = 0$, then (p, x) is a CE. Thus one can think of the fraction $\mu(Q(b_*, \mathcal{E}))/\mu(T)$ as a measure of the departure of the NE from "competitive" behavior. We shall show that for large M and with σ bounded away from 0, this departure becomes small.

<u>THEOREM 2</u>. *Let* b_* *be an active NE of* $\Gamma(\mathcal{E})$, *for some* $\mathcal{E} \in \underset{\sim}{E}_{\sigma,M}$ *with* $0 < \sigma \leqq 1$. *Then*

$$\frac{\mu(Q(b_*, \mathcal{E}))}{\mu(T)} \leqq \frac{C(\sigma)}{M},$$

where $C(\sigma)$ *is independent of* M, $\mathcal{E}$, *and* b_*.

<u>PROOF</u>. We shall keep $\mathcal{E}$ and b_* fixed and write Q for $Q(b_*, \mathcal{E})$. If Q is null we are through. If not, let Q' denote the set* of traders who are *not* interior at b_*. By Lemma 1, $Q' \supset Q$, so Q' is non-null. For any $S \in \mathcal{C}$, denote $\int_S a^t_{m+1} d\mu$ by $\alpha(S)$; thus, $\alpha(T) = \int a_{m+1} = M$. Since the traders in Q' are not holding back any of their cash, there is some good j for which their bids total at least $\alpha(Q')/m$. So

(9) $$p_j \geqq \alpha(Q)/m \int a_j.$$

*These sets are clearly measurable.

On the other hand, $p_j \leqq M/\int a_j$, so the total amount of good j that is purchased by Q' is at least $\alpha(Q') \int a_j/mM$. Hence there is a non-null subset* Q" of Q' such that each trader in Q" purchases at least

$$\frac{\alpha(Q') \int a_j}{mM\mu(Q')} \tag{10}$$

of good j, since otherwise the total amount purchased by Q' would be less than $\mu(Q')$ times (10), a contradiction. Since a_{m+1} is σ-fair, it follows from (8) and (10) that

$$x_j^t \geqq \frac{\sigma \int a_j}{m\mu(T)} \tag{11}$$

for each trader t in Q". Now suppose that t were to decrease his bid b_{*j}^t by a small amount $\Delta > 0$. Then the change in his payoff is given approximately by

$$\Delta u^t \approx u_{m+1}^t(x^t)\Delta - u_j^t(x^t)(\Delta/p_j).$$

Since b_* is an NE, we must have $\Delta u^t \leqq 0$ for almost all $t \in Q"$. This implies, letting $\Delta \to 0$, that

$$p_j \leqq \frac{u_j^t(x^t)}{u_{m+1}^t(x^t)}$$

---

*These sets are clearly measurable.

for almost all $t \in Q''$. Applying Assumption A, we have

$$p_j \leqq B\left(\frac{\sigma \int a_j}{m\mu(T)}\right),$$

using (11). On the other hand, we have

$$p_j \leqq \frac{\sigma M\mu(Q')}{m\mu(T) \int a_j}$$

by (8) and (9), so we can conclude that

$$B\left(\frac{\sigma \int a_j}{m\mu(T)}\right) \geqq \frac{\sigma M\mu(Q')}{m\mu(T) \int a_j},$$

i.e.,

$$\frac{M\mu(Q')}{\mu(T)} \leqq \frac{m \int a_j}{\sigma} B\left(\frac{\sigma \int a_j}{m\mu(T)}\right). \tag{12}$$

Note that $\mu(Q) \leqq \mu(Q')$, and that m, $\int a_j$ and $\mu(T)$ are all independent of the particular choice of M, $\mathcal{E}$, and b_*, except to the extent that the index j may vary. If we let $C(\sigma)$ denote the maximum of the right hand side of (12) for $j = 1, \ldots, m$, then we obtain finally

$$\mu(Q)/\mu(T) \leqq C(\sigma)/M.$$

This completes the proof of Theorem 2.

6. The Finite Type Case. Further Remarks.

Theorem 2 falls short of being an "equivalence" theorem: the possibility remains that no amount of money pumped into the economy will suffice to make the NEs into true CEs. The trouble comes from the existence of small sets of traders whose unusual utilities and endowments keep them from being interior. We can avoid this, however, by assuming that there are only finitely many different kinds of traders.

Let us say that two traders are "of the same type" if they have identical initial endowments (including money) and identical utility functions. Let us call an NE *symmetric* if players of the same type play the same strategies. (It may be remarked that any procedure that establishes the existence of an NE can usually be used also to establish the existence of a symmetric NE.) Let us say that the economy $\mathcal{E}$ is *of finite type* if there is a "typing" partition $\mathfrak{T}$ of T into finitely many sets T_ℓ such that any two members of the same T_ℓ are of the same type. If $\mathcal{E}_0$ is of finite type and if $\mathfrak{T}$ is a typing partition for $\mathcal{E}_0$, then let $\underset{\sim}{E}^{\mathfrak{T}}_{\sigma,M}$ denote the class of all economies obtained from $\mathcal{E}_0$ by changing the money endowment so that it (i) is σ-fair, (ii) has total M, and (iii) preserves the finite type property with respect to $\mathfrak{T}$. Then we have

COROLLARY 1. *There is a constant* M_0 *such that for any* $M \geqq M_0$ *and for any* $\mathcal{E} \in \underset{\sim}{E}^{\mathfrak{T}}_{\sigma,M}$ *every active, symmetric* NE *of* $\Gamma(\mathcal{E})$ *has almost all traders optimal on their respective budget sets.*

To prove this, we merely take M_0 so large that $C(\sigma)\mu(T)/M_0$ is smaller than the smallest nonzero $\mu(T_\ell)$, and apply Theorem 2.

Remark 4

Returning to the general case, it might be pointed out that when something stronger than Assumption A is available we can use the expression (12) to provide more specific information about the possible departure from competitive behavior. In particular, if B is a *constant*, i.e., if there is a global upper bound for the ratio u_j^t/u_{m+1}^t, then we have simply

$$C(\sigma) = \frac{mB \max_j \int a_j}{\sigma},$$

making the bound on $\mu(Q)$ depend only the product σM. In other words, the relative skewness of money no longer matters--only the absolute amounts held by the traders. In fact, since σ may be taken equal to $M_{inf}\ \mu(T)/M$, where M_{inf} denotes the largest number α such that $a_{m+1}^t \geqq \alpha$ for almost all $t \in T$, we have

$$\mu(Q(b_*,\mathcal{E})) = O(1/M_{inf}).$$

Remark 5

We have not established in this paper the *existence* of NEs for either of our two models. In a future paper, however, we shall investigate the convergence properties of the NEs in the finite versions of both models. Specifically, given a "purely competitive"

sequence of finite economies $\{\mathcal{E}_n\}$ which "approaches" a non-atomic economy $\mathcal{E}$ (in the sense of Hildenbrand; see [10], page 139), if p_n is an NE price vector for $\Gamma(\mathcal{E}_n)$ and if $p_n \to p > 0$, then p will be an NE price vector for $\Gamma(\mathcal{E})$. (In this statement, Γ represents either the unbounded credit game of Section 3, with an appropriate penalty function, or the creditless commodity-money game of Section 5.) This result enables us to prove the existence of NEs by first proving it for the finite games $\Gamma(\mathcal{E}_n)$.

Also, in the unbounded credit model, the second part of Theorem 1 enables us to infer the existence of NEs whenever CEs exist, if we are allowed to choose an appropriate penalty function λ or Φ. Conditions for the existence of CEs in a non-atomic trading economy may be found e.g., in [10].

7. Noncooperative Games in "Coalitional Strategic" Form

Let $Z : T \to 2^{\Omega^m}$ be a mapping of t into subsets of Ω^m, where Z(t) is the strategy set of t. We shall assume that Z is measurable and that its graph bounded from above (component-wise) by some integrable function $b : T \to \Omega^m$. We shall imagine that each player t has an "intended strategy" $g(t) \in Z(t)$, but the map $g : T \to \Omega^m$ may not be measurable. How are we to obtain a measurable strategy selection $f : T \to \Omega^m$ with each $f(t) \in Z(t)$, starting from g?

Introduce the notion of a coalition's strategy. For any S in $\mathcal{C}$, this is a measurable function φ^S from S to Ω^m, such that $\varphi^S(t) \in Z(t)$ for each $t \in S$. We can represent it equivalently by a *vector measure* Φ^S on T, with carrier S, given by

$$\Phi^S(R) = \int_{S \cap R} \varphi^S(t) d\mu$$

for any $R \in \mathcal{C}$.

By a *preselection* we shall mean a choice of strategy, in the above sense, by every coalition in $\mathcal{C}$. We may suppose that there is some sort of decision process that arrives at a preselection, starting from the intended strategies g. The exact nature of this process is not important, though we give examples at the end of this section. All we assume for now is that the process gives rise to a preselection.

Given a preselection, there is no problem in determining an outcome for the game for any specific partition $\mathcal{P}$ of T into a finite number of disjoint coalitions $S_1, \ldots, S_p$. But different partitions $\mathcal{P}$ will in general give different outcomes. The "noncooperative" idea

will be embodied in our intention to look only at very fine partitions--passing to the limit in a way that permits the "mesh": max $[\mu(S) : S \varepsilon \mathcal{P}]$ of the partition to go to zero.

We may imagine that the referee hears only measurable instructions. The players in effect band together in order to transmit their moves coherently to the referee, but they do not band together in order to play the game cooperatively in the usual sense.

It is natural now to ask under what circumstances a preselection will lead in the limit to a measurable function b, and hence to a "most nearly noncooperative" mode of coalitional play.

Given a preselection $\Phi = \{\Phi^S : S \varepsilon \mathcal{C}\}$ and a coalition R in $\mathcal{C}$, define $\xi^R : \mathcal{C} \to \Omega^m$ by $\xi_j^R(S) = \Phi_j^S(R)$, $j = 1, \ldots, m$. If the set functions ξ_j^R are of bounded deviation* for all R in $\mathcal{C}$ and all $j = 1, \ldots, m$, then we shall say that Φ is *admissible*. It turns out that if Φ is admissible we can ensure a measurable noncooperative outcome at the limit.

Let $\underset{\sim}{P}$ denote the set of all finite partitions of T. If h is any real-valued function defined on $\underset{\sim}{P}$, define the *directed limit*

$$\overrightarrow{\lim_{\mathcal{P} \varepsilon \underset{\sim}{P}}} h(\mathcal{P})$$

to be the number λ such that, for every $\varepsilon > 0$, there is $\mathcal{P}_\varepsilon \in \underset{\sim}{P}$ such that

*A set function is said to be of *bounded deviation* if it is the difference of two superadditive functions.

$$|h(\mathscr{P}) - \lambda| < \varepsilon$$

holds for every $\mathscr{P} \in \underset{\sim}{P}$ that is a refinement of $\mathscr{P}_\varepsilon$. (Since any two partitions have a common refinement, if a directed limit exists it must be unique; see e.g., [14], p. 26.)

The noncooperative behavior of the preselection Φ at the limit can be now described by the set function $\eta : \mathcal{C} \to \mathbb{R}^m$, where

$$\eta_j(R) = \underset{\mathscr{P}\in\underset{\sim}{P}}{\overrightarrow{\lim}} \left\{ \sum_{\ell=1}^{p} \Phi_j^{S_\ell}(R) \; : \; \{S_1, \ldots, S_p\} = \mathscr{P} \right\}.$$

By Theorem 10 in [15], this limit exists for all $R \in \mathcal{C}$ and all $j = 1, \ldots, m$, provided only that Φ is admissible.

It is obvious that η is finitely additive. We assert that it is in fact countably additive. First recall that since b bounds the graph of Z from above, the inequality

$$\Phi_j^S(R) = \int_{S\cap R} \varphi_j^S(t)d\mu \leqq \int_{S\cap R} b_j(t)d\mu$$

holds for all S and R in $\mathcal{C}$ and all $j = 1, \ldots, m$. Let β represent the vector measure generated by the indefinite integral of b. First observe that, for any $R \in \mathcal{C}$,

$$\sum_{\ell=1}^{p} \varphi_j^{S_\ell}(R) \leqq \sum_{\ell=1}^{p} \beta_j(S_\ell \cap R) = \beta_j(R)$$

where $(S_1, \ldots, S_p)$ is any partition of T. This immediately implies $\eta_j(R) \leqq \beta_j(R)$ for any R and any j. Now suppose $R = \bigcup_{i=1}^{\infty} R_i$, where the R_i's are disjoint. For any k, we have

$$\eta_j(R) - \sum_{i=1}^{k} \eta_j(R_i) = \eta_j(\bigcup_{k+1}^{\infty} R_i)$$

since η_j is finitely additive.

Now β_j is a finite measure, hence $\beta_j(R) = \sum_{i=1}^{\infty} \beta_j(R_i)$, i.e., $\lim_{k\to\infty} \sum_{i=k+1}^{\infty} \beta_j(R_i) = 0$. But

$$\eta_j(\bigcup_{i=k+1}^{\infty} R_i) \leqq \beta_j(\bigcup_{i=k+1}^{\infty} R_i) = \sum_{i=k+1}^{\infty} \beta_j(R_i).$$

It follows that $\lim_{k\to\infty} \eta_j(\bigcup_{k+1}^{\infty} R_i) = 0$, and thus $\eta_j(R) = \sum_{i=1}^{\infty} \eta_j(R_i)$, showing that η is countably additive.

Note that for $j = 1, \ldots, m$, η_j is absolutely continuous with respect to β_j, and each β_j is absolutely continuous with respect to μ. Therefore each η_j is absolutely continuous with respect to μ. Let $f : \mathbb{R} \to \mathbb{R}^m$ be the Radon-Nikodym derivative of η with respect to μ. We then have $\eta = \int f d\mu$. This f represents the measurable choice of strategies of the players derived from the intended non-measurable choice g.

We now give an example of a decision process leading from a set of "intended" strategies $g(t)$, $t \in T$, to an admissible preselection. In this method, each coalition announces that its members all choose the same strategy, given by the least bid in each commodity that any member of the coalition "intends." (This entails the assumption that

the sets $Z(t)$ are comprehensive in Ω^m.) Thus, we have, for any $R \in \mathcal{C}$,

(13) $$\varphi_j^S(t) = \inf \{g_j(t') : t' \in S\},\ j = 1, \ldots, m.$$

Here it is most natural <u>not</u> to exclude sets of measure 0 in taking the infimum, as g is in general not a measurable function. It is easily verified that

$$\Phi^S(R) + \Phi^T(R) \geqq \Phi^{S \cup R}(R)$$

for all $R, S, T \in \mathcal{C}$ with $S \cap T = \emptyset$. In other words, the functions ξ^R defined above are subadditive and hence of bounded deviation, making the preselection Φ admissible as claimed.

To see how this might work, let $m = 1$ and let there be a non-measurable set $A \subset T$ of players who "intend" to play $g(t) = a$, while the complementary set $B = T \setminus A$ "intends" to play $g(t) = b > a$. Denote the inner measures of these sets by $\underline{\mu}(A)$, $\underline{\mu}(B)$. Then we can find sets $A_1, B_1 \in \mathcal{C}$ such that $A_1 \subset A$, $B_1 \subset B$, $\mu(A_1) = \underline{\mu}(A)$, $\mu(B_1) = \underline{\mu}(B)$. Let $C_1 = T \setminus (A_1 \cup B_1)$. Any measurable $R \subset C_1$ having positive measure must contain members of both A and B, so for such R we have

$$\Phi^R(S) = a\mu(R \cap S).$$

From this it follows without difficulty that η is given by

$$\eta(S) = a\mu(S \cap A_1) + b\mu(S \cap B_1) + a\mu(S \cap C_1).$$

In this example, a and b could be replaced by measurable functions of t and a similar argument would go through. Also, in the process (13) itself, we could use "sup" in place of "inf" with the aid of some additional conditions on the $Z(t)$ to assure that $\varphi^S(E)$ will be feasible. In that case, the ξ_j^R would be superadditive rather than subadditive, but still of bounded deviation.

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